Number of spin-I states for three identical particles in a single-j shell

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In this paper we derive the analytical formulas of the number of spin-I states (denoted as D_I) for three identical particles, in a unified form for both fermions and bosons. This is done by using $\bar{n}$ virtual bosons with spin 3/2, where $\bar{n}$ equals $2j-2$ if one studies fermions in a single-j shell or $2l$ if one studies bosons with spin l. We first obtain a reduction rule from U(4) to O(3) for such virtual bosons and thereby derive the formulas of D_I. The formulas thus obtained are proved to be consistent with previous empirical formulas.

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To enumerate the number of spin-I states (denoted as D_I) for n identical particles in a single-j shell is a fundamental practice in nuclear structure theory. A straightforward way to obtain D_I is to subtract the combinatorial number of the angular momentum projection $M = I + 1$ from that with $M = I$, in the m scheme [1]. One may also obtain D_I by the seniority scheme [2] and the generating functions [3,4].

There have been many efforts to obtain algebraic expressions of D_I [5–11]. The first effort was made by Ginocchio and Haxton [5], who obtained a simple formula of D_0 for $n = 4$ in studies of the fractional quantum Hall effect. In Ref. [6], two of the present authors, Zhao and Arima, found empirical formulas for three and four particles and some of five particles. Zamick and Escuderos revisited the Ginocchio-Haxton formula for $I = 0$ with $n = 4$ by a careful consideration of the combinatorial number (in the m scheme) to form $I = j$ with $n = 3$ [7], which equals $D_I(n = 4)$. In Ref. [8] Talmi derived recursion formulas for D_I of n fermions in a j orbit in terms of n, $n-1$, $n-2$ fermions in a $(j-1)$ orbit, and thereby proved the empirical formulas for three fermions in Ref. [6]. In Refs. [9,10] the studies of $n = 3$ and 4 were extended to the number of states with given spin I and isospin T. In Ref. [11], Talmi's recursion formulas [8] was further generalized to boson systems and applied to prove the empirical formula for $n = 5$ bosons given in Ref. [6]. Very interestingly, the number of spin-I states, D_I, was found to be closely related to the sum rules of many six-j and nine-j symbols, and coefficients of fractional parentage [9,12–18].

In Ref. [19], it was proved that D_I for n fermions in a single-j shell or bosons with spin l equals the D_I of another "boson" system with spin $l' = n/2$, the boson number (denoted by $\bar{n}$) of which equals either $2j + 1 - n$ (if one studies D_I for n fermions in a j shell) or $2l$ (if one studies D_I for n spin-l bosons). For convenience of readers, this conclusion is explained as follows. D_I equals the combinatorial number (denoted by C_I) of $M = m_1 + m_2 + \cdots + m_n = I$ (with the requirement $j \geqslant m_1 > m_2 > \cdots > m_n \geqslant -j$ for fermions or $l \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant -l$ for bosons) subtracted by that of M_{I+1}, and this practice (called process A) can be carried out in an equivalent process as follows. We define $I' = I_{\max} - I$, where $I_{\max} = M_{\max} = nj - \frac{n(n-1)}{2}$ for fermions or nl for bosons. Let $P(n,I')$ be the number of partitions of $I' = i_1 + i_2 + \cdots + i_n$, with $2j + 1 - n \geqslant i_1 \geqslant i_2 \geqslant \cdots \geqslant i_n \geqslant 0$ for fermions or $2l \geqslant i_1 \geqslant i_2 \geqslant \cdots \geqslant i_n \geqslant 0$ for bosons, with the convention that $P(n,0) = P(n,1) = 1$. One can prove that $P(n,I')$ equals the combinatorial number of $M = I = I_{\max} - I'$, and thus $D_I = P(n,I') - P(n,I'-1)$. This method is called process B here. Now we denote $P(n,I')$ by using a series of Young diagrams, with the first row i_1 boxes, the second row i_2 boxes, ..., the n row i_n boxes. The number of such Young diagrams is equal to that of their conjugate diagrams with n columns, with the first column i_1 boxes, the second column i_2 boxes, ..., the n column i_n boxes. An example of such one-to-one correspondence is shown in Fig. 1 for $i_1 = 4$, $i_2 = 2$, and $i_3 = 1$. The number of rows for these conjugate diagrams is $2j + 1 - n$ for fermions or $2l$ for bosons. These conjugate diagrams correspond to $P(\bar{n},I')$ for bosons with spin $n/2$. This means $D_I = P(\bar{n},I') - P(\bar{n}, I'-1)$, and we call this method process C. Therefore, D_I of n fermions in a single-j shell or n bosons with spin l can be alternatively studied by using $\bar{n}$ bosons with spin $n/2$. This equivalence also leads to an interesting conclusion that D_I of n fermions in a single-j shell always equals that of n bosons with spin $l = j - \frac{n-1}{2}$. This conclusion was applied to four fermions and bosons in Ref. [19], where D_I of $n = 4$ was derived by studying D_I of d bosons. In Table I we exemplify the processes A, B, and C by using four fermions in a $j = 7/2$ shell, to illustrate *how these three different processes give the same* D_I.

The purpose of this paper is to derive the formulas of D_I for $n = 3$ by the above identity, i.e., D_I of n identical particles equals that of $\bar{n}$ bosons with spin $n/2$. Although the formulas for $n = 3$ are known, they were obtained empirically and proved by induction with respect to j. Therefore it would be desirable if they were derived. These formulas are understood

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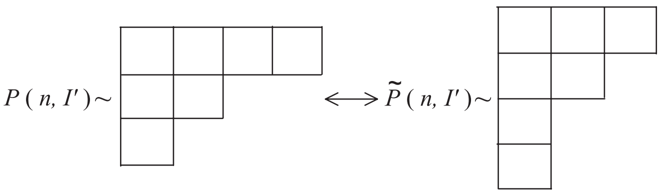

FIG. 1. The Young diagram of $P(n,I')$ and its conjugate Young diagram of $\tilde{P}(n,I') = P(\bar{n},I')$, for $i_1 = 4$, $i_2 = 2$, and $i_3 = 1$. $\bar{n} = 2j + 1 - n$ for fermions or $\bar{n} = 2l$ for bosons. The Young diagram of $P(n,I')$ corresponds to n identical particles (fermions in a single-j shell or bosons with spin l), and $P(\bar{n},I') = \tilde{P}(n,I')$ corresponds to $\bar{n}$ bosons with spin $n/2$.

deeply only if they are analytically obtained in a unified way for both fermions and bosons.

For $n = 3$, however, we should deal with $\bar{n}$ bosons with spin $3/2$ which are not realistic. We call such "bosons" virtual bosons. Below we first come to the reduction rule of symmetric representation from U(4) to O(3). [We note that reduction $U(n) \supset SO(n) \supset SO(3)$ for bosons was studied in Refs. [20–22].] Then we construct the analytical formulas of D_I for $\bar{n}$ virtual bosons with spin $3/2$. Finally we show that the results of this work are consistent with those in Ref. [6].

Let us first denote the creation and annihilation operators of our virtual bosons with spin $3/2$ by $a_m^\dagger$ and a_m ($m =$

TABLE I. Processes A, B, and C for four fermions in a single-j shell ($j = 7/2$). Here $\bar{n} = 2j + 1 - 4 = 4$. In process A we tabulate combinatorial numbers of $M = m_1 + m_2 + m_3 + m_4 = I$ with $7/2 \geqslant m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4 \geqslant -7/2$; in process B, we tabulate combinatorial numbers of $I' = I_{\max} - I = i_1 + i_2 + \cdots + i_n$ with $2j + 1 - n \geqslant i_1 \geqslant i_2 \geqslant \cdots \geqslant i_n \geqslant 0$. In process C, $i_1, i_2, \cdots i_{\bar{n}}$ are given by conjugate partitions in process B. One sees that $C_I = P(n, I_{\max} - I) = P(\bar{n}, I_{\max} - I)$. Therefore the three processes yield the same results of D_I.

A						B						C					$D_{M=I}$
M	C_I	$2m_1$	$2m_2$	$2m_3$	$2m_4$	I'	$P(n,I')$	i_1	i_2	i_3	i_4	$P(\bar{n},I')$	i_1	i_2	i_3	i_4	
8	1	7	5	3	1	0	1	0	0	0	0	1	0	0	0	0	1
7	1	7	5	3	−1	1	1	1	0	0	0	1	1	0	0	0	0
6	2	7	5	1	−1	2	2	1	1	0	0	2	2	0	0	0	1
		7	5	3	−3			2	0	0	0		1	1	0	0	
5	3	7	3	1	−1	3	3	1	1	1	0	3	3	0	0	0	1
		7	5	1	−3			2	1	0	0		2	1	0	0	
		7	5	3	−5			3	0	0	0		1	1	1	0	
4	5	5	3	1	−1	4	5	1	1	1	1	5	4	0	0	0	2
		7	3	1	−3			2	1	1	0		3	1	0	0	
		7	5	−1	−3			2	2	0	0		2	2	0	0	
		7	5	1	−5			3	1	0	0		2	1	1	0	
		7	5	3	−7			4	0	0	0		1	1	1	1	
3	5	5	3	1	−3	5	5	2	1	1	1	5	4	1	0	0	0
		7	3	−1	−3			2	2	1	0		3	2	0	0	
		7	3	1	−5			3	1	1	0		3	1	1	0	
		7	5	−1	−5			3	2	0	0		2	2	1	0	
		7	5	1	−7			4	1	0	0		2	1	1	1	
2	7	5	3	−1	−3	6	7	2	2	1	1	7	4	2	0	0	2
		5	3	1	−5			2	2	2	0		3	3	0	0	
		7	1	−1	−3			3	1	1	1		4	1	1	0	
		7	3	−1	−5			3	2	1	0		3	2	1	0	
		7	3	1	−7			3	3	0	0		2	2	2	0	
		7	5	−3	−5			4	1	1	0		3	1	1	1	
		7	5	−1	−7			4	2	0	0		2	2	1	1	
1	7	5	1	−1	−3	7	7	2	2	2	1	7	4	3	0	0	0
		5	3	−1	−5			3	2	1	1		4	2	1	0	
		5	3	1	−7			3	2	2	0		3	3	1	0	
		7	1	−1	−5			3	3	1	0		3	2	2	0	
		7	3	−3	−5			4	1	1	1		4	1	1	1	
		7	3	−1	−7			4	2	1	0		3	2	1	1	
		7	5	−3	−7			4	3	0	0		2	2	2	1	
0	8	3	1	−1	−3	8	8	2	2	2	2	8	4	4	0	0	1
		5	1	−1	−5			3	2	2	1		4	3	1	0	
		5	3	−3	−5			3	3	1	1		4	2	2	0	
		5	3	−1	−7			3	3	2	0		3	3	2	0	
		7	1	−3	−5			4	2	1	1		4	2	1	1	
		7	1	−1	−7			4	2	2	0		3	3	1	1	
		7	3	−3	−7			4	3	1	0		3	2	2	1	
		7	5	−5	−7			4	4	0	0		2	2	2	2	

3/2, 1/2, −1/2, −3/2). They follow

$$[a_m, a_{m'}] = [a_m^\dagger, a_{m'}^\dagger] = 0, \quad [a_m, a_{m'}^\dagger] = \delta_{mm'}.$$

The 16 bilinear forms, $\{a_m^\dagger a_{m'}\}$, or equivalently $(a^\dagger \times \tilde{a})^{(f)}_\mu$ [$f = 0, 1, 2, 3$, and $\tilde{a}_\mu = (-)^{3/2-\mu} a_{-\mu}$], generate the U(4) algebra; $(a^\dagger \times \tilde{a})^{(f)}_\mu$ with $f = 1$ and 3 generate the Sp(4) algebra; and $(a^\dagger \times \tilde{a})^{(1)}_\mu$ generate the SO(3) algebra. These algebras form a group chain U(4) ⊃ Sp(4) ⊃ SO(3). Let us denote symmetric irreducible representation of U(4) and Sp(4) by $[\bar{n}, \dot{0}] \equiv [\bar{n}, 0, 0, 0]$ and $\langle \bar{n}, 0\rangle$, respectively. By using angular momentum coupling and recoupling techniques [23], or straightforwardly by expanding in terms of Clebsch-Gorden coefficients, one can prove that

$$(a^\dagger \times a^\dagger)^{(f)}_\mu = 0 \quad \text{for} \quad f = 0 \text{ or } 2,$$
$$(a^\dagger \times a^\dagger)^{(3)}_3 \sim (a^\dagger_{3/2})^2, \quad (a^\dagger \times a^\dagger \times a^\dagger)^{(1/2)}_\mu = 0,$$
$$(a^\dagger \times a^\dagger \times a^\dagger)^{(5/2)}_{5/2} \sim a^\dagger_{3/2}(a^\dagger \times a^\dagger)^{(1)}_1,$$
$$(a^\dagger \times a^\dagger \times a^\dagger)^{(9/2)}_{9/2} \sim (a^\dagger_{3/2})^3.$$

Therefore one may use the following four linearly independent operators $A = a^\dagger_{3/2}$, $V = (a^\dagger \times a^\dagger)^{(1)}_1$, $S = (a^\dagger \times a^\dagger \times a^\dagger)^{(3/2)}_{3/2}$, and $U = (a^\dagger \times a^\dagger \times a^\dagger \times a^\dagger)^{(0)}_0$ to construct the basis vectors (up to a normalization constant)

$$|[\bar{n}, 0, 0, 0], \langle \bar{n}, 0\rangle, \alpha, I, I\rangle = A^{n_1} V^{n_2} S^{n_3} U^{n_4} |0\rangle. \quad (1)$$

Because the particle number is a conserved quantity, it is obvious that $\bar{n} = n_1 + 2n_2 + 3n_3 + 4n_4$. Furthermore, because A, V, S, and U are rank 3/2, 1, 3/2, and 0 irreducible tensor operators of O(3) with maximal angular momentum projection onto the z axis, directly using commutation relations of J^2 operator of O(3) with A, V, S, and U, one can prove that the total spin of Eq. (1) is

$$I = \tfrac{3}{2} n_1 + n_2 + \tfrac{3}{2} n_3. \quad (2)$$

For example, we have $[J_z, A^{n_1}] = \frac{3n_1}{2} A^{n_1}$; since the Casimir operator $J^2 = J_- J_+ + J_z(J_z + 1)$, and $[J_+, J_-] = 2J_z$, and $[J_z, J_\pm] = \pm J_\pm$, one has $[J^2, A^{n_1}] = \frac{3n_1}{2}(\frac{3n_1}{2} + 1)A^{n_1}$. Similar commutation relations follow for V^{n_2}, S^{n_3}, and U^{n_4}. In Eqs. (1) and (2), $n_1 = 0, 1, 2, \ldots, \bar{n}$; $n_2 = 0, 1, 2, \ldots, [\bar{n}/2]$; $n_3 = 0, 1$, and $n_4 = 0, 1, 2, \ldots, [\bar{n}/4]$. [ ] means to take the largest integer not exceeding the value inside. α is multiplicity in the reduction Sp(4) → O(3). Since $S^2 \sim A^2 U$, only $n_3 = 0$ or 1 cases should be considered when S and A are used to construct Eq. (1).

Let us define $\lambda = n_1 + 2n_2 + 3n_3$. Equation (2) can be rewritten as

$$\bar{n} = 4n_4 + \lambda, \quad I = \tfrac{3}{2}\lambda - 2n_2 - 3n_3 = \tfrac{1}{2}\lambda + n_1. \quad (3)$$

This is the key to obtaining the D_I for $n = 3$ here. According to Eq. (3), the allowed spin I is given by

$$I = \tfrac{3}{2}\lambda, \ \tfrac{3}{2}\lambda - 2, \ \tfrac{3}{2}\lambda - 3, \ldots, \tfrac{1}{2}\lambda, \quad (4)$$

for a given λ defined in Eq. (3). One sees that there is no $I = \frac{1}{2}$ state for three fermions in a single-j shell, because $\lambda = 1$ presents the $I = \frac{3}{2}$ state, and the $\frac{3}{2}\lambda - 1$ state is missing.

Our process to obtain D_I of $\bar{n}$ virtual bosons with spin 3/2 is explained in three steps:

(i) Let us define

$$k = [\bar{n}/4], \quad \kappa = \bar{n} \mod 4.$$

Apparently, $\kappa = 0, 2$ correspond to bosons, and $\kappa = 1, 3$ correspond to fermions (because $\bar{n} = 2j - 2$ for fermions and $2l$ for bosons). $\lambda = \bar{n} - 4n_4 = 4(k - n_4) + \kappa$.

(ii) According to Eq. (4), for a given spin I, the allowed λ follow

$$\tfrac{2}{3} I \leqslant \lambda \leqslant 2I \quad \text{for} \quad I \leqslant \bar{n}/2,$$
$$\tfrac{2}{3} I \leqslant \lambda \leqslant \bar{n} \quad \text{for} \quad I \geqslant \bar{n}/2, \quad (5)$$

with $\lambda \neq 2(I + 1)/3$. Because $\lambda = \bar{n} - 4n_4 = 4(k - n_4) + \kappa$, we obtain

$$\frac{2I - 3\kappa}{12} \leqslant k - n_4 \leqslant \frac{2I - \kappa}{4} \quad \text{for} \quad I \leqslant \bar{n}/2,$$
$$\frac{2I - 3\kappa}{12} \leqslant k - n_4 \leqslant k \quad \text{for} \quad I \geqslant \bar{n}/2, \quad (6)$$

with $2I \neq 12(k - n_4) + 3\kappa - 2$.

(iii) D_I equals the number of allowed $(k - n_4)$ for a given spin I. From Eq. (6), one obtains that for $I \leqslant \bar{n}/2$,

$$D_I = \left[\frac{2I - \kappa}{4}\right] - \left[\frac{2I - 3\kappa}{12}\right] + \delta_1 - \delta_2, \quad (7)$$

with

$$\delta_1 = \delta_{\tau, 3\kappa}, \quad \delta_2 = \begin{cases} \delta_{\tau, 3\kappa - 2} & \text{for} \quad \kappa = 1, 2, 3, \\ \delta_{\tau, 3\kappa + 10} & \text{for} \quad \kappa = 0. \end{cases}$$

Here τ is equal to ($2I$ mod 12). τ and κ are odd for fermions and even for bosons, respectively. The δ_1 term arises when

TABLE II. Numbers of spin-I states (i.e., D_I) for three identical fermions in a single-j shell, with $j = 3/2, 5/2, 7/2, 9/2$ here. $\bar{n} = 2j - 2$ (odd numbers), $k = [\bar{n}/4]$, $\kappa = \bar{n} \bmod 4$. $n_4 = 0, 1, 2, \ldots, k$, and $\lambda = \bar{n} - 4n_4 = 4(k - n_4) + \kappa$. The allowed values of total spin I are given by $\frac{3}{2}\lambda, \frac{3}{2}\lambda - 2, \frac{3}{2}\lambda - 3, \ldots, \frac{1}{2}\lambda$.

j	$\bar{n}$	k	κ	(n_4,λ)	D_I										
					$2I = 1$	3	5	7	9	11	13	15	17	19	21
3/2	1	0	1	(0,1)	–	1									
5/2	3	0	3	(0,3)	–	1	1	–	1						
7/2	5	1	1	(0,5), (1,1)	–	1	1	1	1	1	–	1			
9/2	7	1	3	(0,7), (1,3)	–	1	1	1	2	1	1	1	1	–	1

TABLE III. Same as Table II except for three identical spin-l bosons, with $l = 1, 2, 3, 4$ here. $\bar{n} = 2l$ (even numbers).

l	$\bar{n}$	k	κ	(n_4,λ)	D_I												
					$I = 0$	1	2	3	4	5	6	7	8	9	10	11	12
1	2	0	2	(0,2)	–	1	–	1									
2	4	1	0	(0,4), (1,0)	1	–	1	1	1	–	1						
3	6	1	2	(0,6), (1,2)	–	1	–	2	1	1	1	1	–	1			
4	8	2	0	(0,8), (1,4), (2,0)	1	–	1	1	2	1	2	1	1	1	1	–	1

$\frac{2I-3\kappa}{12}$ equals an integer. The δ_2 term arises from the condition that $I \neq \frac{3}{2}\lambda - 1$. For $I \geqslant \bar{n}/2$,

$$D_I = k - \left[\frac{2I - 3\kappa}{12}\right] + \delta_1 - \delta_2. \quad (8)$$

Equations (7) and (8) are our final results of D_I. They are in unified form for both fermions and bosons.

In Tables II and III, we tabulate D_I for three fermions in a single-j shell with $j = 3/2, 5/2, 7/2, 9/2$ and three spin-l bosons with $l = 1, 2, 3, 4$, respectively. For the convenience of readers, we also list the corresponding $\bar{n}$, k, and κ as well as the allowed values of n_4 and λ. For each given λ, one finds the allowed values of total spin I, and from which one finally obtains D_I.

We finally prove that Eqs. (7) and (8) are consistent with previous formulas obtained empirically in Ref. [6] for $n = 3$. We exemplify this by using three fermions in a single-j shell. Here $\bar{n} = 2j - 2$ and τ is an odd number below 12 $(1, 3, 5, \ldots, 11)$. For $I \leqslant \bar{n}/2 = j - 1$, Eq. (7) is reduced to

$$\begin{aligned} D_I &= 2I_0 + \left[\frac{\tau - \kappa}{4}\right] - \left[\frac{\tau - 3\kappa}{12}\right] + \delta_1 - \delta_2 \\ &= \begin{cases} 2I_0 & \text{for} \quad \tau = 1 \\ 2I_0 + 1 & \text{for} \quad \tau = 3, 5, 7 \\ 2I_0 + 2 & \text{for} \quad \tau = 9, 11 \end{cases} \\ &= 2I_0 + \left[\frac{\tau + 3}{6}\right] = \left[\frac{2I + 3}{6}\right], \end{aligned} \quad (9)$$

where $I_0 = [2I/12] = [I/6]$. This is Eq. (1) of Ref. [6].

For $I \geqslant \bar{n}/2 = j - 1$, Eq. (8) is reduced to

$$\begin{aligned} D_I &= k - I_0 - \left[\frac{\tau - 3\kappa}{12}\right] + \delta_1 - \delta_2 \\ &= k - I_0 + \left[\frac{3\kappa - \tau}{12}\right] + 1 - \delta_2 \\ &= \left[\frac{3\bar{n} - 2I}{12}\right] + 1 - \delta_2 = \left[\frac{I_{\max} - I}{6}\right] + \delta_I, \end{aligned} \quad (10)$$

where δ_I was defined in Ref. [6]:

$$\delta_I = \begin{cases} 0 & \text{if} (I_{\max} - I) \bmod 6 = 1, \\ 1 & \text{otherwise}. \end{cases}$$

This is Eq. (2) of Ref. [6] for three fermions in a single-j shell.

To summarize, in this paper we derive the analytical formulas for the number of spin-I states, D_I, for three fermions in a single-j shell and three bosons with spin l, by using a reduction rule from the U(4) to the O(3) group chain, U(4) ⊃ Sp(4) ⊃ O(3), for $\bar{n}$ virtual bosons which follow the U(4) symmetry (i.e., spin 3/2). $\bar{n} = 2j - 2$ for fermions and $\bar{n} = 2l$ for bosons. We are able to obtain analytical formulas of three bosons and fermions in a unified form and on a unified footing. We show that the new formulas are consistent with previous empirical formulas.

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