Baryon number conservation and the cumulants of the net proton distribution

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We discuss the modification of the cumulants of the net baryon and net proton distributions due to the global conservation of baryon number in heavy-ion collisions. Corresponding probability distributions and their cumulants are derived analytically. We show that the conservation of baryon number results in a substantial decrease of higher order cumulants. Based on our studies, we propose an observable that is insensitive to the modifications due to baryon number conservation.

I. INTRODUCTION

The phase structure of strong interactions, quantum chromodynamics (QCD), has been studied theoretically and experimentally for many years.

On the theory side, the problem has been investigated within various models and addressed systematically in first-principles lattice QCD (LQCD) calculations. Due to the sign problem, rigorous results from LQCD on the phase structure are presently available only at vanishing net baryon density, where it has been established that QCD exhibits an analytic crossover transition [1] with a pseudo-critical temperature of $T_c \simeq 160$ MeV [2,3].

Exploratory LQCD studies at finite net baryon density [4–7], as well as model results (see, e.g., Ref. [8]), provide some indications for the existence of a critical end point (CEP) of a first-order phase transition at finite temperature and density.

In the laboratory, the properties of hot and dense QCD matter have been studied in heavy-ion collisions. Since the location of the CEP and the first-order coexistence region is not really known from theory, a search of this phase structure requires the exploration of the entire experimentally accessible phase diagram. This is achieved by varying the beam energy and monitoring observables for possible nonmonotonic behavior. A first set of experiments of this nature has been carried out by the NA49 Collaboration [9,10] at the CERN Super Proton Synchrotron (SPS) and is now pursued at somewhat higher energy in the recent beam energy scan program at the Relativistic Heavy Ion Collider (RHIC) [11]. Both the CEP and the first-order phase transition are associated with characteristic fluctuations—long-range ones for the second-order transition at the CEP and possible spinodal instabilities in case of a first-order transition [12,13]. The first measurements by the NA49 Collaboration have concentrated on the variances, or second-order cumulants, of various particle ratios and the transverse momentum as was proposed in Ref. [14]. Their measurements showed hardly any deviation from the expected Poisson fluctuations of a hadron resonance gas (HRG).

Recently it has been realized that higher order cumulants, especially of the baryon number, which serves as an order parameter of the QCD phase transition at finite density, would be more sensitive to the fluctuations associated with the second-order transition, including the CEP [15]. First, because higher order cumulants scale with higher powers of the correlation length, a finite and limited increase of the correlation length as a result of critical slowing down may still be visible. Second, since the baryon number is a conserved quantity its fluctuations are less modified by the final-state interaction in the hadronic phase [16]. Furthermore, model calculations found that the kurtosis, i.e., the ratio of the fourth-over the second-order cumulant, is negative at high baryon chemical potential, close to the chiral crossover line above the CEP [17]. In Ref. [18], the lines where the kurtosis changes sign were obtained from universality arguments in the critical region of the CEP. These theoretical predictions suggest a nonmonotonic behavior of the kurtosis of the baryon number or electric charge distributions as a function of the collision energy if chemical freeze-out happens close to the CEP.

In addition, higher order cumulants also provide sensitive information about the crossover at vanishing baryon density. As shown in Ref. [19], the sixth-order cumulant of the net baryon distribution is negative close to the crossover line. Preliminary LQCD results [20] also indicate negative values of the sixth-order cumulant close to the crossover temperature at zero baryon chemical potential. Thus, by measuring the sixth-order cumulants, one may relate the crossover and the freeze-out line at nonzero chemical potential and provide information on an approximate position of the possible CEP.

When comparing theoretical (model or LQCD) predictions to the experimental data, one has to keep in mind that the singular behavior of fluctuations is predicted in the grand canonical formulation of thermodynamics where conservation laws are imposed only on the average. Consequently, to address the same physics experimentally, one is required to approximately achieve conditions of the grand canonical ensemble, i.e., to study fluctuations in a restricted phase space.[1] This might be done by performing appropriate cuts in the

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[1]Clearly, if *all* particles are observed, the baryon number will not fluctuate.

rapidity and/or transverse momentum of detected particles. Clearly, the smaller the fraction of observed particles the smaller is the effect of global baryon and electric charge conservation. This was also demonstrated in Ref. [21] using the ultrarelativistic quantum molecular dynamics (UrQMD) model. However, care has to be taken that these cuts do not destroy the underlying correlations responsible for the physics one tries to access. Therefore, a fine balance between the need to suppress the effects of conservation laws and the requirement to preserve the dynamical correlations has to be found [16]. The subsequent studies, we believe, will help in achieving this nontrivial task.

In this paper, we will explore to what extent global baryon conservation modifies cumulants of net baryon and net proton distributions as well as their ratios. To this end we consider a system where the only correlations are due to global baryon number conservation. Therefore, we start with Poisson statistics for baryons and antibaryons and, subsequently, enforce baryon number conservation. Since the HRG model in the classical or Boltzmann approximation is also governed by Poisson statistics for the baryons and antibaryons, our results may be directly applied to the HRG, which is commonly used as a theoretical baseline for the analysis of heavy-ion collisions.[2] In other words, our approach is equivalent to a treatment of the HRG in the so-called canonical ensemble [22] with respect to the baryon number. A canonical treatment of the HRG with respect to strangeness has been reported in the literature, e.g., in Refs. [22,23]. The effect of a globally conserved charge on the variance of the charge distribution has been studied in the same framework in Refs. [24,25]. Here we will extend these studies to higher order cumulants, where the effects due to global baryon number conservation are expected to be stronger. Therefore, our results may be considered as an improved HRG prediction for the baryon number cumulants and thus provide a baseline with which measurements should be compared in order to see whether there are *additional* dynamical correlations. Clearly, this baseline can be improved by further imposing electric charge conservation. However, the number of charged particles in high-energy heavy-ion collision is considerably larger than the number of baryons plus antibaryons. Therefore, the corrections due to electric charge conservation will be subleading and only become relevant as the collision energy is reduced. This will be discussed in detail elsewhere.

Ultimately it would be desirable to incorporate the effects of global baryon number conservation into the various models or preferably into LQCD. This is a very difficult task, and, therefore, we believe for the time being the present study will be helpful for the interpretation of the experimental data.

In the next section, we derive an analytical formula for the net baryon (and net proton) probability distribution under constraints imposed by global baryon number conservation. Also the cumulant generating function, which can be used to compute cumulants of any order, is derived. In Sec. III, we consider properties of the cumulants up to the sixth order and

propose a new observable, which is insensitive to the global conservation of baryon number. Comments and conclusions are presented in Sec. IV and Sec. V, respectively.

II. GLOBAL BARYON CONSERVATION

Before we derive the relevant formulas let us remind ourselves what the problem at hand is. On the one hand we need to impose baryon number conservation on a system of baryons and antibaryons following an underlying Poisson distribution. On the other hand we have to model the finite acceptance in an experiment, since for full acceptance the baryon number does not fluctuate. Here, we model the finite acceptances simply by a binomial distribution, noting that in practice this may be more involved.[3]

These two tasks may be done in any order; i.e., one first derives the distribution for all particles subject to baryon number conservation and then folds this distribution with the binomial distribution or vice verse. Here we chose to first separate the system into observed and unobserved particles based on the binomial distribution. This is straightforward since folding a Poisson distribution with a binomial results again in a Poisson distribution. Next we impose baryon number conservation on *all* particles, observed and unobserved.

To get started, let us remind ourselves that the probability distribution of the difference of two independent random variables, each drawn from a Poisson distribution, is the so-called Skellam distribution. Therefore, in our approach as well as in the HRG in the Boltzmann limit, in the absence of baryon number conservation the net baryon number is distributed according to the Skellam distribution (see, e.g., [26,27]). Thus, in the following we will generalize the Skellam probability distribution by imposing global baryon number conservation. For the sake of simplicity, we perform our derivation for the net baryon number, and later we generalize our result to net protons.

Suppose we have on average $\langle N_B \rangle$ baryons and $\langle N_{\bar{B}} \rangle$ antibaryons in the full phase space and an average net baryon number of $B = \langle N_B \rangle - \langle N_{\bar{B}} \rangle$. According to our assumptions, both N_B and $N_{\bar{B}}$ follow a Poisson distribution. In order to model the finite acceptance we next split the full phase space into two subsystems, one representing the measured particles and one representing the unobserved rest of the particles. In the absence of any correlations particles are distributed between two subsystems according to a binomial distribution, where the probability p_B ($p_{\bar{B}}$) to observe a baryon (antibaryon) is simply given by the fraction of the average number of observed baryons (antibaryons) to the average number of baryons (antibaryons) in the full phase space.[4] Consequently, in both subsystems baryons and antibaryons are distributed according to Poisson distributions with appropriate means.

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[2]For baryons and antibaryons the classical (Boltzmann) approximation is justified due to a large ratio of mass to temperature.

[3]The use of a binomial distribution is correct if there are no correlations among baryons and antibaryons other than the global conservation of baryon number, which we take into account explicitly. See Sec. IV for further discussion on this point.

[4]Obviously, $0 \leqslant p_{B,\bar{B}} \leqslant 1$.

As a result, the probability to observe n_1 net baryons in the measured phase space is again given by the Skellam distribution [26,27]

$$P_1(n_1) = \mathcal{N}_1 \left(\frac{p_B \langle N_B \rangle}{p_{\bar{B}} \langle N_{\bar{B}} \rangle} \right)^{n_1/2} I_{n_1}(2z\sqrt{p_B p_{\bar{B}}}), \qquad (1)$$

and analogously in the unmeasured phase space, we have

$$P_2(n_2) = \mathcal{N}_2 \left[\frac{(1-p_B)\langle N_B \rangle}{(1-p_{\bar{B}})\langle N_{\bar{B}} \rangle} \right]^{n_2/2}$$
$$\times I_{n_2}(2z\sqrt{(1-p_B)(1-p_{\bar{B}})}), \qquad (2)$$

where $\mathcal{N}_{1,2}$ are unimportant normalization constants and

$$z = \sqrt{\langle N_B \rangle \langle N_{\bar{B}} \rangle}. \qquad (3)$$

The joint probability to have n_1 net baryons in the observed subsystem and n_2 in the unobserved subsystem is given by

$$P(n_1, n_2) = P_1(n_1) P_2(n_2). \qquad (4)$$

To impose the conservation of baryon number we multiply $P(n_1, n_2)$ by $\delta_{n_1+n_2, B}$ and sum over all values of the unobserved net baryon number, n_2:

$$P_B(n_1) = \mathcal{N} \sum_{n_2} P_1(n_1) P_2(n_2) \delta_{n_1+n_2, B}. \qquad (5)$$

The normalization factor $\mathcal{N}$ is fixed from the condition

$$\sum_{n_1} P_B(n_1) = 1. \qquad (6)$$

Using Graf's addition formula[5] we obtain the net baryon probability distribution

$$P_B(n) = \left(\frac{p_B}{p_{\bar{B}}} \right)^{n/2} \left(\frac{1-p_B}{1-p_{\bar{B}}} \right)^{(B-n)/2}$$
$$\times \frac{I_n(2z\sqrt{p_B p_{\bar{B}}}) I_{B-n}(2z\sqrt{(1-p_B)(1-p_{\bar{B}})})}{I_B(2z)}, \quad (7)$$

where z is given in Eq. (3). A detailed derivation of this result will be shown elsewhere. The cumulant-generating function $g(t)$ for the cumulants c_k,

$$g(t) = \sum_{k=1}^{\infty} c_k \frac{t^k}{k!}, \qquad (8)$$

is given by

$$g(t) = \ln \left(\sum_n P_B(n) e^{nt} \right)$$
$$= \ln \left[\left(\frac{q_+}{q_-} \right)^{B/2} \frac{I_B(2z\sqrt{q_+ q_-})}{I_B(2z)} \right], \qquad (9)$$

where $q_+ = 1 - p_B + p_B e^t$ and $q_- = 1 - p_{\bar{B}} + p_{\bar{B}} e^{-t}$.

Finally, the probability distribution for net protons results from the observation that all baryons other than protons (e.g.,

neutrons) may be considered as unobserved baryons. Thus, to obtain analogs of Eqs. (7) and (9) for net protons, one simply defines the binomial probability as follows:

$$p_B = \frac{\langle n_B \rangle}{\langle N_B \rangle} \rightarrow \frac{\langle n_p \rangle}{\langle N_B \rangle}, \qquad (10)$$

where $\langle n_p \rangle$ is the mean number of observed protons (and analogously for $p_{\bar{B}}$).[6]

III. CUMULANTS

In this section we present the cumulants in the case where $p_B = p_{\bar{B}} = p$. They result from Eq. (9) by taking the appropriate number of derivatives with respect to t and setting t to 0.

As seen from Eq. (9), derivatives will generate Bessel functions of various order. Those can be simplified by using the known properties of the Bessel functions[7] as well as the expression for the average number of baryons and antibaryons in the case of baryon number conservation, $\langle N_B \rangle_C$ and $\langle N_{\bar{B}} \rangle_C$, respectively, which will be discussed in the next section [see Eq. (19)]. For the first three even-order cumulants we obtain

$$c_2 = p(1-p)\langle N \rangle_C, \qquad (11)$$
$$c_4 = c_2 + 3(p^2 q^2 B^2 - c_2^2) + 6pq(2z^2 pq - c_2), \quad (12)$$
$$c_6 = c_4 + 4(c_4 - c_2) - 10(2pq + c_2)(c_4 - c_2)$$
$$\qquad - 30pq(p^2 q^2 B^2 + c_2^2), \qquad (13)$$

where $\langle N \rangle_C = \langle N_B \rangle_C + \langle N_{\bar{B}} \rangle_C$, $q = 1 - p$, and z is defined in Eq. (3). Note that in the limit $p \rightarrow 0$ one obtains $c_6 \approx c_4 \approx c_2 \approx p\langle N \rangle_C$. It is worth mentioning that in case of the Skellam distribution, i.e., disregarding the conservation law, we have $c_6^S = c_4^S = c_2^S = p\langle N \rangle$.

The odd-order cumulants can be expressed by polynomials in p:

$$c_1 = pB, \qquad (14)$$
$$c_3 = c_1(1-p)(1-2p), \qquad (15)$$
$$c_5 = c_3[1 - 12p(1-p)]. \qquad (16)$$

For the Skellam distribution, $c_5^S = c_3^S = c_1^S = pB$. As seen from Eqs. (14)–(16) the odd-order cumulants are linear in B and do not depend on z. Thus, their ratios are uniquely defined in terms of p.

Next let us define the ratio $R_{n,m}$ as

$$R_{n,m} = \frac{c_n}{c_m}. \qquad (17)$$

In Figs. 1 and 2, we show the dependence of the ratios of cumulants, $R_{n,m}$, on p for realistic values of B and $\langle N_{B,\bar{B}} \rangle$. The ratios of even cumulants are symmetric with respect to $p \rightarrow 1 - p$ as seen from Eqs. (11)–(13). We note that for the Skellam distribution the ratios shown in Figs. 1 and 2 are unity. Therefore, we observe substantial corrections due to baryon number conservation.

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[5]Graf's addition formula [28] is given by $\sum_k t^k I_k(x) I_{n-k}(y) = (t\frac{y+tx}{x+ty})^{\frac{n}{2}} I_n(\sqrt{x^2 + y^2 + \frac{1+t^2}{t} xy})$, which for $t = 1$ reduces to $\sum_k I_k(x) I_{n-k}(y) = I_n(x+y)$.

[6]Since there are at least as many neutrons in the system, $\langle n_p \rangle / \langle N_B \rangle \leqslant \frac{1}{2}$.

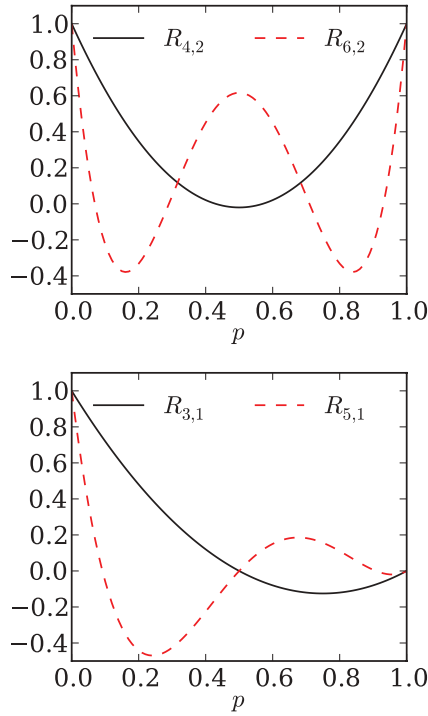[7]$I_k(x) = \frac{x}{2k}[I_{k-1}(x) - I_{k+1}(x)]$.

FIG. 1. (Color online) Ratios of odd- and even-order cumulants as a function of the fraction of measured baryons, p. The parameters are $B = 300$, $\langle N_B \rangle = 400$, and $\langle N_{\bar{B}} \rangle = 100$.

As already mentioned, the ratios of the odd-order cumulants depend only on p. This allows us to construct the following combination:

$$D = R_{5,1} - R_{3,1}\left[1 - \tfrac{3}{4}(1 + \gamma)(3 - \gamma)\right], \qquad (18)$$

such that $D = 0$ for the baryon-conservation-corrected distribution $P_B(n)$, Eq. (7), for any values of p, z, and B. Here, $\gamma = \pm\sqrt{1 + 8R_{3,1}}$. The upper (lower) sign should be taken for $p < 3/4$ ($p > 3/4$).[8] Also, $D = 0$ for the Skellam distribution.

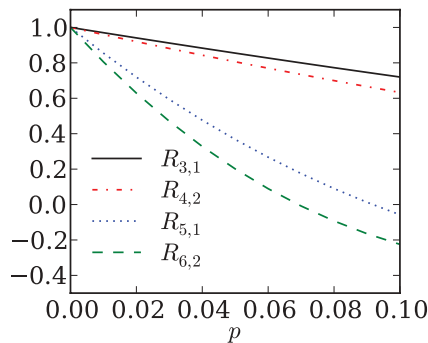[8] For an analysis of experimental data, the case with $p < 3/4$ should be considered.

FIG. 2. (Color online) Ratios of odd- and even-order cumulants as a function of the fraction of measured baryons, p, in the range of values which are of experiment interest. The parameters are $B = 300$, $\langle N_B \rangle = 400$, and $\langle N_{\bar{B}} \rangle = 100$.

Therefore, a deviation of D from zero may indicate physics that is not related to global baryon conservation.

IV. DISCUSSION AND COMMENTS

Several comments are in order regarding our results obtained in the previous sections:

(i) The distribution (7) depends on $z = \sqrt{\langle N_B \rangle \langle N_{\bar{B}} \rangle}$, where $\langle N_B \rangle$ ($\langle N_{\bar{B}} \rangle$) is the total baryon (antibaryon) number present in the Skellam distributions (1) and (2). Thus, $\langle N_B \rangle$ ($\langle N_{\bar{B}} \rangle$) is related to the system without baryon conservation. It is natural to expect that baryon conservation will modify $\langle N_B \rangle$ ($\langle N_{\bar{B}} \rangle$); however, as we argue below this correction is negligible. A straightforward calculation gives

$$\langle N_{B,\bar{B}} \rangle_C = z \frac{I_{B \mp 1}(2z)}{I_B(2z)}, \qquad (19)$$

where the upper (lower) sign corresponds to $\langle N_B \rangle_C$ ($\langle N_{\bar{B}} \rangle_C$), with $\langle N_B \rangle_C - \langle N_{\bar{B}} \rangle_C = B$. Here the subscript $\langle \cdot \rangle_C$ refers to averages obtained with full baryon number conservation. Under the constraint $\langle N_B \rangle - \langle N_{\bar{B}} \rangle = B$, one can express z in terms of $\langle N_{B,\bar{B}} \rangle_C$, and to a very good approximation we find

$$z \approx \sqrt{\langle N_B \rangle_C \langle N_{\bar{B}} \rangle_C}. \qquad (20)$$

Using the properties of the modified Bessel functions, one can show that corrections to Eq. (20) are important only if both B and $\langle N_B \rangle_C \langle N_{\bar{B}} \rangle_C$ simultaneously assume a value of the order of one or smaller. This is never the case in heavy-ion collisions. Relation (20) together with the requirement that $\langle N_B \rangle - \langle N_{\bar{B}} \rangle = B$ ensures that $\langle N_B \rangle \approx \langle N_B \rangle_C$ and $\langle N_{\bar{B}} \rangle \approx \langle N_{\bar{B}} \rangle_C$ to very good precision. The same identities also hold if we only consider protons. Therefore, the formalism developed in the previous section is of a great phenomenological value since it allows us to calculate the effect of baryon number conservation on the probability distribution and its cumulants given experimentally determined average yields.

(ii) We have shown that the odd-order cumulants do not depend on $\langle N_{B,\bar{B}} \rangle$; their ratios are independent of B and uniquely defined by one parameter, the fraction of observed baryons (protons), p. This turns out to be very useful for the phenomenological analysis of experimental data. For example, chiral model calculations at nonzero baryon densities show that both $R_{3,1}$ and $R_{5,1}$ are nontrivial functions of temperature and chemical potential close to the crossover and the CEP. This is demonstrated in Fig. 3, where we present the results obtained in the Polyakov loop-extended quark-meson model [17] for $R_{3,1}$ and $R_{5,1}$. We also show the new observable D [see Eq. (18)], which exhibits strong, temperature-dependent deviations from the baseline of $D = 0$, even for temperatures below the pseudo-critical one, $T < T_{pc}$. Therefore, effects due to a possible phase transition should be accessible in experiment via an analysis of this new observable D.
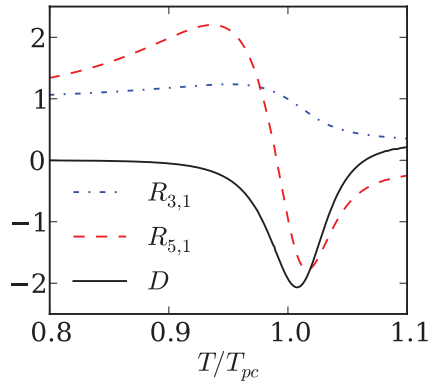
FIG. 3. (Color online) The ratios $R_{3,1}$, $R_{5,1}$, and D as a function of temperature in the Polyakov loop-extended quark-meson model. The calculations are performed along the line of fixed $\mu_B/T \approx 0.5$. T_{pc} is the crossover temperature at the corresponding chemical potential.

(iii) For a given experiment, the parameter p can be roughly estimated. For example, for the STAR experiment [11] the mean number of accepted protons in the most central Au-Au collision at $\sqrt{s} = 200$ GeV in the measured phase space is $\langle n_p \rangle \approx 7$. The total mean number of protons can be estimated from data on dN_p/dy at zero rapidity, $dN_p/dy = 35$ [29], and by assuming a flat rapidity distribution in the range $-3 < y < 3$. Therefore, $\langle N_p \rangle \approx 35 \cdot 6 = 210$. We are, however, interested in the total baryon number, $\langle N_B \rangle$. Therefore, $\langle N_p \rangle$ is to be multiplied by some factor f that takes into account the contribution of neutrons, Λ particles, and other long-living resonances. We estimated this factor in the thermal model. At the temperature $T \approx 166$ MeV corresponding to $\sqrt{s} = 200$ GeV we obtain $f = 2.5$, so that $\langle N_B \rangle \approx 210 \cdot 2.5 = 525$. A similar number for $\langle N_B \rangle$ can be obtained using BRAHMS data [30]. The fraction of measured protons to the total number of baryons is $p \approx 7/525 \approx 0.013$. For this value of p and $B = 350$, we obtain $R_{4,2} \approx 0.95$ and $R_{6,2} \approx 0.77$. There is some uncertainty related to the problem that we should only include those baryons that play a role in the quasi-equilibrium physics. This number is, however, difficult to estimate reliably.

(iv) At sufficiently low energies, $\langle N_B \rangle \gg \langle N_{\bar{B}} \rangle$, and $z \ll B$, the cumulant-generating function (9) reduces to[9]

$$\tilde{g}(t) = B \ln[1 - p(1 - e^t)]. \tag{21}$$

In this case all ratios of the cumulants depend only on the fraction of observed baryons (protons), p. Taking $\langle n_p \rangle = 9$ and $\langle N_B \rangle = 350$ (number of participants in central collisions) we obtain $p \approx 0.026$ and in consequence $R_{4,2} \approx 0.85$ and $R_{6,2} \approx 0.32$.

(v) In this paper we have disregarded other conservation laws, e.g., electric charge conservation, which is expected to play a significant role at low energies. We believe that at a collision energy higher than 10 GeV, energy, momentum, and electric charge conservation can be neglected owing to the high abundance of pions. A detailed investigation of electric charge conservation will be reported elsewhere.

(vi) We also have neglected nonequilibrium effects, effects of interactions, volume fluctuations [31] resulting, e.g., from centrality fluctuations, etc. For example, here we have modeled the acceptance of baryons and antibaryons simply by a binomial distribution. This is only correct if there are no correlations among baryons and antibaryons other than the global baryon number conservation, which we have accounted for explicitly in our calculations. In reality, one could very well imagine that the stopping of the baryons from the colliding nuclei may give rise to correlations and thus fluctuations of the baryon number which are not accounted for by the binomial distribution used in our calculation. These additional fluctuations will contribute to the various cumulants and thus need to be understood in order to extract any signal for critical fluctuations. Consequently, our proposed observable, D, will also deviate from zero, since D is designed to remove only correlations due to global baryon number conservation. This in turn may be utilized to study baryon and antibaryon correlations and fluctuations due to stopping.

(vii) In the present paper we assume that baryon number is conserved globally. However, it is plausible that baryon number is conserved locally (a few units of rapidity). This effect would reduce the effective total number of (anti)baryons $\langle N_{B,\bar{B}} \rangle$ and increase the value of $p_{B,\bar{B}}$ and, consequently, the corrections due to baryon number conservation.

V. CONCLUSIONS

We have studied the effects of global baryon conservation on the cumulants of net baryon and net proton fluctuations. We showed that the cumulants are substantially suppressed if global baryon conservation is taken into account. We also proposed a new observable that is insensitive to global baryon conservation but changes rapidly at the critical end point or the crossover.

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[9]For $2x \ll \sqrt{k+1}$, $I_k(2x) \approx x^k/\Gamma(k+1)$.

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