Spin response and neutrino emissivity of dense neutron matter

G. Shen,[1] S. Gandolfi,[1] S. Reddy,[1,2] and J. Carlson[1]

[1]*Theoretical Division, Los Alamos National Laboratory, New Mexico 87545, USA*
[2]*Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195, USA*
(Received 29 May 2012; published 15 February 2013)

We study the spin response of cold dense neutron matter in the limit of zero momentum transfer and show that the frequency dependence of the long-wavelength spin response is well constrained by sum rules and the asymptotic behavior of the two-particle response at high frequency. The sum rules are calculated using an auxiliary field diffusion Monte Carlo technique and the high-frequency two-particle response is calculated for several nucleon-nucleon potentials. At nuclear saturation density, the sum rules suggest that the strength of the spin response peaks at $\omega \simeq 40$–60 MeV, decays rapidly for $\omega \geqslant 100$ MeV, and has a sizable strength below 40 MeV. This strength at relatively low energy may lead to enhanced neutrino production rates in dense neutron-rich matter at temperatures of relevance to core-collapse supernova.

I. INTRODUCTION

The spin response of dense neutron matter plays an essential role in determining neutrino interaction rates in neutron stars and supernovae [1–5]. (For the effect of spin response on photon interaction with nucleon magnetic moments, see the discussion in Ref. [6].) Since the energy and momentum transfer between neutrinos and matter is small compared to the Fermi energy and momentum, degeneracy and many-body effects can strongly modify interaction rates. The spin response of neutron matter is an intriguing problem in that a nonzero response requires the coupling of spin and space through the tensor and spin-orbit components of the nuclear force.

We study the response in the specific limit of zero temperature and zero momentum transfer, and we discuss how this limiting case will be useful to understand the more general behavior encountered at finite temperatures in neutron stars and supernovas. At zero temperature the spin response can be obtained through a combination of sum rules and a calculation of the high-energy part of the response. The sum rules and the high-energy behavior resolve nuclear interactions with momenta of the order of the pion wavelength, and we use nuclear Hamiltonians previously found to be reliable in describing relevant excitations and their coupling to the ground state in the other contexts.

Bremsstrahlung reactions such as $n + n \rightarrow n + n + \nu + \bar{\nu}$ are an important source of neutrino pair production in dilute neutron matter. When neutrons are nonrelativistic, neutrino emission occurs primarily due to fluctuations of the nucleon spin. Density and current fluctuations are suppressed by the velocity due to particle number and momentum conservation. This dominance of spin fluctuations is a unique feature of nuclear systems because strong noncentral tensor and spin-orbit forces that do not commute with the spin operator lead to enhanced spin fluctuations even in the long-wavelength limit ($\boldsymbol{q} \rightarrow 0$). Its importance in neutrino production rates was first realized in pioneering work by Friman and Maxwell [1]. They calculated the neutrino production rate in the long-wavelength limit using the one-pion-exchange (OPE) potential in leading-order perturbation theory (Born approximation). In subsequent work, Hanhart *et al*. [7] calculated the neutrino production

rate in neutron matter using a low-energy approximation that relates the rate directly to observed nucleon-nucleon phase shifts, obviating the need to rely on either perturbation theory or a specific choice for the nucleon-nucleon potential. While these calculations have provided a useful benchmark, they neglect many-body effects and their regime of validity is restricted to weakly correlated dilute systems.

The inclusion of many-body effects has relied on diagrammatic perturbation theory where specific corrections to long-distance and long-time behavior of nucleon propagation in the medium are incorporated. The finite lifetime of quasiparticles, screening of the weak axial charge, as well as screening of nucleon-nucleon interactions due to particle-hole polarization effects at finite density have been investigated by several authors [5,8–12]. These calculations have shown that these corrections are important and generically tend to decrease neutrino production rates. On the other hand, attempts to include in-medium softening of the pion propagator and corrections to the nucleon propagators and weak vertices [13–15] have shown that the neutrino emissivity can be enhanced. However, these methods neglect terms in many-body perturbation theory and it is presently difficult to estimate associated errors. To overcome this shortcoming, we adopt a different strategy, using a quantum Monte Carlo (QMC) method to compute the three lowest order sum rules, which are described below in Sec. III. We supplement these sum rules with the asymptotic form of the two-particle response valid at high frequency to deduce the distribution of strength of the spin response function at lower energies of relevance to astrophysics.

II. NEUTRINO EMISSIVITY AND THE SPIN STRUCTURE FUNCTION

From the point of view of many-body theory, neutrino interaction rates in the medium can be factored into a product of two terms: (i) the correlation functions of the dense medium and (ii) kinematical factors and coupling constants associated with neutrino currents. The latter are well known and relatively simple functions of the neutrino energy and momentum. In contrast, the spin, density, and current correlation functions

are complex functions of temperature, density, and the energy and momentum transfer because multiparticle dynamics and correlations in the ground state of the strongly interacting system play a critical role.

The dynamic spin structure factor $S_\sigma(\omega, \mathbf{q})$ of neutron matter encodes the linear response of neutron matter to spin fluctuations and is defined as [2]

$$
\begin{aligned}
S_\sigma(\omega, \mathbf{q}) &= \frac{4}{3n} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle \mathbf{s}(t, \mathbf{q}) \cdot \mathbf{s}(0, -\mathbf{q}) \rangle \\
&= \frac{4}{3n} \sum_f \langle 0 | s(\mathbf{q}) | f \rangle \cdot \langle f | s(-\mathbf{q}) | 0 \rangle \delta[\omega - (E_f - E_0)],
\end{aligned}
$$

(2.1)

where $\mathbf{s}(t, q) = V^{-1} \sum_{i=1}^{N} e^{-i\mathbf{q} \cdot r_i(t)} \boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_i$ is the spin operator acting on the ith nucleon at time t. The second line expresses the same response as a sum over final states $|f\rangle$ coupled to the ground state through the time-independent spin operator.

Alternatively, in terms of the field operators, $\mathbf{s}(t, q)$ is the Fourier transform of the spin density operator $\mathbf{s}(x) = \frac{1}{2} \psi^+(x) \boldsymbol{\tau} \psi(x)$ with $\boldsymbol{\tau}$ being the usual Pauli matrix and $\psi(x)$ is the nonrelativistic field operator. The normalization factor $4/3n$, where n is the neutron number density, ensures that the dynamic form factor is canonically normalized such that $S(q \to \infty) = 1$ for the noninteracting Fermi systems and conforms to the standard definitions of the sum rules discussed below in Sec. III.

The rate of neutrino pair production can be expanded in powers of the nucleon velocity and the momentum of the neutrino pair [1]. The neutrino emissivity of neutron matter denoted by Q, and defined as the rate of energy loss due to neutrino pair production per unit volume and per unit time, to leading order in the neutron velocity and neutrino momentum is given by [9]

$$
Q = \frac{C_A^2 G_F^2 n}{20\pi^3} \int_0^\infty d\omega \, \omega^6 e^{-\omega/T} S_\sigma(\omega),
$$

(2.2)

where $G_F = 1.18 \times 10^{-11}$ MeV^{-2} is the Fermi constant of the weak interaction and $C_A = -1.26/2$ is the neutron neutral-current axial coupling constant. Note that, due to the strange-quark contribution to the nucleon spin, C_A may be modified in neutral current processes by a few percent in the energy range of interest to supernova [16]. Here we have not included this modification for simplicity. At low temperature, when $T \ll E_{\mathrm{Fn}}$, where E_{Fn} is the neutron Fermi energy, the neutrino pair momentum $\mathbf{q}$ is small compared to both the Fermi momentum k_{Fn} and the intrinsic momentum scales associated with the strong interaction, and it may be neglected and $S_\sigma(\omega) = S_\sigma(\omega, \mathbf{q} = 0)$. Hence in Eq. (2.2) only $S_\sigma(\omega)$ appears and it is a function of both density and temperature as implied by the ensemble average denoted on the right-hand side of the equation.

III. SUM RULES

The spin response describes the coupling to the ensemble of final states obtained by flipping all the ground-state spins in neutron matter. If spin and space are uncoupled, spin is

a good quantum number and there would be no response at zero momentum transfer. However, the spin-orbit and tensor interactions (acting only in relative p waves and higher in neutron matter) induce a finite expectation value of $\langle S^2 \rangle$ even at $T = 0$ and finite response results. The spin-orbit and tensor interactions are of pion range or less, so they predominantly affect neutrons coupled to spin 1 at a pair separation typical for nearest neighbors at that density. Although there is zero total momentum transfer, the two interacting particles can nevertheless have significant relative momenta in the relevant final states.

The overall strength and energy distribution of the response can be characterized through the relevant sum rules. We employ a QMC method to compute the low-order sum rules that relate moments of $S_\sigma(\omega, \mathbf{q})$ to its ground-state properties. We then combine these sum-rule constraints with asymptotic high-energy behavior expected in the two-particle system to obtain constraints on the distribution of strength $S_\sigma(\omega)$ as a function of ω at $\boldsymbol{q} = 0$. For the same reason, the response in Eq. (2.1) is solely due to the excitation of multiparticle states as single-particle excitations vanish for these kinematics.

Though we ultimately desire information about the spectrum and coupling to the excited states of the system, the moments of the sum rules defined by the relation

$$
S_\sigma^n = \int_0^\infty S_\sigma(\omega, \boldsymbol{q} = 0) \omega^n \, d\omega
$$

(3.1)

are calculable as ground-state properties. The sum rules provide a simple and systematic means to eliminate explicit dependence on the intermediate excited states of the system. The relevant excited-state information is sampled by operators contained in the nuclear Hamiltonian. In this study we use the following sum-rule relations:

$$
S_\sigma^{-1} = \frac{\chi_\sigma}{2n},
$$

(3.2)

$$
S_\sigma^0 = 1 + \lim_{q \to 0} \frac{4}{3N} \sum_{i \neq j}^N \langle 0 | e^{-i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j | 0 \rangle, \quad (3.3)
$$

$$
S_\sigma^{+1} = -\frac{4}{3N} \lim_{q \to 0} \langle 0 | [H_N, \boldsymbol{s}(\mathbf{q})] \cdot \boldsymbol{s}(-\mathbf{q}) | 0 \rangle, \quad (3.4)
$$

where $\chi_\sigma = \partial n_\sigma / \partial \mu_\sigma$ is the spin susceptibility of the interacting ground state $|0\rangle$ of the nuclear Hamiltonian H_N, and n_σ and μ_σ are number density and chemical potential of particles with spin σ ($\pm 1/2$). Our strategy here is to evaluate the right-hand side of Eqs. (3.2), (3.3), and (3.4) using a QMC method and use this information to constrain the behavior of $S(\omega)$ for values of ω relevant to the calculation of neutrino production.

This strategy is not new; in Ref. [8] estimates of the S_σ^0 and S_σ^1 sum rules were used to argue that spin response function must saturate at high density, and in Ref. [17], sum rules were used to estimate the relative importance of multiparticle excitations to the response function in the kinematical regime where $\omega \geqslant q$. Our work improves upon these earlier studies in two respects: (i) we compute and combine for the first time all three sum rules to constrain both low-frequency and high-frequency behavior of $S(\omega, \boldsymbol{q} = 0)$; and (ii) we deduce the high-frequency response or short-time behavior of the two-particle dynamics where they dominate in

the many-body system by direct calculation of the two-particle matrix elements.

To compute the expectation values of operators in the ground state needed to evaluate the sum rules we use a nonrelativistic nuclear Hamiltonian with local two-body potentials of the form

$$H_N = \sum_i^N \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j} \sum_p^4 v_p(r_{ij}) O_{ij}^{(p)}, \qquad (3.5)$$

where $O_{ij}^{p=1,4} = (1, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, S_{ij}, \boldsymbol{L} \cdot \boldsymbol{S})$, $S_{ij} = (3\boldsymbol{\sigma}_1 \cdot \hat{r} \boldsymbol{\sigma}_2 \cdot \hat{r} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ is the tensor operator, and $\boldsymbol{L} \cdot \boldsymbol{S}$ is the spin-orbit operator. We employ the auxiliary field diffusion Monte Carlo (AFDMC) method described in Refs. [18,19] and use the Argonne AV8' form for the two-body interaction as it provides a good description of properties of light nuclei [20]. The AFDMC calculations use auxiliary field quantum Monte Carlo techniques to treat the spin and spatial degrees of freedom in neutron matter. They have been used extensively to calculate the equation of state of neutron matter and also the spin susceptibility [21]. We use AFDMC to compute the sum rules expressed in Eqs. (3.2), (3.3), and (3.4). Note the static structure function S_σ^0 and energy-weighted sum rule S_σ^1 have been previously evaluated for Argonne potentials [22], first based on variational methods [23,24].

The S_σ^{-1} sum rule is calculated by considering the ground state of neutron matter in the presence of a magnetic field as proposed in Ref. [21]. The energy of neutron matter in the presence of a magnetic field is

$$E(p) = E(0) - bP + (1/2)P^2 E''(0), \qquad (3.6)$$

where $E(0)$ is the ground-state energy in the absence of a magnetic field, $P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}$ is the spin polarization, and the spin susceptibility χ_σ is

$$\chi_\sigma = \mu^2 P \frac{1}{E''(0)}. \qquad (3.7)$$

The calculations are performed for zero magnetic field and a finite magnetic field for of order 60 particles in periodic boundary conditions. The system we simulate has a finite number of up and down neutrons, and the magnetic field is chosen in such a way that the finite system is close to the thermodynamic limit as described in Ref. [21]. In a nonsuperfluid system, the calculation of the spin susceptibility yields the S_σ^{-1} sum rule.

We calculate S_σ^0 by computing the spin-dependent pair correlation function and evaluating the structure function at $q = 0$. The spin correlation function is defined by

$$g_\sigma(r) = \frac{1}{2\pi r^2 \rho N} \sum_{i<j} \frac{\langle \psi | \delta(r_{ij} - r) \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j | \psi \rangle}{\langle \psi | \psi \rangle}, \qquad (3.8)$$

where ψ is the ground state of the system. The AFDMC method is useful to compute the expectation values of mixed operators such as $\langle \Psi_T | O | \psi \rangle$. We use a variational Monte Carlo (VMC) method to extrapolate the value of operators that are given by $\langle O \rangle = 2\langle O \rangle_{\text{mix}} - \langle O \rangle_{\text{vmc}}$ as described in Refs. [19,25]. The resulting $g(r)$ is used to obtain the structure function $S_\sigma^0(q)$. We show $g_\sigma(r)$ and $S_\sigma^0(q)$ in Fig. 1. We finally
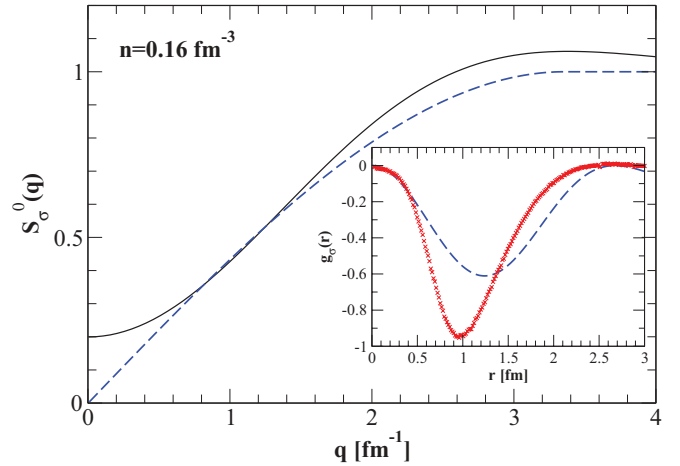

FIG. 1. (Color online) The static structure function $S_\sigma^0(q)$ computed at saturation density. In the inset we show the corresponding spin pair correlation function $g_\sigma(r)$. Free particle results are given as dashed lines.

evaluate the S_σ^0 sum rule by taking the $q \to 0$ limit as indicated in Eq. (3.3).

The energy-weighted sum rule can be calculated by the expectation value of the tensor and spin-orbit interactions when $q = 0$. For the Hamiltonian of Eq. (3.5) we have

$$S_\sigma^{+1} = -\frac{4}{3N} \sum_{i<j} (3\langle v_3(r_{ij}) S_{ij} \rangle + \langle v_4(r_{ij}) \boldsymbol{L} \cdot \boldsymbol{S} \rangle). \qquad (3.9)$$

Because the variational wave function Ψ_T used as input for AFDMC contains neither tensor nor spin-orbit correlations, the most accurate way to obtain these expectation values is by calculating the energy as a function of the spin-orbit and tensor interaction strengths and using the slope of the energy with respect to these couplings to produce the true ground-state expectation values.

These initial calculations are performed with the AV8' NN interaction without any three-nucleon interaction. Based upon simple estimates of the strength of the three-nucleon force, we would expect of order 10%–20% corrections to the sum rules from the three-nucleon interaction. We are exploring this dependence and will report these results separately.

In computing the ground-state properties in AFDMC we neglect the role of pairing and superfluidity. This will restrict our study to the calculation of the neutrino emissivity at temperatures that are large compared to the neutron pairing gaps in neutron matter but still small compared to the Fermi energy. Thus, our results will be applicable to ambient conditions in the supernova but will not apply to old neutron stars where neutron matter is likely to be below the superfluid critical temperature. For $T \ll \Delta$, where $\Delta \approx 1$ MeV is the superfluid gap, the number of quasiparticles is exponentially suppressed and response is vanishingly small. In the vicinity of the critical temperature, Cooper pair breaking and formation, as well as collective modes, can enhance spin fluctuations at a frequency $\omega \approx (1–2)\Delta$ [26]. The spin response function and the neutrino emissivity in the superfluid phase are expected to be qualitatively different and are dominated by the

TABLE I. AFDMC results for the sum rules.

Density (fm^{-3})	S_σ^{-1} (MeV^{-1})	S_σ^0	S_σ^{+1} (MeV)	$\bar\omega_0$ (MeV)	$\bar\omega_1$ (MeV)
$n = 0.12$	0.0057(9)	0.20(1)	8(1)	35(9)	40(8)
$n = 0.16$	0.0044(7)	0.20(1)	11(1)	46(11)	55(8)
$n = 0.20$	0.0038(6)	0.18(1)	14(1)	47(12)	78(10)

pair recombination processes and the decay of finite-energy collective modes [27,28]. It may be possible in the future to examine this regime more critically using techniques similar to those developed here.

The AFDMC results for the sum rules are shown in Table I, where the individual sum rules and average excitation energies defined by $\bar\omega_0 = S_\sigma^0/S_\sigma^{-1}$ and $\bar\omega_1 = S_\sigma^1/S_\sigma^0$ are listed. The density dependence of the S_σ^0 sum rule is quite modest over the range of densities considered.

The spin susceptibilities shown in Table I correspond to $\chi/\chi_F = 0.37, 0.34$, and 0.34 for $\rho = 0.12, 0.16$, and 0.20 fm^{-3}, respectively, where $\chi_F = mk_F/\pi^2$ is the spin susceptibility for a free Fermi gas. At the lowest density this is very similar to results obtained in [21], whereas at the highest density our result is approximately 20% lower for the susceptibility. The difference may lie in the fact that the three-nucleon force used in [21] is repulsive in unpolarized neutron matter, and less so in spin-polarized matter.

The average energies $\bar\omega_0$ and $\bar\omega_1$ are extracted from the sum rules as estimates for the energy of the peak of the response, and their difference is a measure of the width of the distribution. The fact that the calculated $\bar\omega_0$ and $\bar\omega_1$ values are fairly similar indicates a moderately narrow peak in the response. A positive definite response requires $\bar\omega_1 \geqslant \bar\omega_0$. The peaks shift to higher energy with increasing density, as expected. The tables also indicate that the strength distribution gets more diffuse with increasing density, with strength being pushed out to higher energy.

IV. ASYMPTOTIC FORM AT HIGH ENERGY

In order to constrain the low-energy response relevant for astrophysical applications using sum rules we need some knowledge of the behavior of $S_\sigma(\omega)$ at large ω. In this regime the response probes the short-time behavior of the many-body correlation function and on general grounds we can expect this to be dominated by two-particle dynamics. This intuitive expectation can be cast in more formal terms by using the operator product expansion originally developed by Wilson as a standard technique in quantum field theory. The operator product expansion has been used to analyze short-time behavior of the density-density correlation function in a strongly interacting nonrelativistic Fermi gas [29,30]. By adapting this to the spin-density operator, the relevant expansion in this case organizes $S_\sigma(\omega)$ in terms of local operators in inverse powers ω and is given by

$$\int dt\, e^{i\omega t} \int d^3x\, \psi^\dagger \boldsymbol{\sigma} \psi(t, \boldsymbol{R} + \boldsymbol{x}) \psi^\dagger \boldsymbol{\sigma} \psi(0, \boldsymbol{R} - \boldsymbol{x})$$
$$= i W_1(\omega) \mathcal{O}^{(1)}(\boldsymbol{R}) + i W_2(\omega) \mathcal{O}^{(2)}(\boldsymbol{R}) + \cdots, \quad (4.1)$$

where the expectation value of the local operators $\mathcal{O}^{(n)}(\boldsymbol{R})$ depends on the many-body ground state but the Wilson coefficients $W_i(\omega)$ depend only on few-body physics with i incoming and outgoing asymptotic states. For $\boldsymbol{q} = 0$ the Wilson coefficient $W_1(\omega)$ vanishes identically in spin-saturated systems and the leading contribution is due to $W_2(\omega)$. The functional form of $W_2(\omega)$ is determined by the matrix elements of the spin operator between two-body scattering states. This implies that up to an overall constant, which depends only on the ground state, $S_\sigma(\omega)$ at high frequency is determined by the the two-body matrix elements. In general, this will depend sensitively on the short-distance behavior of the two-nucleon interaction and will be model dependent. However, to extract the response at low energy in a model-independent fashion it suffices to use, in the two-body calculation, the same Hamiltonian employed in the calculation of the sum rules in the many-body calculation.

The spin response function $S(q, \omega)$ for two neutrons are evaluated as follows:

$$S(q, \omega) = |\langle \psi_F | \hat{O}_A | \psi_I \rangle|^2 \delta(\omega + E_I - E_F). \quad (4.2)$$

For spin response at $q = 0$, the operator is the sum of spins, $\hat{O}_A = \vec{\sigma}_1 + \vec{\sigma}_2$. ψ_I and ψ_F are the eigenstates of two neutrons in spin-triplet states and we take ψ_I to be the ground state.

We have calculated these matrix elements using the same nuclear Hamiltonian employed in the AFDMC by solving the Schrödinger equation for two neutrons with a simple box boundary condition. These results indicate that the high-frequency behavior denoted as $S_\sigma^{\mathrm{high}}(\omega)$ is determined by two-body physics and has the following asymptotic behavior:

$$S_\sigma^{\mathrm{high}}(\omega) \simeq \left(\frac{\omega_c}{\omega}\right)^i, \quad (4.3)$$

where the density-dependent quantity $\omega_c \simeq 100$–150 MeV for the range of densities considered here, and for the nuclear interaction used we find that $i \approx 9$.

As mentioned earlier the high-frequency response will depend on the model for nucleon-nucleon interactions at short distance. For a correct description of the response at $\omega \geqslant 100$ MeV, the inclusion of two-body currents and explicit pion and Δ isobar degrees of freedom is likely to become important. However, since they are absent in both the many-body and two-body calculation, their consistent omission ensures that we can still obtain useful constraints on $S_\sigma(\omega)$ at lower ω values of interest without these ingredients.

Using the two-body axial currents adjusted to reproduce measured tritium β decay [31], we calculated the contributions to the static spin sum rule of Eq. (4.2) at $q = 0$ due to the most important two-body currents—the axial π-exchange Δ-excitation current and the π-exchange (pair) current. These contributions were found to be a few percent of the total static spin sum rule. Therefore we expect the contribution of two-body currents to the dynamic spin response function at zero momentum transfer to be around a few percent as well.

V. LOW-ENERGY FORMS FOR THE RESPONSE

In the regime where neutron matter behaves like a Fermi liquid, the low-energy form of the response should be

describable in terms of quasiparticles, though the coupling of the ground state to the quasiparticle pairs as well as the quasiparticle interactions may renormalize quantities in the calculated response. At $q = 0$, a low-frequency form for $S_\sigma(\omega)$ has been computed in Refs. [3,5] using the quasiparticle approximation and is given by

$$S_\sigma(\omega) = \frac{N(0)}{n\pi} \frac{\omega\tau_\sigma}{(1 + G_0)^2 + (\omega\tau_\sigma)^2}, \quad (5.1)$$

where the frequency-dependent relaxation time $\tau_\sigma(\omega)$ is the time scale for damping of spin fluctuations, and $N(0)$ is the density of states at the Fermi surface and G_0 is the Landau parameter that encodes the spin susceptibility mentioned earlier [Eq. (3.2)]. This form of the response incorporates collisional broadening and mean-field effects but it is mostly sensitive to $\tau_\sigma(\omega)$. Note that this type of functional form incorporates higher order terms in the scattering and thereby takes into account the Landau-Pomeranchuk-Migdal effect [32–35].

The spin relaxation time τ_σ is related to the quasiparticle scattering amplitude $\mathcal{A}_{\sigma_1,\sigma_2}(\mathbf{k},\mathbf{k}')$ and is given by [5]

$$\frac{1}{\tau_\sigma(\omega + \epsilon_1)} = 2\pi \sum_{2,3,4} |A|^2 \mathcal{F}\delta(\omega + \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$
$$\times \delta(\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_3 - \mathbf{P}_4), \quad (5.2)$$

where

$$|A|^2 = \frac{1}{12} \sum_j \mathrm{Tr}\big[\mathcal{A}_{\sigma_1,\sigma_2}(\mathbf{k},\mathbf{k}')$$
$$\times \sigma_1^j \big[(\sigma_1 + \sigma_2)^j, \mathcal{A}_{\sigma_1,\sigma_2}(-\mathbf{k},\mathbf{k}')\big]\big], \quad (5.3)$$

where the first sum is over the momenta of all initial states of particle 2 and all final states of particle 3 and 4, and $\mathcal{F} = f_2(1 - f_3)(1 - f_4) + (1 - f_2)f_3 f_4$ Pauli blocking factors, where f_i is the Fermi-Dirac distribution for particle i in the reaction $1 + 2 \rightarrow 3 + 4$ with incoming momenta $\mathbf{P}_1$ and $\mathbf{P}_2$ and outgoing momenta $\mathbf{P}_3$ and $\mathbf{P}_4$ and relative momenta $\mathbf{k} = \mathbf{P}_1 - \mathbf{P}_3$, $\mathbf{k}' = \mathbf{P}_1 - \mathbf{P}_4$. The squared matrix element in Eq. (5.3) is a sum over the spin projections $j = 1, 2, 3$ of the spin operators σ_1 and σ_2 acting on nucleons 1 and 2, respectively. In the limit of $\omega \ll E_F$ and $T \ll E_F$, the appropriate average relaxation time from Eq. (5.2) reduces to

$$\frac{1}{\tau_\sigma} = C_\sigma\left[\left(\frac{\omega}{2\pi}\right)^2 + T^2\right], \quad (5.4)$$

where C_σ characterizes the strength of noncentral interactions and depends on the ambient density and T is the temperature. C_σ has been calculated using different models for the nucleon-nucleon potential for a range of densities in Ref. [36]. At nuclear density these authors find that $C_\sigma \simeq 0.22$ MeV^{-1} for OPE, while it is reduced to $C_\sigma \simeq 0.08$ MeV^{-1} for realistic nucleon-nucleon and next-to-next-to-next-to-leading-order chiral perturbation theory (χPT) potentials.

From Eq. (5.4) it follows that the form of the quasiparticle approximation is valid when $\omega \ll (2\pi)^2/C_\sigma$ (and $\omega \ll E_F$) and inserting Eq. (5.4) into Eq. (5.1) we can obtain a low-

frequency form of the zero-temperature structure function,

$$S_\sigma^{\mathrm{low}}(\omega, \mathbf{q} = 0) = \frac{N(0)}{n\pi} \frac{\tilde{C}_\sigma \omega}{1 + (1 + G_0)^2(\tilde{C}_\sigma\omega)^2}, \quad (5.5)$$

where $\tilde{C}_\sigma = C_\sigma/(2\pi)^2$. This form of the structure function satisfies the S_σ^{-1} sum rule by construction but produces divergent results for S_σ^0 and S_σ^1. In the following section we will combine the low- and high-frequency forms in Eq. (5.5) and Eq. (4.3), respectively, with the sum-rule constraints discussed in the preceding section to construct a structure function that can be used in calculations of the neutrino emissivity.

Another commonly used limiting form of $S_\sigma(\omega)$ can be obtained by ignoring any many-particle correlations in the ground state. Here neutrons are distributed as free fermions in the ground state and excitations with $q = 0$ and $\omega \neq 0$ arise as two-particle two-hole states. Following Ref. [7] we denote this as the two-body (2b) response and this is given by

$$S_\sigma^{\mathrm{2b}}(\omega) = \frac{2}{3\pi n} \int \left[\prod_{i=1..4} \frac{d^3 p_i}{(2\pi)^3}\right] (2\pi)^4 \delta^3(\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_3 - \mathbf{P}_4)$$
$$\times \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \omega)\mathcal{F}_2\mathcal{H}, \quad (5.6)$$

$\mathcal{F}_4 = f_1 f_2(1 - f_3)(1 - f_4)$, and $\mathcal{H}$ is related to the square of the matrix element defined in Eq. (5.3). For the nn system where only the spin-triplet two-nucleon state contributes it is explicitly given by

$$\mathcal{H} = \frac{1}{\omega^2} \sum_{M_s M_s'} |\langle 1M_s',\mathbf{p}'|[S,\mathbf{T}_{NN}]|\mathbf{p},1M_s\rangle|^2, \quad (5.7)$$

where S is the total spin and $\mathbf{p}$ ($\mathbf{p}'$) is the relative initial (final) momentum of the two-nucleon system. The matrix element is computed by assuming plane-wave in and out states for the two-nucleon system and thus ignores many-body effects and initial- and final-state correlations. In Ref. [7] it was argued that this result can be expected to be valid up to $\omega \simeq m_\pi$ in the absence of many-body corrections.

VI. CONSTRUCTING $S_\sigma(\omega)$

While it is clear that a unique reconstruction of $S_\sigma(\omega)$ would require an infinite number of moments, here we show that the lowest order sum rules and the asymptotic forms discussed in the preceding section already provide significant constraints. The three sum rules S_σ^{-1}, S_σ^0, and S_σ^{+1} provide a useful sampling of the function at low, intermediate, and high energy, respectively. Their utility in constraining the neutrino emissivity will depend on the redistribution of strength due to finite-temperature effects. We postpone a discussion of these finite-temperature effects to the subsequent section. Here, using as guidance the limiting forms discussed previously, we examine simple ansatze for the functional form of $S_\sigma(\omega)$ at $T = 0$ by imposing sum-rule constraints.

The striking feature of the sum-rule results shown in Table I is that at nuclear density $\bar{\omega}_0 \approx \bar{\omega}_1$ and is comparable to the Fermi energy $E_F = k_F^2/2m$. This suggests that the function $S_\sigma(\omega)$ contains significant strength in the region $\bar{\omega}_0$ to $\bar{\omega}_1$. To properly account for this we study the following simple ansatz

for the frequency dependence of $S_\sigma(\omega)$ given by the form $S_\sigma^{low}(\omega)$ but the with a more complex behavior of $\tau(\omega)$ given by the following forms:

$$\frac{1}{\tau(\omega)} = \left(\tilde{C}_\sigma \omega^2 + \alpha \frac{\omega^{2+n}}{(\omega + \omega_0)^2} \right) \left(\frac{\omega_c}{\omega + \omega_c} \right)^m, \quad (6.1)$$

where the constants α and ω_0 and indices n and m are fit to ensure that the three sum rules and the asymptotic forms are satisfied. At low frequency, this ansatz ensures that out results coincide with the results obtained in Ref. [5] by Lykasov, Pethick, and Schwenk, where only the first term containing $\tilde{C}_\sigma$ contributes. On general grounds (unitarity of scattering amplitudes) at large frequency, pair excitation should be quenched due to the retarded nature of nuclear interaction, and this quenching is naturally incorporated through the asymptotic form discussed in relation to Eq. (4.3).

To better understand the sensitivity of our results to the choice of parametrization, we have also used a simple phenomenological form for the spin response:

$$S_\sigma(\omega) = \alpha \frac{\omega^j}{[1 + (\omega/\omega_c)^i]^4}. \quad (6.2)$$

The high-frequency tail is forced to fall off appropriately by choosing $4i - j = 9$. The parameters α, ω_c, and i are then fitted to the three sum rules. This simple form ensures that the response goes to zero at low frequency, has the correct high-frequency tail, and has a single-peak structure. Comparisons of the two parametrizations provide some information on the reliability of the extracted spin response.

Figure 2 shows the response function obtained by fitting the sum rules and the high-energy response at saturation density using the two different parametrizations, Eqs. (5.5) and (6.2). For comparison, the low-frequency forms of the
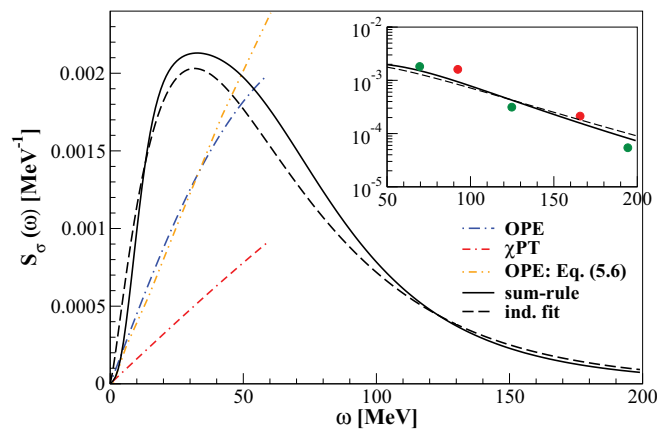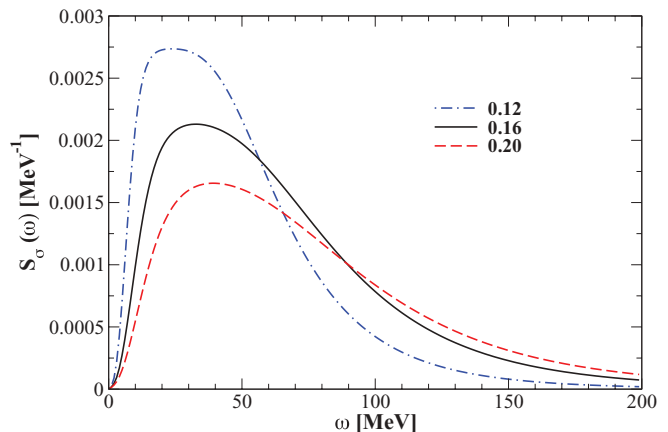
FIG. 3. (Color online) The spin response function $S_\sigma(q = 0, \omega)$ of neutron matter at $\rho = 0.12$, 0.16, and 0.20 fm^{-3} from fits to AFDMC sum-rule results at zero temperature.

structure function obtained in Ref. [36] are shown for the two choices of $\tilde{C}_\sigma$ corresponding to the OPE and χPT potentials discussed earlier. The form of the low-frequency response in Eq. (5.1) is valid only at $\omega \ll E_F$. In the figure we also show the results from the two-body approach [described in Eq. (5.6)] in the Born approximation with OPE. At low frequency $\omega \lesssim E_F/2$, it gives similar results to the quasiparticle picture, then becomes larger at higher frequency since it includes the exact phase-space integrals. The inset compares the fits and the two-particle response at high energy obtained by confining two neutrons in a spherical cavity of radius = 7 fm (red) or 8 fm (green). The asymptotic forms and sum rules force significantly more strength at lower energy than obtained previously.

The simple phenomenological fit [dashed line; Eq. (6.2)] and the fit to the quasiparticle form [solid line; Eqs. (5.5) and (6.1)] produce very similar response functions. In addition to the sum-rule constraints, we are forcing the response to go to zero at low frequency, have a single-peak structure, and to fall off fairly rapidly at high energy as obtained from the two-neutron response. When combined, these considerations place fairly tight constraints on the spin response of neutron matter.

In Fig. 3 we compare the response functions obtained over a range of densities $\rho = 0.12$, 0.16, and 0.20 fm^{-3}. As expected from the sum rules, the peak of the response functions shifts to larger energy with increasing density. The tensor and spin-orbit correlations are naturally of shorter range at the higher densities where the mean interparticle spacing is shorter, and hence the peak shifts to higher energy. The total strength in the response is fairly flat over the regime of densities we consider as obtained in the sum-rule calculations for S_0.

Finally, at higher density the distribution is somewhat broader as ω_1 increases more rapidly with density than ω_0. Both ω_0 and ω_1 increase rapidly, presumably associated with the increasing importance of the shorter range components of the nuclear force at and above saturation density. While we expect this trend to be qualitatively correct, contributions due to three-body forces and from two-body currents are able to play a role in modifying this behavior.

FIG. 2. (Color online) The spin response function $S_\sigma(q = 0, \omega)$ of neutron matter at saturation density obtained by fitting to AFDMC sum rules using two different ansatze (shown as the black solid and dashed curves). The inset compares the fits and the two-particle response at high energy obtained by confining two neutrons in a spherical cavity of radius = 7 fm (red) or 8 fm (green). The linear, low-frequency forms predicted in Ref. [36], labeled as OPE and χPT, are shown for comparison. The dot-dot-dashed curve is obtained by using the two-body approach in Eq. (5.6) with OPE.

VII. EXTENSION TO FINITE TEMPERATURE AND IMPACT ON NEUTRINO PRODUCTION

The AFDMC method we employ is restricted to zero temperature and we have not explicitly computed the temperature corrections to the sum rules. Hence there will be several caveats to consider when using our results in finite-temperature applications such as supernova where $S_\sigma(\omega)$ plays a role in neutrino production rates. To discuss these we first note that there are three fundamental energy scales inherent to our present analysis of the structure function and the neutrino emissivity: (i) the typical energy at which the structure function has significant strength and is given by $\bar\omega_0$ and $\bar\omega_1$; (ii) the energy scale at which the structure function is sampled in the neutrino emissivity, denoted as ω_ν, whose expected value from the expression for Q in Eq. (2.2) is $\omega_\nu \simeq 5T$–$6T$; and (iii) the high-energy scale ω_c at which the asymptotic two-body behavior dominates.

At very low temperature where $\omega_\nu \ll \bar\omega_0$ and $\omega_\nu \ll \bar\omega_1$, the sum rules do not provide useful constraints. Here the low-frequency form of $S_\sigma(\omega)$ given in Eq. (5.5) can be used to calculate the neutrino emissivity with the requirement that $\omega_\nu \ll E_F$ and $\tilde C_\sigma \omega_\nu \ll 1$. In practice, at nuclear density, the condition that $\omega_\nu \ll E_F$ is more restrictive and limits the use of the low-frequency form to the region where $T \leqslant E_F/6$. At intermediate temperature when $\omega_\nu \simeq \omega_0$ or ω_1 and $T \leqslant E_F$, the zero-temperature sum-rule constraints on the form of S_σ^0 and S_σ^1 become relevant. Here the temperature is intermediate and corrections to the $T = 0$ sum rules are expected to be small due to the Pauli principle.

At finite temperature, the dynamic structure factor obeys the detailed balance

$$S_\sigma(-\omega) = \exp\left(-\frac{\omega}{T}\right) S_\sigma(\omega), \qquad (7.1)$$

and this is reflected in Eq. (2.2) for the neutrino emissivity where the exponential term accounts for the fact that neutrino emission corresponds to thermal fluctuations with negative ω. There are residual temperature dependencies in the function $S_\sigma(\omega)$. First, from the fluctuation-dissipation theorem we have

$$S_\sigma(\omega) = -2\left[1 - \exp\left(-\frac{\omega}{T}\right)\right]^{-1} \mathrm{Im}\Pi^R(\omega), \qquad (7.2)$$

where Π^R is the retarded polarization function, which is an odd function of ω and vanishes at $\omega = 0$. To extend to finite temperature the zero-temperature ansatz in Sec. VI needs to be multiplied by the factor $[1 - \exp(-\omega/T)]^{-1}$. A second source of temperature corrections arise from the fact that at low frequency the spin relaxation time $\tau_\sigma^{-1} \simeq C_\sigma T^2$ is dominated by scattering between thermally excited quasiparticles as described in Eq. (6.1). We incorporate this expected behavior by using the finite-temperature expression for τ_σ given in Eq. (6.1) in the low-frequency form given for $S_\sigma(\omega)$ in Eq. (5.5). Other sources of temperature corrections exist such as those arising from transitions in which the excited many-particle states do not decay to the ground state, and these should be investigated in the future. We leave this for future work as it would require the development of finite-temperature QMC techniques.
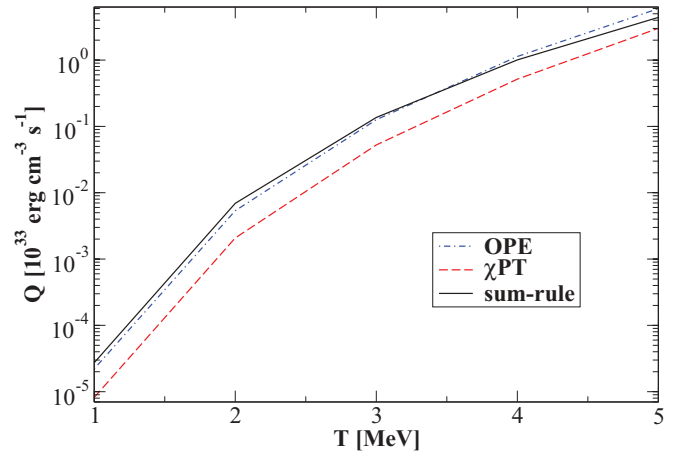

FIG. 4. (Color online) The energy-loss rate Q at various temperature as defined in Eq. (2.2), for OPE, χPT, and our results.

With the aforementioned finite-temperature extensions we employ the $S_\sigma(\omega)$ obtained using the sum-rule constraints to compute the neutrino emissivity. In Fig. 4, the resulting energy-loss rates Q are shown for various temperatures T. The large strength required by the sum rules at intermediate energy leads to a larger neutrino emissivity compared to the simple extrapolation of results obtained in the quasiparticle approximation with only two-particle–two-hole excitations. Our results are almost a factor of 2 larger than either those obtained using χPT in Ref. [36] or those obtained directly from nucleon-nucleon phase shifts in Ref. [7] at $T \leqslant 5$ MeV. We suspect that this enhancement is due to correlations in the ground state that are not captured in the quasiparticle approximation.

VIII. DISCUSSION

Our study of the zero-temperature sum rules suggests that the spin response function of neutron matter has a significant strength at energy between 40 and 60 MeV in the vicinity of nuclear density. This strength should be accessible at temperatures of the order of 5–30 MeV encountered in the supernova environment and could influence the rate of neutrino pair production from nucleon-nucleon processes. Although the zero-temperature sum rules do not directly constrain the finite-temperature response functions needed in the calculation of the neutrino emissivity they provide a useful guidance. For example, they can be used to test predictions obtained using quasiparticle methods at zero temperature, and with some caveats they can be extended to low temperature where response is still dominated by transitions to the ground state.

Comparisons with earlier calculations of the dynamic structure factor indicate that they significantly underpredict the response in the regime where $\omega \simeq E_F/5 - E_F$ for the densities considered. There are several possible resolutions to this discrepancy. The quasiparticle interactions and dispersion relations used in earlier studies may not be adequate as finite density effects are ignored. Similarly, the use of plane-wave states augmented with the T matrix may be too simple to reproduce the coupling between the ground state and the

excitations. Alternatively, our ansatz for the response function may be too simple to capture the complex structure of $S_\sigma(\omega)$.

All of these possibilities can be studied in more detail. To quantify the interplay between increasing phase space and decreasing strength of the two-body interaction with increasing ω we have calculated $S_\sigma(\omega)$ in the standard approach using realistic potentials. Within the QMC approach there are two ways to address these issues. First, the calculation of higher order moments at zero temperature can provide additional constraints and test our ansatze at intermediate energy. Second, extensions to finite temperature will shed light on the importance of transitions not involving the ground state. We hope to pursue these in future work.

[1] B. L. Friman and O. V. Maxwell, Astrophys. J. **232**, 541 (1979).
[2] N. Iwamoto and C. J. Pethick, Phys. Rev. D **25**, 313 (1982).
[3] G. Raffelt, D. Seckel, and G. Sigl, Phys. Rev. D **54**, 2784 (1996).
[4] S. Reddy, M. Prakash, J. M. Lattimer, and J. A. Pons, Phys. Rev. C **59**, 2888 (1999).
[5] G. I. Lykasov, C. J. Pethick, and A. Schwenk, Phys. Rev. C **78**, 045803 (2008).
[6] A. Kopf and G. Raffelt, Phys. Rev. D **57**, 3235 (1998).
[7] C. Hanhart, D. R. Phillips, and S. Reddy, Phys. Lett. B **499**, 9 (2001).
[8] G. Sigl, Phys. Rev. Lett. **76**, 2625 (1996).
[9] S. Hannestad and G. Raffelt, Astrophys. J. **507**, 339 (1998).
[10] E. N. E. van Dalen, A. E. L. Dieperink, and J. A. Tjon, Phys. Rev. C **67**, 065807 (2003).
[11] A. Schwenk, P. Jaikumar, and C. Gale, Phys. Lett. B **584**, 241 (2004).
[12] G. I. Lykasov, E. Olsson, and C. J. Pethick, Phys. Rev. C **72**, 025805 (2005).
[13] D. N. Voskresensky and A. V. Senatorov, Sov. Phys. JETP **63**, 885 (1986).
[14] A. B. Migdal, E. E. Saperstein, M. A. Troitsky, and D. N. Voskresensky, Phys. Rep. **192**, 179 (1990).
[15] D. N. Voskresensky, Lect. Notes Phys. **578**, 467 (2001).
[16] G. Raffelt and D. Seckel, Phys. Rev. D **52**, 1780 (1995).
[17] E. Olsson, P. Haensel, and C. J. Pethick, Phys. Rev. C **70**, 025804 (2004).
[18] K. E. Schmidt and S. Fantoni, Phys. Lett. B **446**, 99 (1999).
[19] S. Gandolfi, A. Y. Illarionov, K. E. Schmidt, F. Pederiva, and S. Fantoni, Phys. Rev. C **79**, 054005 (2009).

[20] B. S. Pudliner, V. R. Pandharipande, J. Carlson, S. C. Pieper, and R. B. Wiringa, Phys. Rev. C **56**, 1720 (1997).
[21] S. Fantoni, A. Sarsa, and K. E. Schmidt, Phys. Rev. Lett. **87**, 181101 (2001).
[22] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995).
[23] A. Akmal and V. R. Pandharipande, Phys. Rev. C **56**, 2261 (1997).
[24] S. Cowell and V. R. Pandharipande, Phys. Rev. C **70**, 035801 (2004).
[25] S. Gandolfi, A. Y. Illarionov, F. Pederiva, K. E. Schmidt, and S. Fantoni, Phys. Rev. C **80**, 045802 (2009).
[26] L. Leinson and A. Perez, arXiv:astro-ph/0606653.
[27] L. B. Leinson, Phys. Rev. C **81**, 025501 (2010).
[28] E. E. Kolomeitsev and D. N. Voskresensky, Phys. Rev. C **84**, 068801 (2011).
[29] E. Braaten, D. Kang, and L. Platter, Phys. Rev. Lett. **104**, 223004 (2010).
[30] D. T. Son and E. G. Thompson, Phys. Rev. A **81**, 063634 (2010).
[31] R. Schiavilla *et al.*, Phys. Rev. C **58**, 1263 (1998).
[32] L. D. Landau and I. Y. Pomeranchuk, Dokl. Akad. Nuak. SSSR **92**, 535 (1953).
[33] L. D. Landau and I. Y. Pomeranchuk, Dokl. Akad. Nuak. SSSR **92**, 735 (1953).
[34] A. B. Migdal, Phys. Rev. **103**, 1811 (1956).
[35] J. Knoll and D. N. Voskresensky, Ann. Phys. (NY) **249**, 532 (1996).
[36] S. Bacca, K. Hally, C. J. Pethick, and A. Schwenk, Phys. Rev. C **80**, 032802 (2009).