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Derivation of the Fermi function in perturbative quantum field theory

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We postulate that the Fermi function should be derived from the amplitude, not from the solution of the Dirac equation, in quantum field theory. Then, we obtain the following results: (1) We give the amplitude and width of the neutron β decay, $n \to p + e^- + \bar{\nu}_e$, to the first order in α. We evaluate it using Feynman parameters. (2) As the result, we confirm the terms, which can be interpreted as the Fermi function expanded to order α. (3) We give the same result using the contour integral. (4) We check that there are no such terms in a similar process, $\bar{\nu}_e + p \to e^+ + n$. (5) We perform the Fermi function expanded to the second order in α using contour integral. (6) The conventional Fermi function affects the convergence of perturbation theory.

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I. INTRODUCTION

To evaluate the β decay rates, for example, $n \to p + e^- + \bar{\nu}_e$, we introduce the Fermi function. It represents the effect of the electromagnetic potential caused by the proton. The electron runs through this potential. This function affects the β spectrum, the decay width, and the lifetime of the parent particle.

The Fermi function has been derived so far as the solution of the Dirac equation in electromagnetic potential [1–4]. In β decay, the decay itself is caused by weak interaction and is treated as the intermediate state, which is represented as the amplitude. The final-state particles are in electromagnetic potential. This effect is factorized as the Fermi function [5]. For the nonrelativistic limit in neutron β decay, it takes the form

$$F_{\rm NR} = \frac{2\pi\alpha/v}{1 - e^{-2\pi\alpha/v}}, \qquad (1)$$

where α is the fine structure constant and v is the electron velocity in the neutron rest frame. To calculate the decay width, the Fermi function is multiplied by the absolute square of the amplitude and integrated over the phase space. This is the same for the loop amplitude. For the sake of simplicity, in most of this paper, we set the parent and daughter nucleons as neutron and proton, respectively.

From a quantum field theoretical point of view, this potential effect is also the interaction of the intermediate state. Generally, the created particles should be considered as the asymptotic fields in the far future. The asymptotic fields are free from the interactions between the created particles. The electromagnetic interaction should be derived from the loop diagrams. Furthermore, this effect should appear in perturbation theory as

$$F_{\rm NR} = 1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{3v^2} - \frac{\pi^4\alpha^4}{45v^4} + \cdots, \qquad (2)$$

order by order with respect to α. References [6,7] mentioned that the Coulomb term appeared in the one-loop correction.

In Refs. [6,7], this term was factored out as a part of the Fermi function following the usual convention. It is appropriate for perturbation theory of order α. However, $F_{\rm NR}$ is actually expanded with respect to α/v rather than α. This affects order α correction as we discuss in Sec. V.

In this paper, we give the Fermi function up to order α^2. In Sec. II, the β-decay amplitude and the decay width to the first order in α are obtained by computing the integral directly using Feynman parameters. In Sec. III, we extract only the Fermi function to the first order in α using the contour integrals. In Sec. IV, we perform the Fermi function to the second order in α. In Sec. V, the conclusion and discussion are given.

II. A ONE-LOOP CALCULATION INTRODUCING THE FEYNMAN PARAMETERS

The tree-level β-decay diagram is depicted in Fig. 1, where the parameters in each set of parentheses represent their momenta. Also, the one-loop diagrams are depicted in Fig. 2 with the field-strength renormalization. According to Ref. [8], the one-loop amplitude is divided into three parts as $iM_{1L} = iM_1 + iM_2 + iM_3$. We calculate only $iM_1 + iM_2$, which does not depends on $\sigma^{\mu\nu}k_\nu$ in the numerator of proton propagator, where $\sigma^{\mu\nu} = i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$. To cancel the infrared divergence, we consider the sum of two bremsstrahlung diagrams, iM_b.

The detailed calculations are given in Appendix A, and the result is

$$d\Gamma = d\Gamma_3 + \frac{1}{\pi}G_F^2\frac{d^3q'}{(2\pi)^3}(1+3C^2)k_M^2 \times \left[1 + \frac{\alpha}{2\pi}\left\{\frac{2\pi^2}{v} + 3\log\frac{m_p}{m_e} - \frac{1}{2} - \frac{4}{v}Li\left(\frac{2v}{1+v}\right)\right.\right.$$

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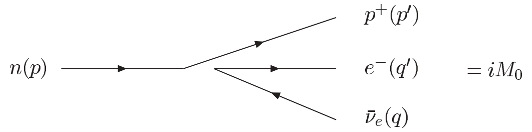

FIG. 1. Tree-level diagram.

$$+4\left(\frac{1}{v}\mathrm{Tanh}^{-1}v-1\right)\left(\frac{k_M}{3E_e}-\frac{3}{2}+\log\frac{2k_M}{m_e}\right)$$
$$+\frac{1}{v}\mathrm{Tanh}^{-1}v\left(2(1+v^2)+\frac{k_M^2}{6E_e^2}-4\mathrm{Tanh}^{-1}v\right)\Bigg\}\Bigg], \quad (3)$$

where $k_M = m_n - m_p - E_e$; G_F, m_n, m_p, m_e, and E_e are the Fermi constant, neutron mass, proton mass, electron mass, and electron energy, respectively; C represents the Gamow-Teller coupling constant relative to the Fermi constant; and $Li(x)$ is the Spence function defined as

$$Li(x) = -\int_0^x dz\frac{\log(1-z)}{z}. \quad (4)$$

k_M can be interpreted as the maximum radiated photon energy for given E_e in the bremsstrahlung diagrams. We set $m_n \simeq m_p \gg E_e, m_e$ and the neutrino mass is zero. $d\Gamma_3$ is a part depending on iM_3, which does not depend on v [8]. We do not calculate $d\Gamma_3$ in this paper. The electron velocity is represented as $v = |\boldsymbol{q}'|/q_0'$ in the neutron rest frame. The first term in the curly brackets of Eq. (3) is inversely proportional to v. This term is interpreted as the Fermi function expanded to the first order in α, which is expressed in Eq. (2).

According to Appendix A, the amplitude is approximately written as

$$iM_0 + iM_1 \ni iM_0\left(1+\frac{\alpha}{4\pi}I_{5a}\right)$$
$$\ni iM_0\left\{1+\frac{\pi\alpha}{2v}+i\frac{\alpha}{2v}\log\left(\frac{4m_e^2}{\mu^2}\frac{v^2}{1-v^2}\right)\right\} \quad (5)$$

for $v \ll 1$, where μ is the photon mass introduced to regulate the infrared divergence. The last term does not affect the one-loop decay width but affects the two-loop one as explained later.

III. EXTRACTING ONE-LOOP FERMI FUNCTION

We derive Eq. (5) again. Here, we use the contour integral. According to Appendix A, I_{5a} contains this term. I_{5a} originates from one-loop diagram in Fig. 2, not from the bremsstrahlung or field-strength renormalization terms. Furthermore, the integrand of I_{5a} does not contain k in its numerator. Therefore, we start from

$$iM_1 \ni \int\frac{d^4k}{(2\pi)^4}\frac{-4e^2M_0p'\cdot q'}{(p'-k)^2-m_p^2+i\epsilon}$$
$$\times\frac{1}{(q'+k)^2-m_e^2+i\epsilon}\frac{1}{k^2-\mu^2+i\epsilon}, \quad (6)$$

where $i\epsilon$ is a convergence factor; e is the electromagnetic coupling constant.

Next, we integrate Eq. (6) over k_0. The k_0 integral can be performed as a contour integral in the complex plane. We close the contour downward, picking up the poles at $k_0 = p_0' + \sqrt{{p_0'}^2+\boldsymbol{k}^2-i\epsilon}$, $k_0 = -q_0' + \sqrt{{q_0'}^2+\boldsymbol{k}^2-2\boldsymbol{q}'\cdot\boldsymbol{k}-i\epsilon}$, and $k_0 = \sqrt{\boldsymbol{k}^2+\mu^2-i\epsilon}$. Since we focus on the terms proportional to α/v, which dominate for $v = |\boldsymbol{q}'|/q_0' \ll 1$, we set $p_0' \gg q_0' \gg |\boldsymbol{q}'|$. The contributions that converge for $v \to 0$ can be ignored. For $|\boldsymbol{k}| \gtrsim q_0'$, the integral over $|\boldsymbol{k}|$ converges even if we set $v \to 0$, though we keep v finite. Hence, we set $q_0' \gg |\boldsymbol{k}|$, and the locations of poles are $k_0 \simeq 2p_0'$, $k_0 \simeq \frac{1}{2q_0'}(\boldsymbol{k}^2 - 2\boldsymbol{q}'\cdot\boldsymbol{k} - i\epsilon)$, and $k_0 \simeq \sqrt{\boldsymbol{k}^2+\mu^2-i\epsilon}$. When we write only the contribution from the second pole explicitly,

$$iM_1 \ni 2e^2q_0'iM_0\int\frac{d^3k}{(2\pi)^3}\frac{1}{\boldsymbol{k}^2+2\boldsymbol{q}'\cdot\boldsymbol{k}-i\epsilon}$$
$$\times\frac{1}{\boldsymbol{k}^2+\mu^2-i\epsilon}+\cdots, \quad (7)$$

where "$+\cdots$" represents the contribution from other poles.

By defining $\boldsymbol{x} = \boldsymbol{k}/|\boldsymbol{q}'|$, $x = |\boldsymbol{x}|$, $\cos\theta = \boldsymbol{q}'\cdot\boldsymbol{k}/(|\boldsymbol{q}'||\boldsymbol{k}|)$, and $\bar{\mu} = \mu/|\boldsymbol{q}'|$, the integrand becomes dimensionless as

$$iM_1 \ni \frac{e^2iM_0}{2\pi^2}\frac{q_0'}{|\boldsymbol{q}'|}\int\frac{d\cos\theta\,dx}{x^2+2x\cos\theta-i\epsilon}\frac{x^2}{x^2+\bar{\mu}^2-i\epsilon}+\cdots$$
$$\ni\frac{\alpha iM_0}{\pi v}\int_0^\infty xdx\frac{\log(x^2+2x-i\epsilon)-\log(x^2-2x-i\epsilon)}{x^2+\bar{\mu}^2-i\epsilon}. \quad (8)$$

In the right-hand side, "$+\cdots$" is ignored since these terms are not proportional to the factor $1/v$. Generally, the dimensionless integrand does not contain the factor v. Then, we can ignore the terms that do not have the factor $1/v$ as a coefficient of the integral.

Here, if the numerator of the integrand has the term that contains k, it gives additional factor $|\boldsymbol{q}'|$ when we take the integrand dimensionless. Then, such a term cannot be the candidate of the Fermi function.

The x integral can be performed as a contour integral in the complex plane. Since the integrand in Eq. (8) is an even function, the amplitude can be written as

$$iM_1 \ni \frac{\alpha}{2\pi v}iM_0\int_{-\infty}^{\infty}\frac{xdx}{x^2+\bar{\mu}^2-i\epsilon}[\log(x^2+2x-i\epsilon)$$
$$-\log(x^2-2x-i\epsilon)]. \quad (9)$$

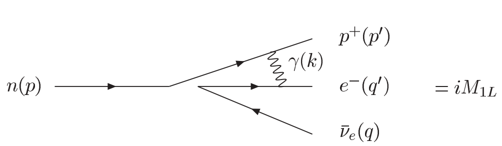

FIG. 2. One-loop diagram.

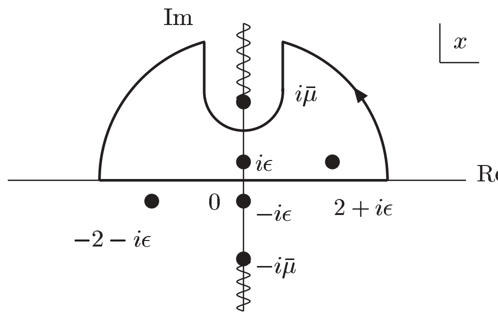

FIG. 3. x contour and the location of the poles.

To apply the residue theorem, we integrate Eq. (9) by parts to form

$$iM_1 \ni \frac{\alpha}{\pi v} iM_0 \int_{-\infty}^{\infty} dx \frac{x^2}{(x^2+2x-i\epsilon)(x^2-2x-i\epsilon)} \times \log(x^2+\bar{\mu}^2-i\epsilon). \tag{10}$$

By applying the residue theorem, the amplitude becomes

$$\begin{aligned} iM_1 &\ni iM_0 \left(\frac{\pi\alpha}{2v} - i\frac{\alpha}{v} \log\frac{\bar{\mu}}{2} \right) \\ &= iM_0 \left\{ \frac{\pi\alpha}{2v} + i\frac{\alpha}{2v} \log\left(\frac{4m_e^2}{\mu^2} \frac{v^2}{1-v^2} \right) \right\}, \end{aligned} \tag{11}$$

where the contour is depicted in Fig. 3.

This result is consistent with Eq. (5), not only the real part but also the imaginary part in the curly brackets.

A. Verification in the scattering process

No more than one charged particles exists at the same time in the scattering process $\bar{\nu}_e + p \to e^+ + n$. Therefore, the electron and proton are not affected by the electromagnetic potential and the amplitude should not contain the α/v terms, which can be interpreted as a part of the Fermi function. Here, we verify it.

According to Ref. [9], the diagram is depicted in Fig. 4.

We extract the terms we are interested in using a similar manner as in the β decay,

$$iM^{(v)} \ni \int \frac{d^4k}{(2\pi)^4} \frac{4e^2 p'\cdot q' M_0'}{(p'-k)^2 - m_p^2 + i\epsilon} \times \frac{1}{(q'-k)^2 - m_e^2 + i\epsilon} \frac{1}{k^2-\mu^2+i\epsilon}, \tag{12}$$

which corresponds to Eq. (6) in β decay. We integrate over k_0. We close the contour upward, picking up the poles at $k_0 \simeq -\frac{1}{2p_0'}(\boldsymbol{k}^2 - i\epsilon)$ and $k_0 \simeq -\frac{1}{2q_0'}(\boldsymbol{k}^2 - 2\boldsymbol{q}'\cdot\boldsymbol{k} - i\epsilon)$ using the same approximation with the β decay. Each residue has the same value with the opposite sign. Then, the amplitude becomes

$$\begin{aligned} iM^{(v)} &\ni 2e^2 q_0' iM_0' \int \frac{d^3k}{(2\pi)^3} \frac{1}{\boldsymbol{k}^2 + 2\boldsymbol{q}'\cdot\boldsymbol{k} - i\epsilon} \frac{1}{\boldsymbol{k}^2+\mu^2-i\epsilon} \\ &\quad - 2e^2 q_0' iM_0' \int \frac{d^3k}{(2\pi)^3} \frac{1}{\boldsymbol{k}^2 + 2\boldsymbol{q}'\cdot\boldsymbol{k} - i\epsilon} \frac{1}{\boldsymbol{k}^2+\mu^2-i\epsilon} \\ &= 0, \end{aligned}$$

where iM_0' is the tree-level amplitude.

As a result, this scattering process does not have the α/v term, which can be interpreted as the part of the Fermi function. This is because the signs on k in the electron and proton propagators are the same. It is just equivalent to the two charged particles, which do not exist at the same time. Also, the one-loop correction of $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$ decay is given in Ref. [10], which does not have the α/v term. These support the idea that the α/v terms in decay correspond to the potential effect.

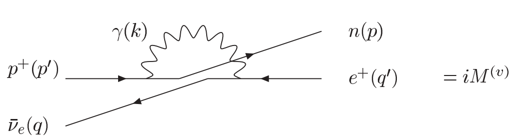

FIG. 4. One-loop diagram.

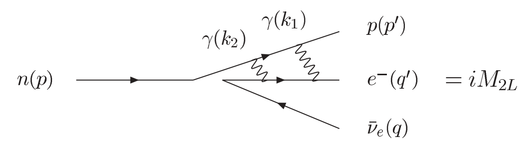

FIG. 5. Two-loop diagram.

IV. TWO-LOOP FERMI FUNCTION

A two-loop ladder diagram is depicted in Fig. 5, where k_1 and k_2 are the photon momenta, respectively. There are some other-two loop diagrams. However, we are only interested in the term proportional to $(\alpha/v)^2$. This term originates only from iM_{2L}. According to Appendix B, after a calculation similar to that of the previous section, the amplitude contains

$$iM_{2L} \ni -iM_0 \frac{\alpha^2}{v^2} \left(-\frac{1}{24}\pi^2 + \frac{1}{2}\log^2\frac{\bar{\mu}}{2} + \frac{i\pi}{2}\log\frac{\bar{\mu}}{2} \right). \tag{13}$$

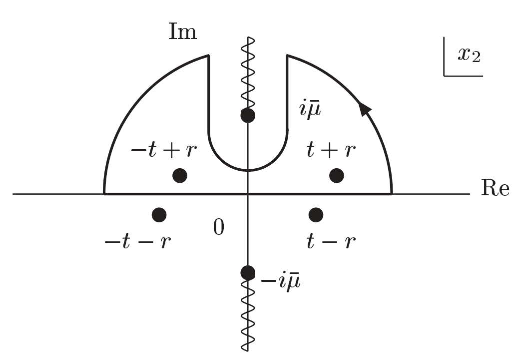

FIG. 6. Contour and poles of $I''(t)$ in the complex x_2 plane.

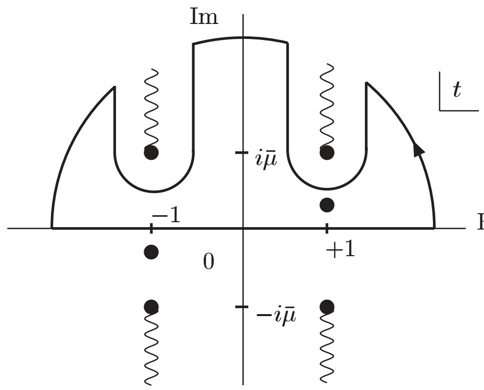

FIG. 7. Contour of the integral in the complex t plane.

Summing this equation, Eq. (11), and the tree-level amplitude, we give

$$\begin{aligned} &iM_0 + iM_{1L} + iM_{2L} \\ &\ni iM_0 \left\{ 1 + \frac{\pi\alpha}{2v} - \frac{i\alpha}{v}\log\frac{\bar{\mu}}{2} \right. \\ &\left. - \frac{\alpha^2}{v^2}\left(-\frac{1}{24}\pi^2 + \frac{1}{2}\log^2\frac{\bar{\mu}}{2} + \frac{i\pi}{2}\log\frac{\bar{\mu}}{2} \right) \right\}. \end{aligned} \tag{14}$$

The absolute square of them is

$$\begin{aligned} &|iM_0 + iM_{1L} + iM_{2L}|^2 \\ &\ni |iM_0|^2 \left(1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{3v^2} \right) + \mathcal{O}(\alpha^3). \end{aligned} \tag{15}$$

The logarithmic terms in Eq. (14) are canceled. For $v \ll 1$, the decay width has the form

$$d\Gamma - d\Gamma_3 \propto 1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{3v^2}. \tag{16}$$

This is consistent with the Fermi function up to order α^2.

V. CONCLUSION AND DISCUSSION

We conclude the main results as follows:

(1) We reviewed the one-loop β decay amplitude to confirm the terms proportional to α/v. It can be interpreted as the part of the Fermi function.

(2) The scattering process $\bar{\nu}_e + p \to e^+ + n$ does not have such terms.

(3) We give the result that the two-loop β decay amplitude has the terms proportional to $(\alpha/v)^2$. These are consistent with the expanded Fermi function up to order α^2.

The α/v term is factored out in Refs. [7,8]. Reference [8] may omit the explanation about this term. Equation (3) differs in the constant in the curly brackets from Refs. [7,8]. We show the process of calculation in Appendix A.

To confirm our conclusion, it is necessary to carry out the Fermi function to higher order. If the systematic calculation will be carried out, we will be able to sum up all the order of contributions.

In a two-loop calculation, we are only interested in the terms proportional to $(\alpha/v)^2$. However, the terms proportional to α^2/v may exist. These terms also affect the decay width. We should consider them for the higher order calculation.

For $\alpha/v \gtrsim 1$, the perturbation up to the finite order does not work. We must sum up all order of α/v. The result should become the full Fermi function written in Eq. (1). Then, we propose the decay width to form

$$\begin{aligned} d\Gamma - d\Gamma_3 = &\frac{1}{\pi} G_F^2 \frac{d^3q'}{(2\pi)^3}(1+3C^2)k_M^2 \left[\frac{2\pi\alpha/v}{1-e^{-2\pi\alpha/v}} \right. \\ &+ \frac{\alpha}{2\pi}\left\{ 3\log\frac{m_p}{m_e} - \frac{1}{2} - \frac{4}{v}Li\left(\frac{2v}{1+v}\right) \right. \\ &+ 4\left(\frac{1}{v}\mathrm{Tanh}^{-1}v - 1\right)\left(\frac{k_M}{3E_e} - \frac{3}{2} + \log\frac{2k_M}{m_e}\right) \\ &\left.\left. + \frac{1}{v}\mathrm{Tanh}^{-1}v\left(2(1+v^2) + \frac{k_M^2}{6E_e^2} - 4\mathrm{Tanh}^{-1}v\right)\right\}\right]. \end{aligned} \tag{17}$$

We can extend this study to other nuclear species by changing the expression in curly brackets and exchanging $\alpha \to Z\alpha$, where Z is the atomic number of the daughter nucleus, since the loop diagrams that contain the photon propagator between parent nucleus and the daughter particles do not give the α/v term as explained in the calculation of the scattering process. Also, we confirmed that the $(\alpha/v)^2$ term does not appear in the corresponding two-loop diagrams. Our study is more important for larger Z.

Our result does not affect the practical use except for $v \lesssim \alpha$. For instance, this result only slightly affects the Kurie plot [11] and the main result of Ref. [12]. However, the theoretical calculation of the nuclear lifetime is changed.

If the Fermi function is derived from the Dirac equation or factored out from the loop correction, the one-loop decay width is expressed as

$$\Gamma \propto F_{\mathrm{NR}}[1 + \mathcal{O}(Z\alpha)] = F_{\mathrm{NR}} + F_{\mathrm{NR}}\mathcal{O}(Z\alpha). \tag{18}$$

On the other hand, our study suggests that

$$\Gamma \propto F_{\mathrm{NR}} + \mathcal{O}(Z\alpha). \tag{19}$$

For instance, Eq. (19) differs from Eq. (18) about 0.1α for neutron β decay and $Z\alpha$ for tritium β decay. Furthermore, for $n \geqslant 2$, the nth order correction contains the $(\alpha/v)^{n-1}$ term after factoring out the Fermi function. It affects the convergence of perturbation theory.

These results suggest that the potential effect named the Fermi function should be considered as a part of the amplitude.

APPENDIX A: THE DETAIL OF ONE-LOOP CALCULATION

The tree-level amplitude is

$$iM_0 = -\frac{iG_F}{\sqrt{2}}\bar{u}(p')(1 - C\gamma^5)u(p)\bar{u}(q')(1-\gamma^5)v(q). \tag{A1}$$

According to Ref. [8], the one-loop amplitude can be separate in three parts as

$$iM_{1L} = iM_1 + iM_2 + iM_3. \tag{A2}$$

iM_1 picks up the factors $(2q' + k)^\mu$ from the electron propagator and $(2p' - k)_\mu$ from the proton propagator. iM_2 picks up the factors $\sigma^{\mu\nu}k_\nu$ from the electron propagator and $(2p' - k)_\mu$ from the proton propagator. iM_3 picks up the remaining factors.

1. $d\Gamma$

The tree-level width is

$$d\Gamma_0 = \frac{1}{8\pi m_n m_p}\frac{d^3q'}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2}\sum_\varepsilon |iM_0|^2\frac{k_M^2}{E_\nu}, \quad \sum_\varepsilon |iM_0|^2 = 32G_F^2 m_n m_p E_e E_\nu(1+3C^2), \tag{A3}$$

where E_ν and m_n are the neutrino energy and the neutron mass, respectively; $\sum_\varepsilon$ represents the spin sums.

The one-loop width is

$$\begin{aligned} d\Gamma &= d\Gamma_b + \frac{1}{2m_n}\left(\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}}\right)\left(\frac{d^3q'}{(2\pi)^3}\frac{1}{2E_{q'}}\right)\left(\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}\right) \\ &\quad\times\frac{1}{2}\sum_\varepsilon\left|iM_0 + \frac{1}{2}(\delta Z_e + \delta Z_p)iM_0 + iM_{1L}\right|^2\delta^{(4)}(p - p' - q - q'), \\ d\Gamma_b &= \frac{1}{2m_n}\left(\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}}\right)\left(\frac{d^3q'}{(2\pi)^3}\frac{1}{2E_{q'}}\right)\left(\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}\right)\left(\frac{d^3k}{(2\pi)^3}\frac{1}{2E_\gamma}\right)\frac{1}{2}\sum_\varepsilon|iM_b|^2\delta^{(4)}(p - p' - q - q' - k), \end{aligned} \tag{A4}$$

where δZ_e and δZ_p are the one-loop electron and proton field strength renormalization, respectively; $d\Gamma_b$ is the term originating from the bremsstrahlung; and $d\Gamma_3$ is the term originating from iM_3.

2. iM_2

iM_2 is written as

$$\begin{aligned} iM_2 &= \frac{2iJ}{(4\pi)^2}\left[\frac{1}{2cq'^2}\left\{\frac{b-c}{1+b-c}\log(b-c) - \frac{b+c}{1+b+c}\log(b+c)\right\} - \frac{i\pi}{(p'+q')^2}\right], \\ J &= -\frac{e^2G}{\sqrt{2}}\bar{u}(p')\gamma^\nu(1 - C_A\gamma_5)u(p)\bar{u}(q')(p'\cdot q' - m_e\not{p}')\gamma_\nu(1-\gamma_5)v(q), \end{aligned} \tag{A5}$$

where $b = p'\cdot q'/q'^2$ and $c = \sqrt{(p'\cdot q')^2 - p'^2q'^2}/q'^2$. The cross term between iM_0 and iM_2 is

$$\sum_\varepsilon iM_0\,(iM_2)^* \simeq \frac{\alpha}{4\pi}\sum_\varepsilon |iM_0|^2 v\left(\log\frac{1+v}{1-v} + \frac{2i\pi E_e v}{m_p}\right). \tag{A6}$$

Then, the decay width takes the form

$$d\Gamma = d\Gamma_0\left\{1 + \frac{\alpha}{2\pi}\mathrm{Re}\left(\sum_{i=1}^{6} I_i + v\log\frac{1+v}{1-v}\right)\right\} + d\Gamma_3 + d\Gamma_b, \tag{A7}$$

where I_i are defined in Appendix A 4.

3. $d\Gamma_b$

The bremsstrahlung amplitude is approximately written as [6]

$$iM_b \simeq eiM_0\left(\frac{q'\cdot\epsilon(k)}{q'\cdot k + i\epsilon} - \frac{p'\cdot\epsilon(k)}{p'\cdot k + i\epsilon}\right) \tag{A8}$$

for small k, where $\epsilon_\mu(k)$ in the numerator represents the polarization vector of the external photon.

The absolute square of this amplitude is

$$\sum_\varepsilon |iM_b|^2 \simeq \sum_\varepsilon |iM_0|^2\frac{e^2}{E_e}\left[\frac{1}{E_e(1 - v\beta w)} + \frac{E_e + k_0}{k_0^2}\frac{v^2(1-\beta^2w^2)}{(1-v\beta w)^2}\right], \tag{A9}$$

where $\beta = |\boldsymbol{k}|/k_0$ and $w = \boldsymbol{k}\cdot\boldsymbol{q}'/(|\boldsymbol{k}||\boldsymbol{q}'|)$. Here, we define $k = |\boldsymbol{k}|$, and

$$I_b \equiv \int_{-1}^{1} dw \int_0^{k_M} dk \frac{k^2}{k_0}\left[\frac{1}{E_e^2(1-v\beta w)} + \frac{E_e+k_0}{E_e}\frac{v^2(1-\beta^2 w^2)}{k_0^2(1-v\beta w)^2}\right]\left(1-\frac{k_0}{k_M}\right)^2$$
$$= 2\left[2+\frac{k_M^2}{12E_e^2} - \frac{1}{v}Li\left(\frac{2v}{1+v}\right) - \frac{1}{v}(\mathrm{Tanh}^{-1}v)^2 + \left(2-\frac{2k_M}{3E_e}-\frac{k_M^2}{12E_e^2}-2\log\frac{2k_M}{\mu}\right)\left(1-\frac{1}{v}\mathrm{Tanh}^{-1}v\right)\right], \quad \text{(A10)}$$

where $\mu = \sqrt{k_0^2 - |\boldsymbol{k}|^2}$ is the photon mass. Then, the bremsstrahlung part is

$$d\Gamma_b = \frac{1}{8\pi m_n m_p}\frac{d^3q'}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2}\sum_{\varepsilon}|iM_0|^2\frac{k_M^2}{E_\nu}\times\frac{\alpha}{2\pi}I_b. \quad \text{(A11)}$$

Therefore, Eq. (A7) becomes

$$d\Gamma - d\Gamma_3 = d\Gamma_0\left[1+\frac{\alpha}{2\pi}\left\{\mathrm{Re}\left(\sum_{i=1}^{6} I_i + v\log\frac{1+v}{1-v}\right) + I_b\right\}\right]. \quad \text{(A12)}$$

According to Appendix A 4, the one-loop decay width finally takes the form

$$d\Gamma - d\Gamma_3 = \frac{1}{\pi}G_F^2\frac{d^3q'}{(2\pi)^3}(1+3C^2)k_M^2\left[1+\frac{\alpha}{2\pi}\left\{\frac{2\pi^2}{v}+3\log\frac{m_p}{m_e}-\frac{1}{2}-\frac{4}{v}Li\left(\frac{2v}{1+v}\right)\right.\right.$$
$$\left.\left.+4\left(\frac{1}{v}\mathrm{Tanh}^{-1}v-1\right)\left(\frac{k_M}{3E_e}-\frac{3}{2}+\log\frac{2k_M}{m_e}\right)+\frac{1}{v}\mathrm{Tanh}^{-1}v\left(2(1+v^2)+\frac{k_M^2}{6E_e^2}-4\mathrm{Tanh}^{-1}v\right)\right\}\right].$$

4. $I_1 \sim I_6$

We define I_i's. Here, $I_1 + I_3$ is derived from $\delta Z_p/2$. Similarly, $I_2 + I_4$ is derived from $\delta Z_e/2$. Also, $I_5 + I_6$ corresponds to iM_{1L}. The results are as follows:

$$\begin{aligned}
I_1 &= \int_0^1 dx(1-x)\log\left[x^2m_p^2+(1-x)\mu^2\right] = -\frac{3}{2}+\log m_p,\\
I_2 &= \int_0^1 dx(1-x)\log\left[x^2m_e^2+(1-x)\mu^2\right] = -\frac{3}{2}+\log m_e,\\
I_3 &= \int_0^1 dx\frac{2x(1-x^2)m_p^2}{x^2m_p^2+(1-x)\mu^2} = -1+\log\frac{m_p^2}{\mu^2},\\
I_4 &= \int_0^1 dx\frac{2x(1-x^2)m_e^2}{x^2m_e^2+(1-x)\mu^2} = -1+\log\frac{m_e^2}{\mu^2}.
\end{aligned} \quad \text{(A13)}$$

$$I_5 = \int_0^1 dx\int_0^{1-x} dy\frac{1}{\Delta - i\epsilon}\{(2-x)p'+yq'\}\{xp'+(2-y)q'\} = I_{5a}+I_{5b}+I_{5c}, \quad \text{(A14)}$$

where $\Delta = m_p^2x^2 + m_e^2y^2 - (2xp'\cdot q' + \mu^2)y + (1-x)\mu^2$,

$$I_{5a} = \int_0^1 dx\int_0^{1-x} dy\frac{4p'\cdot q'}{\Delta - i\epsilon} = \frac{b}{c}\left[\frac{4\pi^2}{3}+2i\pi\log\frac{2c}{c-c'}-\log\frac{2c}{c-c'}\log\frac{b+c}{b-c}+2Li\left(\frac{1+b-c}{1+b+c}\right)\right.$$
$$\left.+2Li\left(\frac{b-c}{b+c}\frac{1+b+c}{1+b-c}\right)+\frac{1}{2}\log^2\left(\frac{1+b+c}{1+b-c}\right)+\frac{1}{2}\log^2\left(\frac{b+c}{b-c}\frac{1+b-c}{1+b+c}\right)\right],$$

$$I_{5b} = \int_0^1 dx\int_0^{1-x} dy\frac{2(p'-q')\cdot(xp'-yq')}{\Delta - i\epsilon} = \frac{-1}{(1+b)^2-c^2}\left[(1+c^2-b^2)\log(b^2-c^2)-2c\left(\log\frac{b+c}{b-c}-2i\pi\right)\right],$$

$$I_{5c} = -\int_0^1 dx\int_0^{1-x} dy\frac{(xp'-yq')^2}{\Delta - i\epsilon} = -\frac{1}{2},$$

where $c' = \sqrt{(p'\cdot q')^2 - p'^2q'^2 - (p'+q')^2\mu^2}/q'^2$,

$$I_6 = \int_0^1 dx\int_0^{1-x} dy\{-2\log(\Delta - i\epsilon)\} = 3 - \log m_e^2 + \frac{1+b}{(1+b)^2-c^2}\log\frac{m_p^2}{m_e^2} + \frac{2c}{(1+b)^2-c^2}\left(-\mathrm{Tanh}^{-1}\frac{c}{b}+i\pi\right).$$

We note here that $b/c = 1/v$, and the ultraviolet divergence is already removed.

The sum of these I_i's is

$$\sum_{i=1}^{6} I_i \simeq -\frac{5}{2} + \log\frac{m_p^3 m_e}{\mu^4} + \frac{2}{v}\log\frac{\mu^2}{m_e^2}\mathrm{Tanh}^{-1}v - \frac{2}{v}Li\left(\frac{2v}{1+v}\right) - \frac{2}{v}(\mathrm{Tanh}^{-1}v)^2 + \frac{2\pi^2}{v} + \frac{2i\pi}{v}\log\left(\frac{4m_e^2}{\mu^2}\frac{v^2}{1-v^2}\right).$$

The last two terms diverge for $v \to 0$. The latter one, which contains μ, does not affect the one-loop decay width. However, this term has a nontrivial, important role in the two-loop calculation, as explained in Appendix B.

APPENDIX B: TWO LOOP CALCULATION USING THE CONTOUR INTEGRALS

We first note that we are only interested in the terms proportional to $(\alpha/v)^2$ and we ignore the others. The two-loop ladder amplitude contains

$$\begin{aligned} iM_{2L} &= i\frac{e^4}{2}\frac{8}{\sqrt{2}}G_F\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\bar{u}(q')\frac{\gamma^{\mu_1}(\not{p}'+\not{k}_1+m_e)\gamma^{\mu_2}(\not{p}'+\not{k}_1+\not{k}_2+m_e)\gamma^{\rho}P_L}{(k_1^2+2q'\cdot k_1+i\epsilon)\{(k_1+k_2)^2+2q'\cdot(k_1+k_2)+i\epsilon\}}v(q) \\ &\quad\times\bar{u}(p')\frac{\gamma_{\mu_1}(\not{p}'-\not{k}_1+m_p)\gamma_{\mu_2}(\not{p}'-\not{k}_1-\not{k}_2+m_p)\gamma_\rho P_L'}{(k_1^2-2p'\cdot k_1+i\epsilon)\{(k_1+k_2)^2-2p'\cdot(k_1+k_2)+i\epsilon\}}u(p)\frac{1}{k_1^2-\mu^2+i\epsilon}\frac{1}{k_2^2-\mu^2+i\epsilon} \\ &\ni i\frac{4e^4}{\sqrt{2}}G_F\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{4q'^{\mu_1}q'^{\mu_2}\bar{u}(q')\gamma^\rho P_L v(q)}{(k_1^2+2q'\cdot k_1+i\epsilon)\{(k_1+k_2)^2+2q'\cdot(k_1+k_2)+i\epsilon\}} \\ &\quad\times\frac{4p'_{\mu_1}p'_{\mu_2}\bar{u}(p')\gamma_\rho P_L'u(p)}{(k_1^2-2p'\cdot k_1+i\epsilon)\{(k_1+k_2)^2-2p'\cdot(k_1+k_2)+i\epsilon\}}\frac{1}{k_1^2-\mu^2+i\epsilon}\frac{1}{k_2^2-\mu^2+i\epsilon}, \end{aligned} \tag{B1}$$

where $P_L = (1-\gamma^5)/2$ and $P_L' = (1-C\gamma^5)/2$.

By separating the integrations over k_1^0 and k_2^0, the amplitude is written as

$$iM_{2L} \ni i\frac{4e^4}{\sqrt{2}}G_F\int\frac{d^3k_1}{(2\pi)^4}\int\frac{d^3k_2}{(2\pi)^4}\times 4q'^{\mu_1}q'^{\mu_2}\bar{u}(q')\gamma^\rho P_L v(q)\times 4p'_{\mu_1}p'_{\mu_2}\bar{u}(p')\gamma_\rho P_L'u(p)\times D_a, \tag{B2}$$

where

$$\begin{aligned} D_a &= \int dk_1^0 dk_2^0\,\frac{1}{k_1^2-\mu^2+i\epsilon}\,\frac{1}{k_2^2-\mu^2+i\epsilon}\times\frac{1}{k_1^2+2q'\cdot k_1+i\epsilon}\,\frac{1}{(k_1+k_2)^2+2q'\cdot(k_1+k_2)+i\epsilon} \\ &\quad\times\frac{1}{k_1^2-2p'\cdot k_1+i\epsilon}\,\frac{1}{(k_1+k_2)^2-2p'\cdot(k_1+k_2)+i\epsilon}. \end{aligned} \tag{B3}$$

We close the integration contour downward. The term we focus on is derived from the pole of $[(k_1+k_2)^2+2q'\cdot(k_1+k_2)+i\epsilon]^{-1}$ for k_1^0 integral and then the pole of $[k_1^2-2p'\cdot k_1+i\epsilon]^{-1}$ for k_2^0 integral. By applying the approximation similar to one-loop calculation, D_a contains

$$D_a \ni -\frac{\pi^2}{p_0'^2}\frac{1}{\boldsymbol{k}_1^2+\mu^2-i\epsilon}\,\frac{1}{\boldsymbol{k}_2^2+\mu^2-i\epsilon}\,\frac{1}{B-i\epsilon}\,\frac{1}{A-i\epsilon}, \tag{B4}$$

where

$$\begin{aligned} A &= (\boldsymbol{k}_1+\boldsymbol{k}_2)^2 + 2\boldsymbol{q}'\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2), \\ B &= \boldsymbol{k}_1^2 + 2\boldsymbol{q}'\cdot\boldsymbol{k}_1. \end{aligned} \tag{B5}$$

Then, the amplitude contains

$$\begin{aligned} iM_{2L} &\ni -i\frac{4e^4}{\sqrt{2}}G_F q_0'^2\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\times 4\bar{u}(q')\gamma^\rho P_L v(q)\times\bar{u}(p')\gamma_\rho P_L'u(p)\times\frac{1}{\boldsymbol{k}_1^2+\mu^2-i\epsilon}\,\frac{1}{\boldsymbol{k}_2^2+\mu^2-i\epsilon}\,\frac{1}{B-i\epsilon}\,\frac{1}{A-i\epsilon} \\ &= 4e^4 q_0'^2 iM_0\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\,\frac{1}{\boldsymbol{k}_1^2+\mu^2}\,\frac{1}{\boldsymbol{k}_2^2+\mu^2}\,\frac{1}{B-i\epsilon}\,\frac{1}{A-i\epsilon}. \end{aligned}$$

When we write $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ in spherical polar coordinates,

$$\begin{aligned} iM_{2L} \ni& \frac{4e^4 q_0'^2 iM_0}{(2\pi)^4}\int dk_1 d\cos\theta_1 dk_2 d\cos\theta_2 \frac{k_1^2}{k_1^2+\mu^2}\frac{k_2^2}{k_2^2+\mu^2}\frac{1}{B-i\epsilon}\frac{1}{A-i\epsilon} \\ =& \frac{4e^4 q_0'^2 iM_0}{(2\pi)^4}\int dk_1 dk_2 d\cos\theta_1 \frac{k_1^2}{k_1^2+\mu^2}\frac{k_2^2}{k_2^2+\mu^2}\frac{1}{k_1^2+2q'k_1\cos\theta_1-i\epsilon} \\ &\times\int d\cos\theta_2 \frac{1}{k_1^2+k_2^2+2q'k_1\cos\theta_1+2|\boldsymbol{q}'+\boldsymbol{k}_1|k_2\cos\theta_2-i\epsilon}, \end{aligned} \tag{B6}$$

where $k_1=|\boldsymbol{k}_1|$, $k_2=|\boldsymbol{k}_2|$, $q'=|\boldsymbol{q}'|$, $\cos\theta_1=\boldsymbol{q}'\cdot\boldsymbol{k}_1/(q'k_1)$, and $\cos\theta_2=(\boldsymbol{q}'+\boldsymbol{k}_1)\cdot\boldsymbol{k}_2/(|\boldsymbol{q}'+\boldsymbol{k}_1|k_2)$. By performing the $\cos\theta_2$ integral,

$$\begin{aligned} iM_{2L} \ni& \frac{4e^4 q_0'^2 iM_0}{(2\pi)^4}\int dk_1 dk_2 d\cos\theta_1 \frac{k_1^2}{k_1^2+\mu^2}\frac{k_2^2}{k_2^2+\mu^2}\frac{1}{k_1^2+2q'k_1\cos\theta_1-i\epsilon} \\ &\times\frac{1}{2|\boldsymbol{q}'+\boldsymbol{k}_1|k_2}\{\log((k_2+|\boldsymbol{q}'+\boldsymbol{k}_1|)^2-q'^2-i\epsilon)-\log((k_2-|\boldsymbol{q}'+\boldsymbol{k}_1|)^2-q'^2-i\epsilon)\}, \end{aligned} \tag{B7}$$

where $|\boldsymbol{q}'+\boldsymbol{k}_1|=\sqrt{k_1^2+2q'k_1\cos\theta_1+q'^2}$.

By defining $\boldsymbol{x}_1=\boldsymbol{k}_1/q'$, $\boldsymbol{x}_2=\boldsymbol{k}_2/q'$, $x_1=|\boldsymbol{x}_1|$, $x_2=|\boldsymbol{x}_2|$, $\hat{\boldsymbol{q}}'=\boldsymbol{q}'/q'$, $\bar{\mu}=\mu/q'$, $r=\sqrt{1+i\epsilon}$ to make the integrand dimensionless and performing the x_2 integral,

$$\begin{aligned} iM_{2L} \ni& \frac{2e^4 q_0'^2 iM_0}{q'^2(2\pi)^4}\int dx_1 d\cos\theta_1 \frac{x_1^2}{x_1^2+\bar{\mu}^2}\frac{1}{x_1^2+2x_1\cos\theta_1-i\epsilon}\frac{1}{|\hat{\boldsymbol{q}}'+\boldsymbol{x}_1|} \\ &\times\int dx_2\frac{x_2}{x_2^2+\bar{\mu}^2}\{\log((x_2+|\hat{\boldsymbol{q}}'+\boldsymbol{x}_1|)^2-r^2)-\log((x_2-|\hat{\boldsymbol{q}}'+\boldsymbol{x}_1|)^2-r^2)\}. \end{aligned} \tag{B8}$$

When we substitute $t=\sqrt{x_1^2+2x_1\cos\theta_1+1}=|\hat{\boldsymbol{q}}'+\boldsymbol{x}_1|$, the amplitude is

$$iM_{2L}\ni\frac{8\alpha^2 iM_0}{v^2(2\pi)^2}\int_0^\infty dx_1\frac{x_1}{x_1^2+\bar{\mu}^2}\int_{|x_1-1|}^{x_1+1}dt\frac{1}{t^2-r^2}\times\int_0^\infty dx_2\frac{x_2}{x_2^2+\bar{\mu}^2}\{\log((x_2+t)^2-r^2)-\log((x_2-t)^2-r^2)\}.$$

Here, we define

$$\begin{aligned} I''(t) &= \int_0^\infty dx_2\frac{x_2}{x_2^2+\bar{\mu}^2}[\log\{(x_2+t)^2-r^2\}-\log\{(x_2-t)^2-r^2\}], \\ I &= \frac{1}{2}\int_0^\infty dx_1\frac{x_1}{x_1^2+\bar{\mu}^2}s\int_{|x_1-1|}^{x_1+1}dt\frac{1}{t^2-r^2}I''(t). \end{aligned} \tag{B9}$$

Then, the amplitude is written as

$$iM_{2L}\ni\frac{4\alpha^2 iM_0}{v^2\pi^2}I. \tag{B10}$$

We now calculate $I''(t)$. Since it is an even function, we can change the integrating interval to form

$$I''(t)=\frac{1}{2}\int_{-\infty}^\infty dx_2\frac{x_2}{x_2^2+\bar{\mu}^2}[\log\{(x_2+t)^2-r^2\}-\log\{(x_2-t)^2-r^2\}]. \tag{B11}$$

After integrating by parts, it becomes

$$\begin{aligned} I''(t) =& \frac{1}{4}\left[\log\left(x_2^2+\bar{\mu}^2\right)[\log\{(x_2+t)^2-r^2\}-\log\{(x_2-t)^2-r^2\}]\right]_{-\infty}^{\infty} \\ &+\int_{-\infty}^\infty dx_2\log\left(x_2^2+\bar{\mu}^2\right)\frac{t\left(x_2^2-t^2+r^2\right)}{[(x_2+t)^2-r^2][(x_2-t)^2-r^2]}. \end{aligned} \tag{B12}$$

The first term is 0. We evaluate the x_2 integral along the contour shown in Fig. 6. By applying the residue theorem and changing the variable $x_2 = iz$, we have

$$\begin{aligned} I''(t) &= 2\pi i \mathrm{Res}(t+r) + 2\pi i \mathrm{Res}(-t+r) - \int_{i\bar{\mu}}^{i\bar{\mu}+i\infty} dx_2 (-2\pi i) \frac{t\left(x_2^2 - t^2 + r^2\right)}{[(x_2+t)^2 - r^2][(x_2-t)^2 - r^2]} \\ &= \frac{i\pi}{2}[\log\{(t+r)^2 + \bar{\mu}^2\} - \log\{(t-r)^2 + \bar{\mu}^2\}] - 2\pi \int_{\bar{\mu}}^{\infty} dz \frac{t(-z^2 - t^2 + r^2)}{[(t+iz)^2 - r^2][(t-iz)^2 - r^2]}. \end{aligned} \tag{B13}$$

After performing the integral over z, it becomes

$$I''(t) = i\pi[\log(\bar{\mu} - ir - it) - \log(\bar{\mu} - ir + it)]. \tag{B14}$$

Changing the order of integration, we obtain

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty dx_1 \int_{|x_1-1|}^{x_1+1} dt \frac{x_1}{x_1^2 + \bar{\mu}^2} \frac{1}{t^2 - r^2} I''(t) = \frac{1}{2}\int_0^\infty dt \int_{|t-1|}^{t+1} dx_1 \frac{x_1}{x_1^2 + \bar{\mu}^2} \frac{1}{t^2 - r^2} I''(t) \\ &= \frac{1}{4}\int_0^\infty dt [\log\{(t+1)^2 + \bar{\mu}^2\} - \log\{(t-1)^2 + \bar{\mu}^2\}] \frac{1}{t^2 - r^2} I''(t). \end{aligned} \tag{B15}$$

Since the integrand is an even function, this expression becomes

$$I = \frac{i\pi}{8}\int_{-\infty}^\infty dt [\log\{(t+1)^2 + \bar{\mu}^2\} - \log\{(t-1)^2 + \bar{\mu}^2\}] \frac{1}{t^2 - r^2} \times [\log(\bar{\mu} - ir - it) - \log(\bar{\mu} - ir + it)]. \tag{B16}$$

Applying the residue theorem to I, we evaluate the t integral along the contour shown in Fig. 7. We define the contribution of the contour around the cut that starts from point t as $\mathrm{Cut}(t)$. This expression becomes

$$I = 2\pi i \mathrm{Res}(r) - \mathrm{Cut}(1 + i\bar{\mu}) - \mathrm{Cut}(-1 + i\bar{\mu}), \tag{B17}$$

where

$$\begin{aligned} 2\pi i \mathrm{Res}(r) &= -\frac{\pi^2}{4}\log^2\frac{\bar{\mu}}{2} - \frac{i\pi^2}{8}\log\frac{\bar{\mu}}{2}, \\ -\mathrm{Cut}(1 + i\bar{\mu}) &= \frac{\pi^2}{8}\left(\frac{3}{8}\pi^2 + \frac{1}{2}i\pi\log\frac{\bar{\mu}}{2} + \frac{3}{2}\log^2\frac{\bar{\mu}}{2}\right), \\ -\mathrm{Cut}(-1 + i\bar{\mu}) &= -\frac{\pi^2}{8}\left(\frac{7}{24}\pi^2 + \frac{1}{2}i\pi\log\frac{\bar{\mu}}{2} + \frac{1}{2}\log^2\frac{\bar{\mu}}{2}\right). \end{aligned} \tag{B18}$$

Our result then becomes

$$iM_{2L} \ni -iM_0\frac{\alpha^2}{v^2}\left(-\frac{1}{24}\pi^2 + \frac{1}{2}\log^2\frac{\bar{\mu}}{2} + \frac{i\pi}{2}\log\frac{\bar{\mu}}{2}\right). \tag{B19}$$

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