Relativistic correction of the quarkonium melting temperature with a holographic potential

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The relativistic correction of the anti de Sitter space/conformal field theory implied potential model for a quarkonium state is examined. For the typical range of the coupling strength appropriate to heavy-ion collisions, we find the correction is significant in size for J/Ψ.

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I. INTRODUCTION

The phase structure of quantum chromodynamics (QCD) remains an active field of research. At sufficiently high temperature, hadronic matter will evolve into a quark-gluon plasma (QGP), which has been explored experimentally by relativistic heavy-ion collisions (RHIC). Quarkonium melting is an important signal of this new phase [1].

A quarkonium is a bound state composed of a heavy quark q and its antiparticle $\bar{q}$. It is found that the level spacings between the ground states and the excitations of the quarkonium are much smaller than that of normal hadrons and that the pair are very tightly bounded [2] in vacuum. Theoretically, there are two approaches to study the quarkonium: lattice QCD and potential models [3]. From lattice QCD, we can calculate the spectral function numerically via the quarkonium correlators and identify the quarkonium states with the resonance peaks [4–8]. The potential model relies on the small velocity($v \ll 1$) of the constituent quarks. By solving a nonrelativistic Schrödinger equation with a temperature-dependent effective potential, we can determine the energy levels and thereby the threshold temperature when the bound state dissolves [9–14]. The potential model will be applied in the present work.

Anti de Sitter space/conformal field theory (AdS/CFT) correspondence is a powerful tool to explore the strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma. The equation of state, viscosity ratio, etc. extracted from AdS/CFT show remarkable agreement with lattice QCD or experimental data from RHIC. It would be interesting to extend the comparison to a wide range of other quantities, for instance heavy quark melting, which are calculable in both ways to assess whether the super Yang-Mills plasma serves as an important reference model of the QGP phase of QCD. This is the primary motivation of the research reported here and in a previous paper [15].

In the previous work [15], we examined heavy quarkonium melting within the potential model with the AdS/CFT implied potential function (holographic potential). We found that the holographic potential can be approximated by a truncated Coulomb potential to great accuracy. With the typical values of the 't Hooft coupling constant, $\lambda \equiv \sqrt{N_c g_{\rm YM}^2}$, considered in the literature [16],

$$5.5 < \lambda < 6\pi, \tag{1.1}$$

our melting temperatures are systematically lower, though not far from the lattice prediction [32]. On the other hand, an estimate of the velocity of the constituent quarks inside the bound state indicates that the nonrelativistic approximation may be marginal, especially for J/Ψ. This led us to examine the relativistic corrections of the holographic potential with the aid of a two-body Dirac equation (TBDE).

While the holographic potential alone is sufficient in the nonrelativistic limit, it does not provide all information necessary for the relativistic corrections even to the order v^4 term. Except for the correction brought about by the relativistic kinetic energy, the spin-orbital coupling and the Darwin term depend on how the holographic potential is introduced in the two-body Dirac equation. In addition, the gravity dual of spin-dependent forces is not available in the literature. Therefore our result remains incomplete at this stage. We would like to comment that the same issues exist for the relativistic corrections of the heavy quark potential extracted from lattice QCD simulations.

In the next section, the work reported in [15] will be reviewed and the melting temperature beyond the truncated Coulomb approximation is presented. The corrections to the melting temperature are computed in Sec. III through a Foldy-Wouthuysen (FW) transformation of a two-body Dirac Hamiltonian. The kinetic energy contribution and the contribution from the Darwin and the spin-orbit coupling are calculated separately with the latter simply by replacing the perturbative Coulomb potential in the two-body Dirac Hamiltonian with the holographic heavy quark potential. Section IV concludes the paper.

II. THE HOLOGRAPHIC POTENTIAL MODEL

In the conventional potential model of QCD, the nonrelativistic wave function of a heavy quarkonium satisfies the Schrödinger equation

$$\left[-\frac{1}{2\mu}\nabla^2 + U(r,T)\right]\psi = -E(T)\psi, \tag{2.1}$$

where $E(T)$ is the binding energy and $U(r,T)$ is identified with the internal energy of a static pair of q and $\bar{q}$ in QGP and

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is related to the free energy $F(r,T)$ via

$$U(r,T) \; = -T^2\left[\frac{\partial}{\partial T}\left(\frac{F(r,T)}{T}\right)\right]_r. \tag{2.2}$$

The free energy $F(r,T)$ can be extracted from the expectation value of a Wilson loop operator that consists of a pair of Wilson lines (Polyakov loops) operator according to

$$e^{-\frac{1}{T}F(r,T)} = \frac{\mathrm{tr}\langle W^\dagger(L_+)W(L_-)\rangle}{\mathrm{tr}\langle W^\dagger(L_+)\rangle\langle W(L_-)\rangle}, \tag{2.3}$$

where $L_\pm$ stands for the Wilson line running in the Euclidean time direction at spatial coordinates $(0,0,\pm\frac{1}{2}r)$ and is closed with the periodicity $\beta = \frac{1}{T}$. We have

$$W(L_\pm) = Pe^{-i\oint_{L_\pm} dx^\mu A_\mu(x)}. \tag{2.4}$$

where spatial coordinates of $L_\pm$ are $(0,0,\pm\frac{1}{2}r)$. The lattice QCD simulation of the expectation value (2.3) can be found in Refs. [10,17].

In case of super Yang-Mills, the holographic principle places the Wilson lines $L_\pm$ on the boundary ($y \to \infty$) of the 5D AdS-Schwarzschild metric [18]:

$$ds^2 = \pi^2T^2y^2(f dt^2 + d\vec{x}^2) + \frac{1}{\pi^2T^2y^2f}dy^2, \tag{2.5}$$

where $f = 1 - \frac{1}{y^4}$ and $d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2$ with the ansatz $x_1 = x_2 = 0$ and $x_3 = \pm\frac{1}{2}r$.

The free energy $F(r,T)$ of the corresponding super Yang-Mills theory at large N_c and large 't Hooft coupling is proportional to the minimum area of the worldsheet in the AdS bulk bounded by L_+ and L_- and is given parametrically by [18–20]

$$\begin{aligned} F(r,T) &= T\min(I,0),\\ r &= \frac{2q}{\pi T}\int_{y_c}^\infty \frac{dy}{\sqrt{(y^4-1)\left(y^4-y_c^4\right)}}, \end{aligned} \tag{2.6}$$

where

$$I = \sqrt{\lambda}\left[\int_{y_c}^\infty dy\left(\sqrt{\frac{y^4-1}{y^4-y_c^4}}-1\right)+1-y_c\right] \tag{2.7}$$

and the parameter $y_c \in (1,\infty)$. Eliminating y_c between (2.6) and (2.7), we find that

$$F(r,T) = -\frac{\alpha}{r}\Phi(\rho)\theta(\rho_0-\rho), \tag{2.8}$$

where $\alpha \doteq 0.2285\sqrt{\lambda}$, $\rho = \pi T r$, $\rho_0 = 0.7541$, and $\Phi(\rho)$ is a screening factor. The corresponding internal energy is

$$\begin{aligned} U(r,T) = -\frac{\alpha}{r}\Bigg[&\Phi(\rho) - \rho\left(\frac{d\Phi}{d\rho}\right)_{y_c}\\ &-\rho\left(\frac{d\Phi}{dy_c}\right)_\rho\left(\frac{dy_c}{d\rho}\right)\Bigg]\theta(\rho_0-\rho) \end{aligned} \tag{2.9}$$

and will be substituted into the Schrödinger equation (2.1).

The small-ρ expansion of $\Phi(\rho)$ is

$$\Phi(\rho) = 1 - \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3}\rho + \frac{3\Gamma^8\left(\frac{1}{4}\right)}{640\pi^6}\rho^4 + O(\rho^8). \tag{2.10}$$

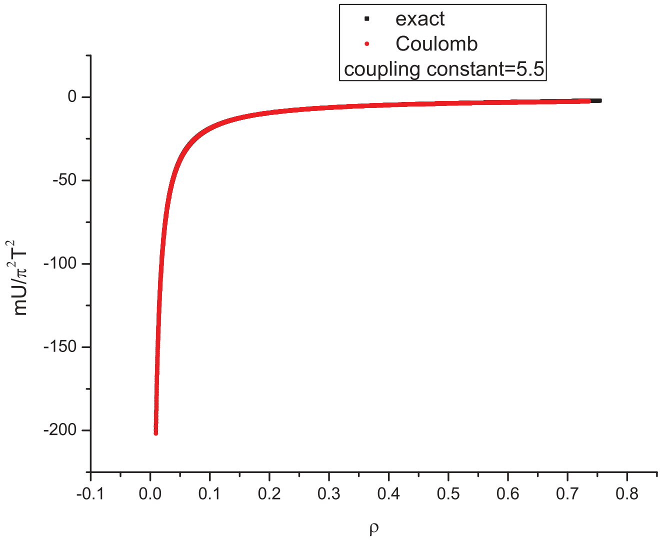

FIG. 1. (Color online) Comparison of the two-term series approximation to the exact Φ. The vertical axis is the reduced internal potential, which is convenient for us to do the nonrelativistic calculation later, where m is the mass of the particle. The red dots correspond to the Coulomb case, and the black dots correspond to the exact case.

Within the screening radius ρ_0, the first two terms of the series (2.10) approximate the exact Φ well as is shown in Fig. 1. If we keep only the first two terms, the screening radius $\rho_0 \simeq 0.7359$ and $U(r,T)$ becomes a truncated Coulomb potential

$$U = -\frac{\alpha}{r}\theta(\rho_0-\rho) \tag{2.11}$$

under the approximation.

We define the melting temperature T_d as the temperature at which the binding energy falls to zero, i.e., $E(T_d) = 0$, and the corresponding radial Schrödinger equation reads [15]

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} - \left[\frac{l(l+1)}{\rho^2} + V\right]R = 0, \tag{2.12}$$

where the reduced potential $V = \frac{mU}{\pi^2T^2}$ is dimensionless.

The truncated Coulomb potential approximation was employed in [15] and the melting temperature of the bound state of the lth partial wave and the nth radial quantum number is given by

$$T_d = \frac{4\alpha\rho_0 m}{\pi x_{nl}^2}, \tag{2.13}$$

with x_{nl} the nth nonzero root (ascending order) of the Bessel function $J_{2l}(x)$. The corresponding radial wave function reads

$$R(r) = \frac{1}{\sqrt{\rho}}J_{2l+1}\left(x_{nl}\sqrt{\frac{\rho}{\rho_0}}\right) \tag{2.14}$$

for $\rho \leqslant \rho_0$ and $R(r) = \mathrm{const.}/r^{l+1}$ for $\rho > \rho_0$.

In this work, we have calculated the melting temperature with the exact holographic potential (2.9). The comparison with that obtained from the truncated Coulomb potential in [15] for J/Ψ is shown in Table I, where we choose $m =$

TABLE I. Comparison between the melting temperatures in MeV for $1s$, $2s$, and $1p$ states with the exact holographic potential and with the truncated Coulomb potential in the nonrelativistic limit. The two sets of values are very close, which confirmed that the truncated Coulomb approximation is excellent.

	T_d ($\lambda = 5.5$)		T_d ($\lambda = 6\pi$)	
	Exact	Truncated Coulomb	Exact	Truncated Coulomb
1s	142	143	262	265
2s	27	27	50	50
1p	31	31	57	58

1.65 GeV for the mass of c quarks.[1] From the comparison of these two results we confirmed that the truncated Coulomb approximation is a good approximation and we shall stay with this approximation for the rest of the paper.

III. THE RELATIVISTIC CORRECTION OF THE HOLOGRAPHIC POTENTIAL

As is mentioned in Sec. I, the velocity of the heavy quarks is not low enough, so the relativistic correction may be significant, especially for J/Ψ. To explore this correction, one has to go beyond the Schrödinger equation (2.1) and switch to the two-body Dirac equation [21–24]

$$i\frac{\partial \Psi}{\partial t} = H\Psi. \tag{3.1}$$

In the center-of-mass frame, the Hamiltonian of the two-body Dirac equation is

$$H = \vec{\alpha}_1 \cdot \vec{p} + \beta_1 \cdot m - \vec{\alpha}_2 \cdot \vec{p} + \beta_2 \cdot m + U, \tag{3.2}$$

where $\vec{\alpha}_1 = \vec{\alpha} \otimes I$, $\vec{\alpha}_2 = I \otimes \vec{\alpha}$, $\beta_1 = \beta \otimes I$, $\beta_2 = I \otimes \beta$, $\vec{\alpha}$ and β are the usual 4×4 Dirac matrices, $\vec{p} = -i\vec{\nabla}$, and U is the interaction potential between the two particles. We shall take the ansatz by identifying U here with the holographic potential in the last section and other possible implementations will be discussed in the next section. The Hamiltonian H is a 16×16 matrix. A quarkonium state corresponds to a bound state of H with the eigenvalue $2m - E(T) < 2m$, which goes to $2m$ at the melting temperature, i.e., $E(T) = 0$. Since we are interested in the leading order relativistic correction of the melting temperature for the quarkonium, we have to expand the Hamiltonian to the order v^4. The sorting of the order in v follows from the rules that $\frac{\vec{p}^2}{m} \sim U \sim v^2$ and $\vec{\nabla} \sim \frac{1}{r} \sim mv$. Also the expectation values of $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are of the order v.

[1]The mass we used here is the same as the mass used in [15], but for the calculation of the relativistic correction in next section, we refit the mass of the c quark. We set the mass of J/Ψ in vacuum for our relativistic case equal to the nonrelativistic limit case here, and we got a mass of the c quark of 1649.998 MeV for our relativistic case. Here we would like to thank Prof. Pengfei Zhuang for his valuable suggestions.

In analogy to the one-body Foldy-Wouthuysen transformation [25], we introduce the unitary operator

$$\mathcal{U} = e^{iS_2'}e^{iS_1'}e^{iS_2}e^{iS_1}, \tag{3.3}$$

where

$$S_1 = -\frac{i}{2m}\beta_1 \cdot O_1, \tag{3.4}$$

$$S_2 = -\frac{i}{2m}\beta_2 \cdot (-O_2), \tag{3.5}$$

$$S_1' = -\frac{i}{2m}\beta_1 \cdot O_1', \tag{3.6}$$

$$S_2' = -\frac{i}{2m}\beta_2 \cdot (-O_2'), \tag{3.7}$$

with

$$O_1 = \vec{\alpha}_1 \cdot \vec{p}, \quad O_2 = \vec{\alpha}_2 \cdot \vec{p}.$$

The transformed Hamiltonian reads

$$\begin{aligned} H_{\rm FW} &= \mathcal{U} H \mathcal{U}^\dagger \\ &= (\beta_1 + \beta_2)\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + U + \frac{1}{4m^2}\nabla^2 U \\ &\quad + \frac{1}{4m^2 r}\frac{dU}{dr}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot \vec{L}, \end{aligned} \tag{3.8}$$

where higher order terms in v have been dropped. The details of the transformation are shown in the Appendix. The nonrelativistic wave function, $\Psi = \Psi_{s_1 s_2}(\vec{r}_1 - \vec{r}_2)$, with subscripts s_1 and s_2 labeling the spin components of the two quarks, corresponds to the sector with $\beta_1 = \beta_2 = 1$. (This wave function can be expanded in a series of products of the orbital wave functions of the preceding section and the spin wave functions.) We may stay within this sector for the first-order perturbation of the v^4 terms of (3.8) with the effective Hamiltonian $H_{\rm eff.} = H_0 + H_1$, where

$$H_0 = 2m + \frac{\vec{p}^2}{m} + U$$

corresponds to the nonrelativistic limit, and

$$\begin{aligned} H_1 &= -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2}\nabla^2 U + \frac{1}{4m^2 r}\frac{dU}{dr}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot \vec{L} \\ &= -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2}\nabla^2 U + \frac{1}{4m^2 r}\frac{dU}{dr}(J^2 - L^2 - S^2) \end{aligned} \tag{3.9}$$

is the relativistic correction. We have introduced the total spin $\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$ and the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ in the last step. The contribution of H_1 is somewhat similar to that for the fine structure of the hydrogen atom. The first term represents the relativistic correction to the kinetic energy, the second term is the Drawin term, and the third term is the spin-orbit coupling, which can be decomposed into spin singlet and spin triplet channels.

Perturbatively, we may write $E(T) = E_0(T) + \delta E(T)$ and $T = T_0 + \delta T$, where $E_0(T)$ is the nonrelativistic binding energy in (2.1), T_0 is the melting temperature given by (2.13), and $\delta E(T)$ and δT are the v^4 corrections. We have $E_0(T_0) = 0$. Expanding the melting condition $E(T) = 0$ to the order v^4, we obtain the formula for δT, i.e.,

$$\delta T = \delta_1 T + \delta_2 T = -\frac{\delta_1 E(T_0) + \delta_2 E(T_0)}{\left(\frac{\partial E_0}{\partial T}\right)_{T_0}}, \quad (3.10)$$

where

$$\delta_1 E(T_0) = -\left\langle \frac{\vec{p}^4}{4m^3} \right\rangle,$$
$$\delta_2 E(T_0) = \left\langle \frac{1}{4m^2}\nabla^2 U \right\rangle + \left\langle \frac{1}{4m^2 r}\frac{dU}{dr}(J^2 - L^2 - S^2) \right\rangle, \quad (3.11)$$

and

$$\left(\frac{\partial E_0}{\partial T}\right)_{T=T_0} = \left\langle \frac{\partial H_0}{\partial T} \right\rangle = \left\langle \frac{\partial U}{\partial T} \right\rangle, \quad (3.12)$$

and the average

$$\langle O \rangle \equiv \frac{\int d^3\vec{r}\,\psi^*(\vec{r})O(\vec{r})\psi(\vec{r})}{\int d^3\vec{r}\,\psi^*(\vec{r})\psi(\vec{r})}, \quad (3.13)$$

with $\psi(\vec{r})$ the nonrelativistic wave function. The reason for our separating the contribution from p^4, $\delta_1 T$, and that from the Darwin and spin-orbital terms, $\delta_2 T$, is the uncertainty in the representation of the holographic potential in (3.2), which does not impact the p^4 correction. We will come to this point in the next section. In the limit of zero binding energy, we find $\langle \frac{\vec{p}^4}{4m^3} \rangle = \langle \frac{1}{4m} U^2 \rangle$. Under the truncated Coulomb approximation,

$$\nabla^2 U = 4\pi\alpha\pi^3 T^3 \delta^3(\vec{\rho})\theta(\rho_0 - \rho) + \frac{\alpha\pi^3 T^3}{\rho}\delta'(\rho - \rho_0). \quad (3.14)$$

In terms of the radial wave function $R_l(r)$ of $\psi(\vec{r})$,

$$\left\langle -\frac{\vec{p}^4}{4m^3} \right\rangle = -\frac{\alpha^2}{4m\pi T}\int_0^{\rho_0} R_l(\rho)^2 d\rho,$$
$$\left\langle \frac{1}{4m^2}\nabla^2 U \right\rangle = \frac{\alpha}{4m^2}\{|R_l(0)|^2 - 2R_l(\rho_0)R_l'(\rho_0)\rho_0 - R_l^2(\rho_0)\},$$
$$\left\langle \frac{1}{4m^2 r}\frac{dU}{dr} \right\rangle = \frac{\alpha}{4m^2}\int_0^{\rho_0}\frac{d\rho}{\rho}R^2(\rho) + \frac{\alpha}{4m^2}R^2(\rho_0),$$
$$\left\langle \frac{\partial U}{\partial T} \right\rangle = \frac{\alpha\pi}{\pi^3 T^3}R_l^2(\rho_0)\rho_0^2. \quad (3.15)$$

For the ns state, we find that

$$\delta_1 T = \frac{\pi\alpha T_0^2}{4m\rho_0}\left[\frac{1}{J_1^2(x_{n0})} - \frac{J_0^2(x_{n0})}{J_1^2(x_{n0})} - 1\right],$$
$$\delta_1 T + \delta_2 T = -\frac{\pi\alpha T_0^2}{4m\rho_0}\left[\frac{J_0^2(x_{n0})}{J_1^2(x_{n0})} + 1 + \frac{2}{x_{n0}^2}\right]. \quad (3.16)$$

For the np states, we can also get analytical expressions which are more lengthy.

The numerical values of the corrected temperature $T_0 + \delta_1 T$ and $T_0 + \delta_1 T + \delta_2 T$ in MeV for $1s$, $2s$, and $1p$ states of $J/\Psi(c\bar{c})$ and $\Upsilon(b\bar{b})$ are listed in Table II.

TABLE II. Melting temperatures in MeV with relativistic corrections. The upper panel corresponds to the results of $T_0 + \delta_1 T$ and the lower one corresponds to $T_0 + \delta_1 T + \delta_2 T$. For the lower panel, we denote the states as nL_J^{2S+1} in the first column, where n is the main quantum number, L is the orbit angular momentum quantum number, S is the spin quantum number, and J is the total angular momentum quantum number. Since the spin-orbit coupling term vanishes for ns states, the spin singlet and spin triplet are degenerate. This, however, is not the case with a nonzero orbital angular momentum, such as np states included in the table.

	$c\bar{c}$		$b\bar{b}$	
	$\lambda = 5.5$	$\lambda = 6\pi$	$\lambda = 5.5$	$\lambda = 6\pi$
$1s$	162.54	387.54	478.76	1139.11
$2s$	29.15	62.75	85.67	184.44
$1p$	32.04	62.14	94.18	182.66
$1s_0^1$	130.79	188.65	385.63	555.58
$1s_1^3$	130.79	188.65	385.63	555.58
$2s_0^1$	26.71	48.16	79.15	142.59
$2s_1^3$	26.71	48.16	79.15	142.59
$1p_1^1$	31.53	61.33	93.54	180.79
$1p_0^3$	32.65	68.48	96.85	201.80
$1p_1^3$	32.09	64.90	95.20	191.30
$1p_2^3$	30.96	57.76	91.89	170.29

IV. DISCUSSION

In summary, we have explored the leading relativistic correction to the melting temperature of a heavy quarkonium state through a FW-like transformation of the two-body Dirac Hamiltonian with the AdS/CFT implied potential. Among the contributions we considered, the p^4 correction of the kinetic energy, being negative, enhances the binding but the Darwin term does the opposite and dominates. Consequently, the melting temperature of the s state is lowered, leaving the corrected values further below the lattice result. This disagreement can be attributed to the short screening length $r_0 = \frac{\rho_0}{\pi T}$, about 0.25 fm at $T = 200$ MeV, of the AdS/CFT potential and the sharp cutoff nature of the screening. For J/Ψ, the magnitude of the correction ranges from 8% for $\lambda = 5.5$ to 30% for $\lambda = 6\pi$, indicating significant relativistic effects toward the high end of the domain (1.1). The kink in the holographic free energy (2.8), and consequently the cutoff in the holographic potential (2.11), has been questioned in the literature [26,27]. The authors of [26] argued that the potential should never cross zero and approaches zero from below exponentially as $r \to \infty$ if an exchange of the supergravity mode is included. This modification would certainly enhance the binding and reduce the contribution of the Darwin term in the relativistic correction. Then the melting temperature would be raised. Alternatively, a complex potential was obtained in [27] by including the contribution of a complex saddle point in the path integral with the Nambu-Goto action. Its real part, similar to the previous proposal, does not cross zero and approaches zero from below following a power law $O(r^{-4})$, and thereby it strengthens the binding. But the imaginary part triggers the melting once the wave function extends beyond the onset radius of the imaginary part. The net result requires more careful investigation.

The holographic potential between moving q and $\bar{q}$ through a thermal medium has been calculated in the literature The case with a center-of-mass motion alone has been addressed in Refs. [28,29] and the effect of a rotation has been considered in [30,31]. It is suggested in Ref. [28] that the screening radius, $r_s \sim (1 - V^2)^{\frac{1}{4}}/\pi T$, is reduced with increasing velocity. As a crude estimate, the melting corresponds to a screening radius less than or equal to the radius of the quarkonium and therefore the potential alone with the center-of-mass velocity will decrease the melting temperature. On the other hand, if the center-of-mass momentum $\vec{P}$ is of the same order of magnitude as the relative one, $\vec{p}$, the expectation value of the 1st term on the right-hand side (RHS) of (3.9) will be replaced by

$$-\left\langle\frac{\left|\frac{\vec{P}}{2}+\vec{p}\right|^4+\left|\frac{\vec{P}}{2}-\vec{p}\right|^4}{8m^3}\right\rangle = -\frac{P^4}{64m^3} - \frac{\langle p^4\rangle}{4m^3} - \frac{5}{12m^3}P^2\langle p^2\rangle \tag{4.1}$$

for an s state. Unlike the nonrelativistic limit, the center-of-mass momentum couples with the relative momentum through the last term on the RHS of (3.9) and acts in the same direction as the p^4 term to raise the melting temperature. It would be interesting to extend the analysis in this paper to work out the details of the competition.

The melting temperature of a heavy quarkonium has also been determined via holographic spectral analysis by the authors of [32]. In contrast to (2.13), their melting temperature, $T_d = 2.17m/\sqrt{\lambda}$, is inversely proportional to $\sqrt{\lambda}$. It follows that $T_d(c\bar{c}) = 1.53$ GeV (826 MeV) and $T_d(b\bar{b}) = 4.50$ GeV (2.43 GeV) for $\lambda = 5.5$ (6π), which are higher than the lattice results.

The potential model, though physically more transparent than the spectral function approach, does not provide complete v^4 corrections with the holographic potential extracted from the Wilson loop alone. The same deficiency applies to the relativistic correction based on the lattice heavy quark potential alone. As the Wilson loop for a non-Abelian theory involves multigluon exchanges, its form in the two-body Dirac Hamiltonian (3.2) may not be adequate unless single gluon exchange serves as a reasonable approximation. A more general form of the interaction in (3.2) without violating charge conjugation symmetry is to replace U by

$$(\Lambda_{++} + \Lambda_{--})U + (\Lambda_{+-} + \Lambda_{-+})U' = U_+ + \beta_1\beta_2 U_-, \tag{4.2}$$

where the projection operator $\Lambda_{\pm 1,\pm 1} \equiv \frac{1\pm\beta_1}{2}\frac{1\pm\beta_2}{2}$, U' is the potential between qq or $\bar{q}\bar{q}$, which is unknown from the holograhpic dual, and $U_\pm = \frac{U\pm U'}{2}$. $U' = U$ in the previous section. Repeating the steps of the FW transformation in the Appendix, we find that the perturbing Hamiltonian (3.9) is replaced by

$$H_1 = -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2}\nabla^2 U_+ + \frac{1}{4m^2 r}\frac{dU_+}{dr}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\vec{L} + \frac{1}{8m^2}\{\{\vec{\sigma}_1\cdot\vec{\nabla}, \{\vec{\sigma}_1\cdot\vec{\nabla}, U_-\}\} + \{\vec{\sigma}_2\cdot\vec{\nabla}, \{\vec{\sigma}_2\cdot\vec{\nabla}, U_-\}\}, \tag{4.3}$$

with $\{\ldots\}$ an anticommutator. This will modify the potential part of (3.9). Only the correction from the kinetic energy, p^4, term of (3.9) is robust, which raises the melting temperature.

In addition to the holographic potential considered in this work, there should be spin-dependent ones that split degeneracies between spin singlets and spin triplets (e.g., between η_c and J/Ψ). The gravity duals of the latter are unknown in the literature. A first-principles derivation of the spin-dependent forces [33] associates them with the expectation value of Wilson loops with operator insertions, $\langle \mathrm{tr}\mathcal{W}_{\mu\nu}(L_+)^\dagger \mathcal{W}_{\rho\lambda}(L_-)\rangle$, where

$$\mathcal{W}_{\mu\nu}(L) = PF_{\mu\nu}(x)e^{-i\int_L dx^\mu A_\mu}, \tag{4.4}$$

with $F_{\mu\nu}$ the Yang-Mills field strength and x a point along L. $\mathcal{W}_{\mu\nu}(L)$ is obtained from Eqs. (2.4) by a small distortion of L at x. Within the AdS/CFT framework, it corresponds to the perturbation of the Nambu-Goto action of the worldsheet underlying the holographic potential under a small distortion of its boundary. It is a challenging boundary value problem and we hope to report our progress along this line in future.

Finally, we would like to comment on a phenomenological formulation of the two-body Dirac equation [34], which has been applied recently to the same problem addressed in this work [35]. It amounts to dividing the heavy quark potential of Cornell type into a linearly confining term and a single-gluon Coulomb term. While it is legitimate in vacuum in the weak-coupling limit because of Lorentz invariance, a direct application to a medium beyond weak coupling remains to be justified, given the different screening properties of the electric and magnetic gluons. Comparing Ref. [35] and our works, we see that both approaches rely on the static potential extracted from the expectation value of the Wilson loop with one from the lattice QCD and the other from the gravity dual. While the approximation in Ref. [35] mixed up different orders in the velocity, we attempted a systematic expansion to the order v^4. The relativistic correction of the kinetic term, p^4, raises the melting temperature in both cases, as expected. But the spin-dependent contribution is hard to compare since its gravity dual is incomplete for the purpose. After all, a question raised by the general formulation in Ref. [33] is whether the static potential extracted either from lattice QCD or from the gravity dual is sufficient to explore a heavy quarkonium beyond the nonrelativistic limit.

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APPENDIX

In this Appendix, we shall fill in the details of the Foldy-Wouthuysen transformation for the two-body Dirac equation which we have done in Sec. IV.

Let us recall the Foldy-Wouthuysen transformation of the one-body Dirac Hamiltonian, $H = \vec{\alpha}\cdot\vec{p} + \beta m + V$. For a four-component spinor with velocity $v \ll 1$ and a positive energy, one can work in the standard Dirac representation where the upper two components correspond to the nonrelativistic limit, referred to as the large components, while the lower two components are suppressed by a power of v, referred to as the small components. An operator is even (odd) if it is diagonal (off-diagonal) with respect to large and small components. For example, β is even and $\vec{\alpha}$ is odd. The Foldy-Wouthuysen transformation amounts to successive unitary transformations that push the odd operators to higher orders in v.

The Foldy-Wouthuysen transformation can be easily generalized to the two-body case, the Hilbert space of which is spanned by the direct products of two one-body spinors. To the leading order relativistic correction, we stop at the v^4 terms, ignoring all higher order terms. Consider the two-body Dirac Hamiltonian (3.2)

$$H = \vec{\alpha}_1\vec{p} + \beta_1 m - \vec{\alpha}_2\vec{p} + \beta_2 m + U \tag{A1}$$

in terms of

$$\begin{aligned} &\vec{\alpha}_1\vec{p} = O_1, \quad \vec{\alpha}_2\vec{p} = O_2, \\ &H = \underbrace{\beta_1 m + O_1}+\underbrace{\beta_2 m - O_2}+U = H_1 + H_2 + U, \end{aligned} \tag{A2}$$

where H_1 corresponds to the first underbrace, and H_2 corresponds to the second underbrace. It is easy to prove the commutation relations $[O_1, O_2] = [\beta_1, \beta_2] = [O_1, \beta_2] = [O_2, \beta_1] = 0$, and $O_1 \sim O_2 \sim v$.

1. The first transformation

We select

$$S_1 = -\frac{i}{2m}\beta_1 O_1,$$
$$S_2 = -\frac{i}{2m}\beta_2(-O_2),$$

so the transformed Hamiltonian turns into $H' = \underbrace{e^{iS_2}\overbrace{e^{iS_1}He^{-iS_1}}e^{-iS_2}}$. We calculate the mid overbrace first:

$$\begin{aligned} e^{iS_1}He^{-iS_1} &= \underbrace{e^{iS_1}H_1e^{-iS_1}}+\underbrace{e^{iS_1}H_2e^{-iS_1}}+\underbrace{e^{iS_1}Ue^{-iS_1}} \\ &= \overline{H_1} + \overline{H_2} + \bar{U}, \end{aligned} \tag{A3}$$

where $\overline{H_1}$ corresponds to the first underbrace, $\overline{H_2}$ corresponds to the second, and $\bar{U}$ corresponds to the third. We have

$$\begin{aligned} \overline{H_1} &= H_1 + [iS_1, H_1] + \frac{1}{2!}[iS_1, [iS_1, H_1]] + \frac{1}{3!}[iS_1, [iS_1, [iS_1, H_1]]] \\ &\quad + \frac{1}{4!}[iS_1, [iS_1, [iS_1, [iS_1, H_1]]]] + \cdots, \end{aligned} \tag{A4}$$

$$\begin{aligned} &[iS_1, H_1] = -O_1 + \frac{1}{m}\beta_1 O_1^2 \\ &\frac{1}{2!}[iS_1, [iS_1, H_1]] = -\frac{1}{2m}\beta_1 O_1^2 - \frac{1}{2m^2}O_1^3, \quad \frac{1}{3!}[iS_1, [iS_1, [iS_1, H_1]]] = \frac{1}{6m^2}O_1^3 - \frac{1}{6m^3}\beta_1 O_1^4, \\ &\frac{1}{4!}[iS_1, [iS_1, [iS_1, [iS_1, H_1]]]] \sim \frac{1}{24m^3}\beta_1 O_1^4, \quad \overline{H_1} = \beta_1 m + \frac{1}{2m}\beta_1 O_1^2 - \frac{1}{8m^3}\beta_1 O_1^4 - \frac{1}{3m^2}O_1^3 + O(v^5). \end{aligned} \tag{A5}$$

Similarly,

$$\overline{H_2} = H_2 + [iS_1, H_2] + \frac{1}{2!}[iS_1, [iS_1, H_2]] + \frac{1}{3!}[iS_1, [iS_1, [iS_1, H_2]]] + \frac{1}{4!}[iS_1, [iS_1, [iS_1, [iS_1, H_2]]]] + \cdots, \tag{A6}$$

$$\begin{aligned} &[iS_1, H_2] = \frac{1}{2!}[iS_1, [iS_1, H_2]] = \frac{1}{3!}[iS_1, [iS_1, [iS_1, H_1]]] = \cdots = 0, \quad \overline{H_2} = H_2 = \beta_2 m - O_2, \\ &\bar{U} = U + \frac{1}{2m}\beta_1[O_1, U] - \frac{1}{8m^2}[O_1, [O_1, U]]. \end{aligned} \tag{A7}$$

It follows that

$$\begin{aligned} e^{iS_1}He^{-iS_1} &= \overline{H_1} + \overline{H_2} + \bar{U} \\ &= \beta_1 m + \beta_2 m - O_2 + U + \frac{1}{2m}\beta_1 O_1^2 - \frac{1}{8m^2}[O_1, [O_1, U]] - \frac{1}{8m^3}\beta_1 O_1^4 + \frac{1}{2m}\beta_1[O_1, U] - \frac{1}{3m^2}O_1^3 + O(v^5) = \overline{H}, \end{aligned} \tag{A8}$$

where we mark $e^{iS_1}He^{-iS_1}$ as $\overline{H}$ for convenience. Then

$$H' = e^{iS_2}\overline{H}e^{-iS_2} \tag{A9}$$

$$= \overline{H} + [iS_2, \overline{H}] + \frac{1}{2!}[iS_2, [iS_2, \overline{H}]] + \frac{1}{3!}[iS_2, [iS_2, [iS_2, \overline{H}]]] + \frac{1}{4!}[iS_2, [iS_2, [iS_2, [iS_2, \overline{H}]]]] + \cdots. \tag{A10}$$

Since

$$[iS_2,\overline{H}]=O_2+\frac{1}{m}\beta_2O_2^2-\frac{1}{2m}\beta_2[O_2,U]-\frac{1}{4m}\beta_1\beta_2[O_2,[O_1,U]],$$
$$\frac{1}{2!}[iS_2,[iS_2,\overline{H}]]=-\frac{1}{2m}\beta_2O_2^2+\frac{1}{2m^2}O_2^3-\frac{1}{8m^2}[O_2,[O_2,U]],$$
$$\frac{1}{3!}[iS_2,[iS_2,[iS_2,\overline{H}]]]=-\frac{1}{6m^2}O_2^3-\frac{1}{6m^3}\beta_2O_2^4,\quad \frac{1}{4!}[iS_2,[iS_2,[iS_2,[iS_2,\overline{H}]]]]\sim\frac{1}{24m^3}\beta_2O_2^4, \tag{A11}$$

we find that

$$H'=\left(\beta_1m+U+\frac{1}{2m}\beta_1O_1^2-\frac{1}{8m^2}[O_1,[O_1,U]]-\frac{1}{8m^3}\beta_1O_1^4+\frac{1}{2m}\beta_1[O_1,U]-\frac{1}{3m^2}O_1^3\right)\Longrightarrow \text{mark}:H_1'$$
$$+\left(\beta_2m+\frac{1}{2m}\beta_2O_2^2-\frac{1}{8m^2}[O_2,[O_2,U]]-\frac{1}{8m^3}\beta_2O_2^4-\frac{1}{2m}\beta_2[O_2,U]+\frac{1}{3m^2}O_2^3\right)\Longrightarrow \text{mark}:H_2'$$
$$-\frac{1}{4m^2}\beta_1\beta_2[O_2,[O_1,U]]+O(v^5), \tag{A12}$$

$$H_1'=\beta_1m+\underbrace{U+\frac{1}{2m}\beta_1O_1^2-\frac{1}{8m^2}[O_1,[O_1,U]]-\frac{1}{8m^3}\beta_1O_1^4}+\underbrace{\frac{1}{2m}\beta_1[O_1,U]-\frac{1}{3m^2}O_1^3}=\beta_1m+U_1'+O_1', \tag{A13}$$

$$H_2'=\beta_2m+\underbrace{\frac{1}{2m}\beta_2O_2^2-\frac{1}{8m^2}[O_2,[O_2,U]]-\frac{1}{8m^3}\beta_2O_2^4}-\underbrace{\left(\frac{1}{2m}\beta_2[O_2,U]-\frac{1}{3m^2}O_2^3\right)}=\beta_2m+U_2'+O_2', \tag{A14}$$

$$H'=H_1'+H_2'-\frac{1}{4m^2}\beta_1\beta_2[O_2,[O_1,U]], \tag{A15}$$

where the first underbrace in the expression of $H_1'(H_2')$ corresponds to $U_1'(U_2')$, and the second corresponds to $O_1'(O_2')$.

2. The second transformation

We select

$$S_1'=-\frac{i}{2m}\beta_1O_1',\quad S_2'=-\frac{i}{2m}\beta_2(-O_2'),$$

where $O_1'\sim O_2'\sim v^3$.

We have

$$H''=e^{iS_2'}e^{iS_1'}H'e^{-iS_1'}e^{-iS_2'}=e^{iS_2'}\underbrace{e^{iS_1'}H_1'e^{-iS_1'}}e^{-iS_2'}+e^{iS_2'}\underbrace{e^{iS_1'}H_2'e^{-iS_1'}}e^{-iS_2'}-\frac{1}{4m^2}e^{iS_2'}e^{iS_1'}\beta_1\beta_2[O_2,[O_1,U]]e^{-iS_1'}e^{-iS_2'}$$
$$=e^{iS_2'}\overline{H_1'}e^{-iS_2'}+e^{iS_2'}\overline{H_2'}e^{-iS_2'}-\frac{1}{4m^2}e^{iS_2'}e^{iS_1'}\beta_1\beta_2[O_2,[O_1,U]]e^{-iS_1'}e^{-iS_2'}, \tag{A16}$$

where the first underbrace in the second line corresponds to $\overline{H_1'}$, and the second corresponds to $\overline{H_2'}$. Applying the results of the first transformation, we obtain

$$\overline{H_1'}=H_1'+[iS_1',H_1']+O(v^5)=\beta_1m+U_1'+\frac{1}{2m}\beta_1[O_1',U_1'], \tag{A17}$$

$$e^{iS_2'}\overline{H_1'}e^{-iS_2'}=\overline{H_1'}+[iS_2',\overline{H_1'}]+O(v^5)=\beta_1m+U_1'+\frac{1}{2m}\beta_1[O_1',U_1']-\frac{1}{2m}\beta_2[O_2',U_1'], \tag{A18}$$

$$\overline{H_2'}=H_2'+[iS_1',H_2']+O(v^5)=\beta_2m+U_2'-O_2'+\frac{\beta_1}{2m}[O_1',U_2']-\frac{\beta_1}{2m}[O_1',O_2'], \tag{A19}$$

$$e^{iS_2'}\overline{H_2'}e^{-iS_2'}=\overline{H_2'}+[iS_2',\overline{H_2'}]+O(v^5)=\beta_2m+U_2'-\frac{1}{2m}\beta_2[O_2',U_2']+\frac{1}{2m}\beta_1[O_1',U_2']-\frac{1}{2m}\beta_1[O_1',O_2']. \tag{A20}$$

Coming to the last term in H'', $-\frac{1}{4m^2}\underbrace{e^{iS_2'}\overbrace{e^{iS_1'}\beta_1\beta_2[O_2,[O_1,U]]e^{-iS_1'}}e^{-iS_2'}}$, we do the calculation of the overbrace first:

$$e^{iS_1'}\beta_1\beta_2[O_2,[O_1,U]]e^{-iS_1'}=\beta_1\beta_2[O_2,[O_1,U]]+[iS_1',\beta_1\beta_2[O_2,[O_1,U]]]+O(v^5)$$
$$=\left(\beta_1\beta_2[O_2,[O_1,U]]+\frac{1}{4m^2}[\beta_1[O_1,U],[O_2,[O_1,U]]]\right)\Longrightarrow mark:A. \tag{A21}$$

By considering $O_1, O_2 \sim v$ and $O_1', O_2' \sim v^3$ we get

$$e^{iS_2'} A e^{-iS_2'} = A + O(v^5) \tag{A22}$$

$$= -\frac{1}{4m^2}\beta_1\beta_2[O_2,[O_1,U]]. \tag{A23}$$

Then

$$\begin{aligned} H'' &= \beta_1 m + U_1' + \beta_2 m + U_2' - \frac{1}{4m^2}\beta_1\beta_2[O_2,[O_1,U]] \\ &= \beta_1 m + \beta_2 m + U + \frac{1}{2m}\beta_1 O_1^2 + \frac{1}{2m}\beta_2 O_2^2 - \frac{1}{8m^2}[O_1,[O_1,U]] - \frac{1}{8m^2}[O_2,[O_2,U]] \\ &\quad - \frac{1}{8m^3}\beta_1 O_1^4 - \frac{1}{8m^3}\beta_2 O_2^4 - \frac{1}{4m^2}\beta_1\beta_2[O_2,[O_1,U]]. \end{aligned} \tag{A24}$$

Substituting the explicit expressions of O_1 and O_2, we end up with

$$\begin{aligned} &O_1^2 = O_2^2 = \vec{p}^2, \quad O_1^4 = O_2^4 = \vec{p}^4, \quad [O_1,[O_1,U]] = -\nabla^2 U - \frac{2}{r}\frac{\partial U}{\partial r}\vec{\Sigma}_1\vec{L}, \\ &[O_2,[O_2,U]] = -\nabla^2 U - \frac{2}{r}\frac{\partial U}{\partial r}\vec{\Sigma}_2\vec{L}, \quad [O_2,[O_1,U]] = -\alpha_{1i}\alpha_{2j}\nabla_i\nabla_j U, \\ &H'' = \beta_1\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + \beta_2\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + U + \frac{1}{4m^2}\nabla^2 U \\ &\quad + \frac{1}{4m^2}\frac{1}{r}\frac{\partial U}{\partial r}(\vec{\Sigma}_1 + \vec{\Sigma}_2)\vec{L} + \frac{1}{4m^2}\beta_1\beta_2\alpha_{1i}\alpha_{2j}\nabla_i\nabla_j U. \end{aligned} \tag{A25}$$

The last term in (A25), though of the order of v^4, is a direct product of two odd operators and therefore does not contribute to the first-order perturbation considered in this paper.

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