Parton distributions in the presence of target mass corrections

F. M. Steffens,[1] M. D. Brown,[2] W. Melnitchouk,[3] and S. Sanches[1]

[1]*IF–USP, Cidade Universitaria São Paulo, São Paulo, Brazil*
[2]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Jefferson Lab, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA*

We study the consistency of parton distribution functions in the presence of target mass corrections (TMCs) at low Q^2. We review the standard operator product expansion derivation of TMCs in both x and moment space, and present the results in closed form for all unpolarized structure functions and their moments. To avoid the unphysical region at $x > 1$ in the standard TMC analysis, we propose an expansion of the target mass corrected structure functions order by order in M^2/Q^2, and assess the convergence properties of the resulting forms numerically.

I. INTRODUCTION

The application of the operator product expansion (OPE) to the phenomenological study of quantum chromodynamics (QCD) has been very successful in the determination of the quark and gluon substructure of the nucleon. The OPE allows the formal separation of cross sections for high-energy processes such as deep-inelastic scattering (DIS) into perturbatively calculable partonic cross sections and nonperturbative contributions parametrized by parton distribution functions (PDFs). The factorization of the cross section becomes especially clean in the Bjorken limit, where the energy ν and four-momentum squared Q^2 transferred to a nucleon with mass M both become infinite, with the ratio $x = Q^2/2M\nu$ fixed.

In the Bjorken limit the DIS process becomes dominated by scattering at light-cone space-time distances $z_\mu z^\mu \sim 0$, with the expansion made in terms of products of singular and nonsingular terms around the light cone. The singularities are isolated in the perturbative Wilson coefficients, while the nonsingular terms are all the possible operators allowed by the underlying quantum field theory. The coefficient of the operators of lowest twist (where twist is defined as the dimension minus the spin of the operator) contains the most singular terms. Operators in the expansion with higher twist are less singular, and at large Q^2 are suppressed by powers of $1/Q^2$.

While this framework has met with considerable success in describing data at high $Q^2 \gg M^2$ and large final-state hadron masses $W^2 = M^2 + Q^2(1-x)/x$, many recent high-precision experiments [1] have been performed at lower energies, with Q^2 down to ≈ 1–2 GeV^2, where the use of the asymptotic Bjorken limit formalism is more questionable. In addition to the strong coupling constant α_s becoming large, at low Q^2 the higher twist power corrections, which describe nonperturbative multiparton correlations, become increasingly important. Furthermore, even at leading twist, there are corrections arising from purely kinematic effects associated with finite values of $Q^2/\nu^2 = 4M^2x^2/Q^2$, usually termed target mass corrections (TMCs) [2–6].

To perform reliable perturbative QCD based analyses which include data in the low Q^2 region, a careful treatment of the subleading $1/Q^2$ corrections is essential, and global PDF analyses [7–11] have only recently begun to take such effects

systematically into account. Studies of quark-hadron duality [12,13] have also strongly suggested that data at low W can be described (to within $\sim$10%–15%) by leading twist parton distributions. A more basic question, however, is whether one can consistently define leading twist parton distributions in the presence of TMCs, that can be valid at low Q^2 over the entire range of x.

The first analysis to tackle this question was by Georgi and Politzer (GP) [2], who proposed taking TMCs into account by defining distributions at low Q^2 in terms of the Nachtmann scaling variable, ξ [14,15],

$$\xi = \frac{2x}{1+\rho}, \quad \text{with} \quad \rho = \sqrt{1+4\mu x^2} \quad \text{and} \quad \mu = \frac{M^2}{Q^2}. \tag{1}$$

This leads to a specific prescription for removing TMCs from measured structure functions that has been used extensively in the literature [5].

Unfortunately, problems with the standard TMC prescription were soon realized [16–21] in the behavior of the target mass corrected structure functions in the vicinity of $x \approx 1$. In particular, functions expressed in terms of ξ over the interval $0 \leqslant \xi \leqslant 1$ necessarily extend into the unphysical region between the elastic limit $\xi = \xi_0 \equiv \xi(x=1)$ and $\xi = 1$ for any finite value of Q^2 [2]. This not only violates the conservation of energy and momentum, but also makes structure functions nonzero at $x = 1$, at odds with the expectation that leading twist functions should vanish at the elastic point [12,22].

As a possible remedy, De Rujula *et al.* [3,4] noted that in the threshold region analyses of data should not be performed in terms of leading twist structure functions alone, without also incorporating the effects of higher twist operators. They argued that a nonuniformity in the limits as $n \to \infty$ and $Q^2 \to \infty$ renders the entire approach untenable at very low W, when higher twists exceed $\sim nM^2/Q^2$ for the nth structure function moment.

Attempts were also made by Tung and collaborators [19,20] to phenomenologically remove the threshold problem by utilizing an *ansatz* to smoothly merge the moments in the perturbative region at large Q^2 with their correct threshold behavior in the $n \to \infty$ limit, although such a prescription is

not unique. Steffens and Melnitchouk [23] extended this approach by proposing threshold-dependent distributions which exactly satisfy threshold kinematics at all Q^2, at the expense of sacrificing the universality of PDFs in the presence of TMCs.

Other approaches based on collinear factorization, starting with the seminal work of Ellis, Furmanski and Petronzio [24], avoid the inversion of moments by implementing TMCs directly in momentum space within the parton model [25–28]. These, too, however, suffer from prescription dependence [25–28], or do not extend to all orders in $1/Q^2$ [24]. In addition, even though they do not invoke distributions at $x > 1$, all these formulations nevertheless retain the problem of nonvanishing structure functions at $x = 1$.

Given the desire to maximally utilize the recent precision structure function measurements at large x [1,7–11,29], as well as those planned for the near future [30,31], there is a pressing need to address the question of TMCs and the consistency of parton distributions with mass corrections at finite Q^2. A more reliable treatment of the high-x region at moderate Q^2 is important not only in providing a better understanding of the quark structure of the nucleon in the deep valence region [32,33], it is also vital for constraining cross sections at collider energies through the evolution to lower x at higher Q^2 values [34,35].

In this paper we revisit the problem of kinematic thresholds and PDF definitions in the OPE approach to TMCs, elucidating its shortcomings, and proposing an alternative method that addresses some of the problems inherent in the standard TMC formulation. In Sec. II we review the standard TMC approach, outlining the OPE derivation of target mass corrected moments and their inversion to x space. We demonstrate explicitly the conflict of the usual inversion procedure with energy-momentum conservation, and illustrate its consequences for the x dependence of the structure functions as well as their moments. We propose a new method to compute TMCs in Sec. III, based on inversion of the moments order by order in M^2/Q^2, without having to introduce the Nachtmann scaling variable ξ, and study the convergence of the series numerically. The advantages and limitations of this method are summarized in Sec. IV. Further technical details of the TMC derivations of moments and structure functions are provided in Appendices A and B, respectively.

II. TARGET MASS CORRECTIONS IN THE OPE

In this section we begin by summarizing the basic formulas for inclusive cross sections and structure functions, before outlining the main steps in the derivation of TMCs from the operator product expansion. We present results for the complete set of leading twist moments of unpolarized structure functions, and discuss their inversion to obtain the x dependence at nonzero M^2/Q^2.

In the one-boson exchange approximation, the differential cross section for a lepton scattering from a nucleon target is given (in the target rest frame) by

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4}\frac{E'}{ME}\, \eta\, L_{\mu\nu} W^{\mu\nu}, \tag{2}$$

where Ω is the scattered lepton solid angle, and E and E' are the initial and final electron energies, respectively. The lepton tensors $L_{\mu\nu}$ and the coefficients η depend on the type of boson exchanged (γ, γZ, Z, or $W^\pm$) [36]. Denoting the initial and final lepton momenta by k and k', respectively, and the momentum transferred to the nucleon by $q = k - k'$, the hadronic tensor is given by the commutator of electroweak current operators J_μ,

$$W^{\mu\nu} = \frac{1}{2\pi}\int d^4z\, e^{iq\cdot z}\langle N|[J^\mu(z), J^\nu(0)]|N\rangle \tag{3}$$

$$= -g^{\mu\nu}F_1 + \frac{p^\mu p^\nu}{p\cdot q}F_2 - i\epsilon^{\mu\nu\lambda\sigma}\frac{p_\lambda p_\sigma}{2p\cdot q}F_3 + \frac{2q^\mu q^\nu}{Q^2}F_4$$

$$+ \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}F_5, \tag{4}$$

where F_i ($i = 1$–5) are the structure functions of the nucleon, usually expressed in terms of the variables x and $Q^2 = -q^2$, and we adopt the convention $\epsilon_{0123} = 1$ [36]. The structure functions F_1 and F_2 are accessible in charged lepton or neutrino scattering through a product of vector currents, while F_3 requires the interference of vector and axial vector currents. The vector structure functions F_4 and F_5 are accessible in neutrino scattering but are suppressed by lepton masses, m_l^2/M^2; for completeness, however, we include them in this analysis. The hadronic tensor $W_{\mu\nu}$ is also related to the imaginary part of the virtual forward Compton scattering amplitude,

$$T^{\mu\nu} = i\int d^4z\, e^{iq\cdot z}\langle N|T(J^\mu(z)J^\nu(0))|N\rangle, \tag{5}$$

with

$$W^{\mu\nu} = \frac{1}{\pi}\,\mathrm{disc}\, T^{\mu\nu}. \tag{6}$$

In the following we derive expressions for the amplitude $T^{\mu\nu}$ and use Eq. (6) to extract the results for the structure functions.

A. Moments of structure functions

The standard derivation of TMCs in the OPE in the twist-2 approximation begins with the Compton scattering amplitude $T_{\mu\nu}$, which can in general be written as [2]

$$T^{\mu\nu} = \sum_{k=1}^{\infty}\Bigg(-g^{\mu\nu}q_{\mu_1}q_{\mu_2}C_1^{2k} + g_{\mu_1}^\mu g_{\mu_2}^\nu Q^2 C_2^{2k}$$

$$- i\epsilon^{\mu\nu\alpha\beta}g_{\alpha\mu_1}q_\beta q_{\mu_2}C_3^{2k} + \frac{q^\mu q^\nu}{Q^2}q_{\mu_1}q_{\mu_2}C_4^{2k}$$

$$+ \left(g_{\mu_1}^\mu q^\nu q_{\mu_2} + g_{\mu_1}^\nu q^\mu q_{\mu_2}\right)C_5^{2k}\Bigg)q_{\mu_3}$$

$$\cdots q_{\mu_{2k}}\frac{2^{2k}}{Q^{4k}}\, A_{2k}\, \Pi^{\mu_1\cdots\mu_{2k}}, \tag{7}$$

where

$$\Pi^{\mu_1\cdots\mu_{2k}} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\cdots g\, p\cdots p\}_{k,j}\,(p^2)^j \tag{8}$$

and $\{g \cdots g \, p \cdots p\}_{k,j}$ represents the (symmetric) sum of $(2k)!/[2^j j!(2k-2j)!]$ distinct products of the form $g^{\mu_{i_1}\mu_{i_2}} \cdots g^{\mu_{i_{2j-1}}\mu_{i_{2j}}} p^{\mu_{2j+1}} \cdots p^{\mu_{2k}}$ resulting from permutations of the indices $\mu_1, \ldots, \mu_{2k}$. The Wilson coefficients C_i^{2k} are calculated perturbatively, while the factors A_{2k} are matrix elements of local twist-2 operators $\mathcal{O}^{\mu_1 \cdots \mu_{2k}}$ [37],

$$\langle N | \mathcal{O}^{\mu_1 \cdots \mu_{2k}} | N \rangle = A_{2k} \, p^{\mu_1} \cdots p^{\mu_{2k}} - \text{traces}, \qquad (9)$$

which parametrize the nonperturbative structure of the nucleon. In the case of the flavor singlet operator, for example, one has

$$\mathcal{O}_{\text{sing}}^{\mu_1 \cdots \mu_{2k}} = \bar{\psi} \gamma^{\{\mu_1} D^{\mu_2} \cdots D^{\mu_{2k}\}} \psi - \text{traces}, \qquad (10)$$

where the braces $\{\cdots\}$ denote symmetrization with respect to the indices $\mu_1, \ldots, \mu_{2k}$.

The Cornwall-Norton moments $M_i^{(n)}$ of the structure functions F_i are defined by

$$M_i^{(n)}(Q^2) = \begin{cases} \int_0^1 dx \, x^{n-1} F_i(x, Q^2) & \text{if} \quad i = 1, 3, 4, 5 \\ \int_0^1 dx \, x^{n-2} F_i(x, Q^2) & \text{if} \quad i = 2, L, \end{cases} \qquad (11)$$

where the longitudinal structure function F_L is given by

$$F_L = (1 + 4\mu x^2) F_2 - 2x F_1, \qquad (12)$$

with the corresponding coefficient function $C_L^n = C_2^n - C_1^n$. A straightforward but tedious calculation gives for each of the moments [2,38]

$$M_1^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \left(\frac{1}{2} C_1^{n+2j} + \frac{j}{(n+2j)(n+2j-1)} C_2^{n+2j} \right) A_{n+2j}, \qquad (13a)$$

$$M_2^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \frac{n(n-1)}{(n+2j)(n+2j-1)} C_2^{n+2j} \, A_{n+2j}, \qquad (13b)$$

$$M_L^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \left(C_L^{n+2j} + \frac{4j}{(n+2j)(n+2j-1)} C_2^{n+2j} \right) A_{n+2j}, \qquad (13c)$$

$$M_3^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \frac{n}{n+2j} C_3^{n+2j} \, A_{n+2j}, \qquad (13d)$$

$$M_4^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \left(\frac{j(j-1)}{(n+2j)(n+2j-1)} C_2^{n+2j} + \frac{1}{4} C_4^{n+2j} - \frac{j}{(n+2j)(n+2j-1)} C_5^{n+2j} \right) A_{n+2j}, \qquad (13e)$$

$$M_5^{(n)}(Q^2) = \sum_{j=0}^{\infty} \mu^j \binom{n+j}{j} \frac{n}{n+2j} \left(-\frac{j}{n+2j-1} C_2^{n+2j} + \frac{1}{2} C_5^{n+2j} \right) A_{n+2j} \qquad (13f)$$

where the binomial symbol $\binom{a}{b} = a!/[b!(a-b)!]$. Further details of the derivation of Eqs. (13) are given in Appendix A. Note that the expression for the $M_1^{(n)}$ moment is the same as that in Ref. [38] once the differences between the corresponding operator definitions are taken into account [39].

Up to this point the effects of the target mass on the structure function moments are rigorously derived within the OPE formalism. To proceed beyond Eqs. (13) and determine the TMC effects on the x dependence of the structure functions themselves requires additional assumptions, which inevitably introduces some model dependence in the calculation, as we discuss next in the following section.

B. Parton distributions with TMCs

In the absence of color interactions, the matrix elements A_{2k} in Eq. (9) should not depend on any scale apart from the factorization scale. With this in mind, the products $C_i^{2k} A_{2k}$ in Eq. (7) can be written in terms of parton distribution functions

f_i as

$$C_i^{2k} A_{2k} = \int_0^1 dy \, y^{2k-1} f_i(y), \qquad (14)$$

where for ease of notation we suppress the dependence in C_i^{2k} and f_i on the scale Q^2, which arises from perturbative QCD corrections. The functions f_i are defined such that in the massless limit ($\mu \to 0$) one has

$$\begin{aligned} & \{ F_1^{(0)}, \ F_2^{(0)}, \ F_L^{(0)}, \ F_3^{(0)}, \ F_4^{(0)}, \ F_5^{(0)} \} \\ & = \{ \tfrac{1}{2} f_1, \ x f_2, \ x(f_2 - f_1), \ f_3, \ \tfrac{1}{4} f_4, \ \tfrac{1}{2} f_5 \}, \quad (15) \end{aligned}$$

where $F_i^{(0)} \equiv \lim_{\mu \to 0} F_i$ is the massless limit of the physical structure function F_i. Note that our notation for the parton distribution functions f_i differs from that in Refs. [2,27], whose distributions effectively correspond to $f_i(x)/x$.

The functions f_i can in principle be identified with the PDFs measured in deep-inelastic or other high-energy scattering processes. (For simplicity we omit the flavor dependence of the structure functions, including their electroweak couplings,

which can be incorporated straightforwardly with the distributions f_i.) Following the derivation of GP [2], the structure functions at finite Q^2 can be inverted using the inverse Mellin transform,

$$F_i(x, Q^2) = \begin{cases} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \, x^{-n} \, M_i^{(n)}(Q^2) & \text{if } i = 1, 3, 4, 5 \\ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \, x^{-n+1} \, M_i^{(n)}(Q^2) & \text{if } i = 2, L. \end{cases} \tag{16}$$

Using Eqs. (13) and (14), the x dependence of the structure functions can then be determined in terms of the functions f_i, as outlined in Appendix B [2,27],

$$F_1(x, Q^2) = \frac{1}{2(1 + \mu\xi^2)} f_1(\xi) - \mu x^2 \frac{\partial}{\partial x}\left(\frac{g_2(\xi)}{1 + \mu\xi^2}\right), \tag{17a}$$

$$F_2(x, Q^2) = x^2 \frac{\partial^2}{\partial x^2}\left(\frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}\right), \tag{17b}$$

$$F_L(x, Q^2) = -\frac{x}{1 + \mu\xi^2} f_1(\xi) + 2\mu x^3 \frac{\partial}{\partial x}\left(\frac{g_2(\xi)}{1 + \mu\xi^2}\right) + (1 + 4\mu x^2)x^2 \frac{\partial^2}{\partial x^2}\left(\frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}\right), \tag{17c}$$

$$F_3(x, Q^2) = -x \frac{\partial}{\partial x}\left(\frac{h_3(\xi)}{1 + \mu\xi^2}\right), \tag{17d}$$

$$F_4(x, Q^2) = \frac{1}{4(1 + \mu\xi^2)} f_4(\xi) + \mu x^2 \frac{\partial}{\partial x}\left(\frac{g_5(\xi)}{1 + \mu\xi^2}\right) + \mu^2 x^3 \frac{\partial^2}{\partial x^2}\left(\frac{\xi^2 g_2(\xi)}{1 - \mu^2\xi^4}\right), \tag{17e}$$

$$F_5(x, Q^2) = -\frac{x}{2} \frac{\partial}{\partial x}\left(\frac{h_5(\xi)}{1 + \mu\xi^2}\right) - \mu x^2 \frac{\partial^2}{\partial x^2}\left(\frac{\xi g_2(\xi)}{1 - \mu^2\xi^4}\right), \tag{17f}$$

where the functions h_i and g_i are given by

$$h_i(\xi) = \int_\xi^1 du \, \frac{f_i(u)}{u}, \tag{18}$$

$$g_i(\xi) = \int_\xi^1 du \, h_i(u). \tag{19}$$

Note that the expression for the F_4 structure function in Ref. [27] contains "ξh_5" instead of "g_5" in the second term of Eq. (17e). Equations (17) define the complete set of unpolarized structure functions in the standard treatment of TMCs in the OPE. As was noted already in Ref. [3], however, the standard results lead to problems in the limit as $x \to 1$, which we shall focus on in the remainder of this section.

C. Consistency of the standard TMCs?

When taking the moments of the calculated x-dependent structure functions in the presence of TMCs, one should for consistency recover the expressions for the moments in Eqs. (13). To be specific, we investigate this here for the F_2 structure function, Eq. (17b), but the same arguments can be applied to all the other structure functions. From the definition

of the moments in Eq. (11), the nth moment of F_2 can be written as

$$M_2^{(n)}(Q^2) = \int_0^1 dx \, x^n \frac{\partial^2}{\partial x^2}\left(\frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}\right) \tag{20a}$$

$$= \left[x^n \frac{\partial}{\partial x}\left(\frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}\right)\right]_{x=0}^1$$

$$- \left[nx^{n-1} \frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}\right]_{x=0}^1$$

$$+ n(n-1) \int_0^1 dx \, x^{n-2} \frac{x g_2(\xi)}{\xi(1 + \mu\xi^2)}, \tag{20b}$$

where integration by parts has been performed twice. Changing variables from x to ξ, and using the fact that the kinematic maximum value of ξ is given by ξ_0, the moment becomes

$$M_2^{(n)}(Q^2) = \frac{4\mu^2\xi_0^3}{(1 + \mu\xi_0^2)^3} g_2(\xi_0) + \frac{1 - \mu\xi_0^2}{(1 + \mu\xi_0^2)^2} \frac{\partial g_2(\xi)}{\partial\xi}\bigg|_{\xi=\xi_0}$$

$$- \frac{n}{(1 - \mu^2\xi_0^4)} g_2(\xi_0) + n(n-1)$$

$$\times \sum_{j=0}^\infty \mu^j \binom{n+j}{j} \int_0^{\xi_0} d\xi \, \xi^{n+2j-2} g_2(\xi), \tag{21}$$

where we have also used $dx/d\xi = (1 + \mu\xi^2)/(1 - \mu\xi^2)^2$, together with the relation

$$\frac{1}{(1 - \mu\xi^2)^{n+1}} = \sum_{j=0}^\infty \mu^j \binom{n+j}{j} \xi^{2j}. \tag{22}$$

Now, consider the last term in Eq. (21) involving the integral of the function $g_2(\xi)$. From the definition of the parton distributions in Eq. (14), one can write

$$\frac{1}{(n + 2j)(n + 2j - 1)} C_2^{n+2j} A_{n+2j}$$

$$= \int_0^{\xi_0} d\xi \xi^{n+2j-2} g_2(\xi) + \int_{\xi_0}^1 d\xi \, \xi^{n+2j-2} g_2(\xi). \tag{23}$$

However, because the function f_2 [and hence its integrals as in Eqs. (18) and (19)] has no reason to vanish in the region $\xi_0 < \xi < 1$, the second term in Eq. (23) is in general nonzero. The same is true for the first three terms in Eq. (21), and as a consequence one does not recover exactly the original expression, Eq. (13b).

On the other hand, if the parton distributions were to vanish in the region $\xi_0 < \xi < 1$, the moments would have to depend on ξ_0,

$$C_i^n A_n(\xi_0) = \int_0^1 d\xi \, \xi^{n-1} f_i(\xi; \xi_0)$$

$$\implies \frac{dA_n(\xi_0)}{d\xi_0} = \int_0^1 d\xi \, \xi^n \frac{df_i(\xi; \xi_0)}{d\xi_0} \neq 0, \tag{24}$$

where we explicitly label the dependence of the functions f_i on ξ and ξ_0. This result suggests two immediate problems:

(i) universal (process-independent) parton distributions would no longer exist at finite Q^2; and (ii) the separation between short and long distances on the light cone, as embodied in the OPE, would no longer be possible.

If the condition that the structure functions vanish for $\xi > \xi_0$ is not imposed, one is then faced with the prospect of energy-momentum not being conserved. In fact, if the upper limit of integration in Eq. (20a) were extended from $x = 1$ to $x = 1/(1 - \mu)$, the first three terms of Eq. (21) would be identically zero, and extending the integration in the fourth term to $\xi = 1$, Eq. (13b) would be recovered. Consequently, the consistency of the GP prescription [2], and in most subsequent TMC treatments, requires the violation of energy-momentum conservation. It thus appears that a general consequence of defining parton distributions at finite Q^2 in the presence of TMCs is that one must choose between two less-than-ideal options: either keeping a universal parton distribution and violating energy-momentum conservation, or conserving energy and momentum but working with process-dependent distributions.

We can assess the numerical significance of the $\xi > \xi_0$ region by evaluating the lowest ($n = 2$) moment $M_2^{(2)}$ using a simple form for the parton distribution,

$$x f_2(x) = \frac{35}{32} \sqrt{x} \, (1 - x)^3, \qquad (25)$$

chosen to approximately reproduce a typical valence quark distribution, normalized such that $\int_0^1 dx \, f_2(x) = 1$. The moment shown in Fig. 1 is computed from Eq. (13b) for several values of j (from the leading term only, $j = 0$, up to the inclusion of the first four terms, $j < 4$), and is compared with directly integrating $F_2(x, Q^2)$ over x from 0 to 1 (or equivalently up to $\xi = \xi_{max} = \xi_0$), using Eqs. (17b). As noted above, this procedure does not recover the formal result for the moment, Eq. (13b), which is reflected in the nonconvergence
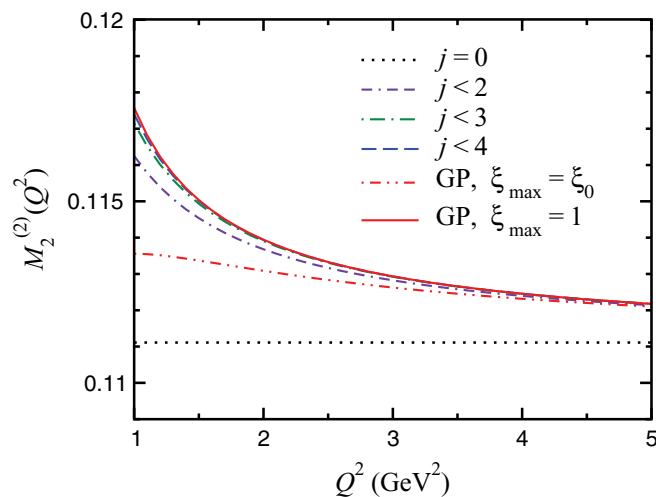

FIG. 1. (Color online) $n = 2$ moments of the F_2 structure function, illustrating the convergence of the series in Eq. (13b) for $j = 0$ (dotted), $j < 2$ (dot-dash-dashed), $j < 3$ (dot-dashed) and $j < 4$ (dashed), compared with the standard TMC result from GP [2] using Eq. (17b) with the upper limit of integration $\xi_{max} = \xi_0$ (21) (dot-dot-dashed) and $\xi_{max} = 1$ (solid).

of the moments with increasing j to the standard TMC result from GP [2] in Eq. (17b) (dot-dot-dashed curve in Fig. 1). On the other hand, if the missing integration range as expressed in the second term of Eq. (23) is kept, the convergence of the moments is recovered (solid curve in Fig. 1), although at the expense of effectively integrating beyond $x = 1$ (to $\xi_{max} = 1$).

The problem encountered here is at the core of the parton interpretation of the matrix elements A_n. The approach of GP attempts to maintain a partonic interpretation at finite Q^2 by introducing a new scaling variable ξ [14,15]. However, as shown in Eq. (21), this leads to inconsistencies in the extracted x dependence of the structure functions and their moments.

A possible way to avoid the problematic $\xi \sim \xi_0$ region is, ironically, to not introduce the Nachtmann scaling variable ξ in the first place. This can be realized by performing the inversion of the moments order by order in μ, rather than summing over all powers of μ during the inversion. As we shall see in the next section, this allows us to work with universal twist-2 distribution functions, while simultaneously preserving energy-momentum conservation. The only drawback of this approach is that the region of x and Q^2 where parton distributions can be formulated consistently in the presence of TMCs will be somewhat restricted.

III. SERIES EXPANSION OF INVERTED MOMENTS

In the course of inverting the moments to obtain the structure functions, the binomial theorem is used to perform the integration by absorbing combinatorial factors involving the integration variable n (see Appendix B). Instead of this standard procedure, in this section we describe how the moments can be inverted term by term by absorbing the combinatorial factor into derivatives, which gives rise to novel series expansions for each of the structure functions. We illustrate this procedure for the case of the F_2 structure function, with the derivation of the other structure functions following similarly.

For the jth term in the series expansion for the F_2 moment in Eq. (13a), $M_{2,j}^{(n)}$, the contribution to the structure function is given by the inverse Mellin transform

$$F_{2,j}(x, Q^2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \, x^{1-n} \, M_{2,j}^{(n)}(Q^2). \qquad (26)$$

Using integration by parts to write

$$C_2^{n+2j} A_{n+2j} = (n + 2j)(n + 2j - 1)$$
$$\times \int_0^1 dy \, y^{n+2j-2} \, g_2(y, Q^2), \qquad (27)$$

the contribution to $F_{2,j}$ can then be expressed in the form

$$F_{2,j}(x, Q^2) = \mu^j \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \int_0^1 dy \, \frac{(n + j)!}{j!(n - 2)!}$$
$$\times x^{1-n} \, y^{n+2j-2} \, g_2(y). \qquad (28)$$

Next, we can observe that

$$\frac{(n+j)!}{(n-2)!} x^{-n+1} = (-x)^{2+j} \frac{\partial^{2+j}}{\partial x^{2+j}} x^{-n+1} \qquad (29)$$

for all n on the imaginary axis except the origin, so that

$$F_{2,j}(x, Q^2) = \mu^j \frac{(-x)^{2+j}}{j!} \frac{\partial^{2+j}}{\partial x^{2+j}} \int_0^1 dy \, x \, y^{2j-2} \, g_2(y) \frac{1}{2\pi i}$$
$$\times \int_{-i\infty}^{i\infty} dn \left(\frac{y}{x}\right)^n. \qquad (30)$$

Finally, making use of the δ-function representation in Eq. (B3) we arrive at the desired result,

$$F_{2,j}(x, Q^2) = \frac{(-x)^{2+j}}{j!} \mu^j \frac{\partial^{2+j}}{\partial x^{2+j}} [x^{2j} \, g_2(x)]. \qquad (31)$$

This result can also be obtained by noting that, instead of Eq. (29), we can write

$$\frac{(n+j)!}{(n-2)!} y^{n+2j-2} = y^{2j} \frac{\partial^{2+j}}{\partial y^{2+j}} y^{n+j}. \qquad (32)$$

Substituting this into Eq. (28) then leads to

$$F_{2,j}(x, Q^2)$$
$$= \mu^j \frac{1}{j!} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \int_0^1 dy x^{-n+1} y^{2j} \frac{\partial^{2+j}}{\partial y^{2+j}} [y^{n+j} g_2(y)]$$
$$= \mu^j \frac{x^2}{j!} \int_0^1 dy y^{2j} g_2(y) \frac{\partial^{2+j}}{\partial y^{2+j}} [y^j \delta(y-x)], \qquad (33)$$

using again the δ-function representation (B3). Now, because the δ function is a distribution, one defines its derivative (in analogy with integration by parts of regular functions) as

$$\int dz \, \delta(z) \, \phi'(z) = -\int dz \, \delta'(z) \, \phi(z) \qquad (34)$$

for a given function ϕ. Applying this definition $(2+j)$ times to Eq. (33), we find

$$F_{2,j}(x, Q^2) = \mu^j \frac{(-1)^{2+j}}{j!} x^2$$
$$\times \int_0^1 dy \frac{\partial^{2+j}}{\partial y^{2+j}} [y^{2j} g_2(y)] \, y^j \, \delta(y-x), \qquad (35)$$

which gives a result identical to that in Eq. (31).

Either of these two methods may be applied to the moments of the other structure functions to obtain the complete expressions for the unpolarized TMC structure functions in terms of power series in M^2/Q^2, summing over all values of j,

$$F_1(x, Q^2) = x \sum_{j=0}^{\infty} \mu^j \frac{(-x)^j}{j!} \frac{\partial^j}{\partial x^j}$$
$$\times \left[x^{2j-2} \left(\frac{1}{2} x f_1(x) + j g_2(x) \right) \right], \qquad (36a)$$

$$F_2(x, Q^2) = x^2 \sum_{j=0}^{\infty} \mu^j \frac{(-x)^j}{j!} \frac{\partial^{2+j}}{\partial x^{2+j}} [x^{2j} g_2(x)], \qquad (36b)$$

$$F_L(x, Q^2) = x^2 \sum_{j=0}^{\infty} \mu^j \frac{(-x)^j}{j!} \frac{\partial^j}{\partial x^j}$$
$$\times [x^{2j-2}(x f_2(x) - x f_1(x) + 4 j g_2(x))], \qquad (36c)$$

$$F_3(x, Q^2) = \sum_{j=0}^{\infty} \mu^j \frac{(-x)^{1+j}}{j!} \frac{\partial^{1+j}}{\partial x^{1+j}} [x^{2j} h_3(x)], \qquad (36d)$$

$$F_4(x, Q^2) = x \sum_{j=0}^{\infty} \mu^j \frac{(-x)^j}{j!} \frac{\partial^j}{\partial x^j} \left[x^{2j-2} \left(j(j-1) g_2(x) \right. \right.$$
$$\left. \left. + \frac{1}{4} x f_4(x) - j g_5(x) \right) \right], \qquad (36e)$$

$$F_5(x, Q^2) = \sum_{j=0}^{\infty} \mu^j \frac{(-x)^{1+j}}{j!} \frac{\partial^{1+j}}{\partial x^{1+j}}$$
$$\times \left[x^{2j-1} \left(-j g_2(x) + \frac{1}{2} x h_5(x) \right) \right]. \qquad (36f)$$

Note that the Nachtmann variable ξ does not enter in Eqs. (36), and all the functions f_i, g_i, and h_i are expressed as functions of x only. As required, the $j = 0$ term in the expansion of F_i is simply the massless limit structure function, $F_i^{(0)}$.

The advantage of this formulation is that it explicitly avoids the problems encountered with the consistency of the inversion in the GP approach discussed in Sec. II C. Indeed, direct integration of the structure functions in Eq. (36) leads to the correct expressions for the moments in Eqs. (13).

To examine the convergence of the series in Eqs. (36), we show in Fig. 2(a) the first few terms in the expansion of F_2, starting with the leading order, $j = 0$, term and up to the first five terms in the series, $j < 5$. For illustration, we use the simple massless limit function in Eq. (25), and compare the result with the standard TMC calculation from GP [2] in Eq. (17b). The results show that the convergence at $Q^2 = 1 \text{ GeV}^2$ is fairly rapid for $x \lesssim 0.5$, with just the first two or three terms already giving a target mass corrected function that does not change noticeably with inclusion of higher orders.

It is noteworthy that in this range one is already well within the nucleon resonance region, traditionally taken to be $W < 2$ GeV, from which data are typically excluded in global PDF analyses. This can be more clearly seen in Fig. 2(b), where the structure function is shown as a function of W. The convergence of the TMCs is well under control down to values as low as $W \approx 1.3$ GeV, just above the peak of the first resonance region dominated by the $\Delta(1232)$ resonance. At smaller W, or higher x, the higher order terms display oscillatory behavior as one approaches the nucleon elastic point, $x = 1$ (or $W = M$). For the particular form of f_2 chosen in Eq. (25), $x f_2 \sim (1-x)^3$, the first three terms in the series $(j < 3)$ vanish as $x \to 1$, while the contributions for $j \geqslant 3$ diverge at $x = 1$. The target mass corrected function from Eq. (17b) (labeled "GP" in Fig. 2) is finite at $x = 1$ and indeed extends into the unphysical region $W < M$.

The large-x oscillatory behavior is significantly dampened by the time one reaches $Q^2 = 5 \text{ GeV}^2$, with the first three terms $(j < 3)$ converging well up to $x \approx 0.8$, as shown in
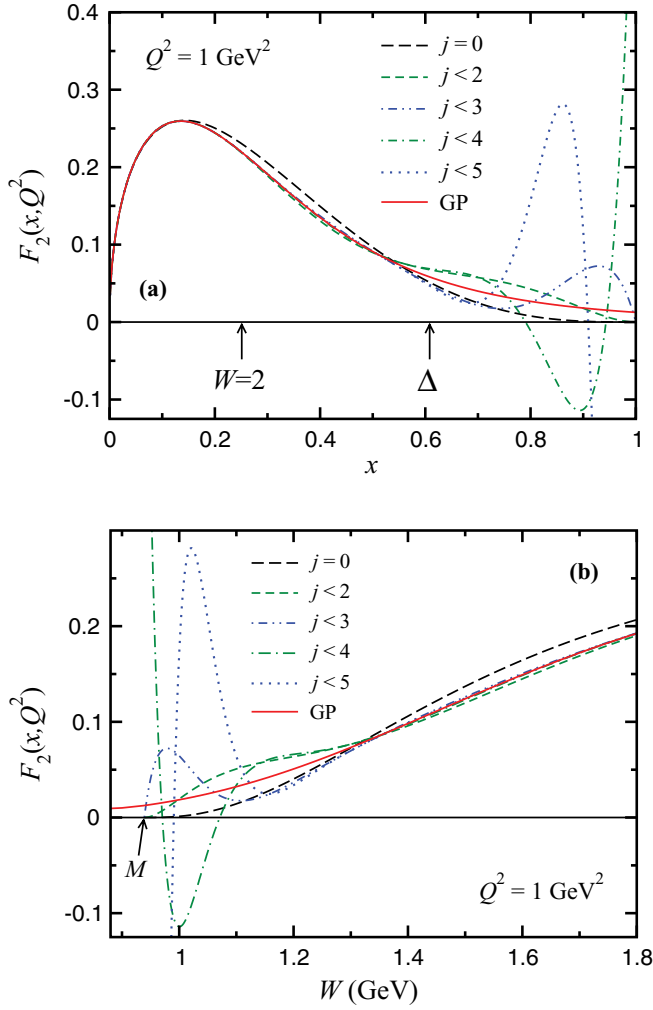
FIG. 2. (Color online) Target mass corrected F_2 structure function at $Q^2 = 1$ GeV^2 from Eq. (36b), showing the convergence with increasing j, and compared with the standard TMC result from GP [2] using Eq. (17b), shown as a function of (a) Bjorken x, and (b) the hadronic final state mass W. The arrows in (a) indicate the locations of the resonance region ($W = 2$ GeV) and the Δ resonance, while the arrow in (b) denotes the elastic limit, $W = M$.

FIG. 3. (Color online) Ratio of target mass corrected F_2 structure function at $Q^2 = 5$ GeV^2 from Eq. (36b) for various j ($j = 0$ up to $j < 5$) to the standard TMC result from GP [2] using Eq. (17b), shown as a function of (a) Bjorken x, and (b) the hadronic final state mass W. The arrows in (a) indicate the locations of the resonance region ($W = 2$ GeV) and the Δ resonance.

Fig. 3(a) for the ratio of F_2 to the GP target mass corrected function, Eq. (17b). At this Q^2 this corresponds to values of $W \gtrsim 1.4$ GeV, illustrated in Fig. 3(b), which again is well outside of the range where DIS data are typically used in global PDF analyses. Note that the vanishing of ratio of the leading order term, $j = 0$, to the full GP result as $x \to 1$ reflects the nonzero value of the GP TMC function at $x \geqslant 1$.

One may ask whether the series converges over the entire range of x at finite Q^2 if a sufficiently large number of terms is included in the sum over j. For the trial distribution (25) used here, Fig. 4 shows the result of summing up to ≈ 100, 200, or 300 terms for $Q^2 = 2$ GeV^2. For $x \lesssim 0.73$ (or $W \gtrsim 1.27$ GeV) fairly good convergence is observed, while for $x \gtrsim 0.74$ (or $W \lesssim 1.25$ GeV) the addition of a large number of terms is needed to push the convergence of the target mass corrected function to significantly higher x values. It is interesting to observe that the inclusion of additional odd or even terms in
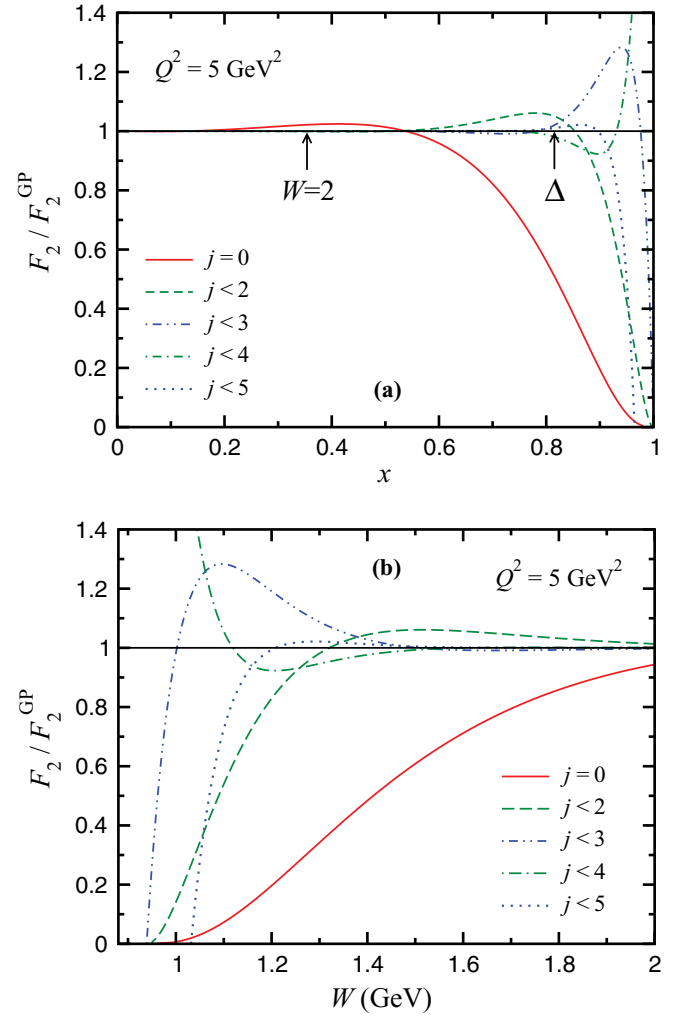
j gives alternating negative and positive divergent behaviors, respectively. The systematics of this convergence with Q^2 and W will be discussed in more detail in Ref. [40]. What is clear, however, is that a severe limitation on the computation of TMC exists at very high x for small values of Q^2, although this occurs deep in the resonance region at low W.

IV. CONCLUSION

The problem of target mass corrections to deep-inelastic structure functions is almost as old as the theory of QCD itself. With the focus of most structure function analyses being on the perturbative region where subleading $1/Q^2$ effects can be neglected, the study of TMCs remained largely dormant for several decades. The advent of new, high-precision data in the resonance-scaling transition region at high x and low Q^2 has brought the problem of TMCs back to the fore, giving rise
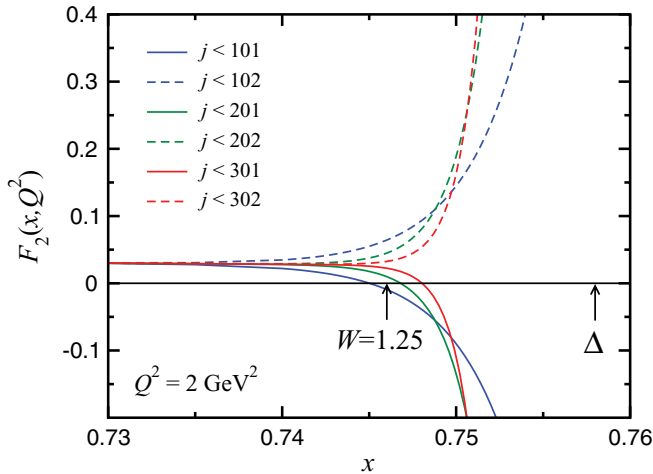
FIG. 4. (Color online) Convergence of the series expansion for the target mass corrected F_2 structure function for large values of j ($j < 101$, 102, 201, 202, 301, and 302), at $Q^2 = 2\,\text{GeV}^2$. The arrows indicate the values of x at which the final state mass corresponds to the Δ resonance and to $W = 1.25$ GeV. Note the limited x range on the ordinate.

to greater urgency to the need for resolution of the remaining open issues with respect to their implementation.

In this paper we have sought to illustrate the inherent problem with the standard TMC formulation, already evident in the pioneering work of Georgi and Politzer [2], in the treatment of the $x \approx 1$ region and the inversion of the structure functions from their moments. In particular, we have critically analyzed the definition of PDFs in the presence of TMCs, and discussed the consequences of the violation of energy and momentum conservation in the standard TMC analysis. Historically it has been argued [3,4] that the problem in the threshold region exists because at low Q^2 the higher twist contributions cannot be neglected. We do not disagree that higher twists are essential for describing low energy structure function data; we believe, however, that one ought to maintain consistency of leading twist functions at any x, regardless of how large the higher twists may be at a given Q^2.

We contend that the introduction of the Nachtmann variable ξ, which appears naturally in the standard TMC implementation, does not lead to self-consistent parton distributions that are valid at all x. In fact, our analysis suggests that strictly speaking PDFs cannot be defined consistently at any finite Q^2 when the mass of the target is incorporated. What is feasible, however, is to compute the x dependence of the TMC structure functions in terms of standard PDFs as a series whose convergence can be studied as a function of x and Q^2.

To this end, we have derived formulas for the entire set of unpolarized structure functions, as a series in M^2/Q^2, involving PDFs and their derivatives. The virtue of this approach is that the resulting TMC functions can be consistently inverted from their moments, without ever encountering unphysical regions of kinematics or violating energy and momentum conservation. Moreover, it allows us to systematically study the regions of x and Q^2 where TMCs can be reliably applied. Using a simple trial function, we have illustrated the

convergence of our scheme numerically for the case of the F_2 structure function. Rapid convergence is observed for most of the range of x, with the first two or three terms saturating the sum well into the nucleon resonance region. For Q^2 values as low as 1 GeV^2, we find that the convergence of the TMC series expansion is under control down to $W \approx 1.3$ GeV, which is almost in the vicinity of the Δ resonance peak.

At smaller W, or higher x for fixed Q^2, rapid oscillations ensue as one approaches the elastic scattering limit, and beyond $W \approx 1.3$ GeV it becomes prohibitively difficult to tame these with a finite number of higher order terms. At $Q^2 = 2$ GeV^2, for example, even summing over $\sim$300 terms allows for a smooth TMC function up to $x \approx 0.73$ (or $W \approx 1.27$ GeV). Fortunately, such low W values are well outside the range typically encountered in perturbative QCD analyses of DIS data, and in practice will not pose any serious restrictions.

Our results therefore lend greater support to global PDF fits which incorporate low-W data, currently down to $W^2 = 3$ GeV^2 in some analyses [7–9], but with more ambitions plans to extend the range further into the traditional resonance region. Provided higher twist and other subleading corrections are tractable, our method for accounting for TMCs will introduce only minimum theoretical uncertainty into global analyses. A more detailed discussion of the utility of the present approach, as well as its application to spin-dependent structure functions, will be presented in a forthcoming publication [40].

APPENDIX A: DERIVATION OF STRUCTURE FUNCTION MOMENTS

In this Appendix we illustrate the derivation of moments of structure functions in the presence of TMCs, using as an example the F_1 structure function (the F_1 case contains some more general features that are not present in the F_2 derivation discussed in Secs. II and III). The results for the other structure functions follow in a similar manner.

We begin by finding T_1, the coefficient of $-g^{\mu\nu}$ in Eq. (7), which has contributions from both the C_1^{2k} and C_2^{2k} terms. For the C_1^{2k} term, for fixed $k \in \mathbb{N}$, $j \in \{0, \ldots, k\}$, and each term of $\{g \cdots g\, p \cdots p\}_{k,j}$, a total of j q_{μ_i} factors will have their indices raised by j $g^{\mu_i \mu_l}$ metric tensors. The result will then contract with j q_{μ_l}'s to give a factor of $(q^2)^j$. The remaining $(2k - 2j)$ q_{μ_i} factors will contract with the $(2k - 2j)$ p^{μ_i} factors to give $(p \cdot q)^{2k-2j}$. Since there are $(2k)!/[2^j\, j!\,(2k - 2j)!]$ terms in $\{g \cdots g\, p \cdots p\}_{k,j}$, we find

that

$$\sum_{k=1}^{\infty} \left(- g^{\mu\nu} q_{\mu_1} q_{\mu_2} C_1^{2k} \right) q_{\mu_3} \cdots q_{\mu_{2k}} \frac{2^{2k}}{Q^{4k}} A_{2k} \Pi^{\mu_1 \cdots \mu_{2k}}$$

$$= -g^{\mu\nu} \sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \frac{(2k)!}{2^j j! (2k-2j)!} (p^2 q^2)^j (p \cdot q)^{2k-2j} \frac{2^{2k}}{(Q^2)^{2k}} C_1^{2k} A_{2k}. \tag{A1}$$

The $g_{\mu_1}^{\mu} g_{\mu_2}^{\nu} Q^2 C_2^{2k}$ term of Eq. (7) contributes to the coefficient of $-g^{\mu\nu}$ due to the identity $g_{\mu_1}^{\mu} g_{\mu_2}^{\nu} g^{\mu_1 \mu_2} = g^{\mu\nu}$. For fixed $k \in \mathbb{N}$ and $j \in \{0, \ldots, k\}$, we seek the terms of $\{g \cdots g \, p \cdots p\}_{k,j}$ which include a factor of $g^{\mu_1 \mu_2}$. The number of such terms is determined by the number of ways to distribute the indices $\mu_3, \ldots, \mu_{2k}$ among $(j-1)$ g's and $(2k-2j)$ p's without creating duplicate products. From the $(2k)!/[2^j j! (2k-2j)!]$ ways to distribute the indices $\mu_1, \ldots, \mu_{2k}$ over j g's and $(2k-2j)$ p's without creating duplicates, relabeling indices $j \to j-1$ and $k \to k-1$ gives $(2k-2)!/[2^{j-1}(j-1)!(2k-2j)!]$ terms of $\{g \cdots g \, p \cdots p\}_{k,j}$ which contain $g^{\mu_1 \mu_2}$ for $k \in \mathbb{N}$, $j \in \{1, \ldots, k\}$. (Note that there are no terms of $\{g \cdots g \, p \cdots p\}_{k,0}$ that contain $g^{\mu_1 \mu_2}$.) As for the C_1^{2k} terms, we then find

$$\sum_{k=1}^{\infty} \left(g_{\mu_1}^{\mu} g_{\mu_2}^{\nu} Q^2 C_2^{2k} \right) q_{\mu_3} \cdots q_{\mu_{2k}} \frac{2^{2k}}{Q^{4k}} A_{2k} \Pi^{\mu_1 \cdots \mu_{2k}} = g^{\mu\nu} \sum_{k=1}^{\infty} \sum_{j=1}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \frac{(2k-2)!}{2^{j-1}(j-1)!(2k-2j)!}$$

$$\times (p^2)^j (q^2)^{j-1} (p \cdot q)^{2k-2j} \frac{2^{2k}}{(Q^2)^{2k}} Q^2 C_2^{2k} A_{2k} + \text{terms not involving } g^{\mu\nu}. \tag{A2}$$

Now, the term $-i\epsilon^{\mu\nu\alpha\beta} g_{\alpha\mu_1} q_{\beta} q_{\mu_2} C_3^{2k} + (q^{\mu} q^{\nu}/Q^2) q_{\mu_1} q_{\mu_2} C_4^{2k} + (g_{\mu_1}^{\mu} q^{\nu} q_{\mu_2} + g_{\mu_1}^{\nu} q^{\mu} q_{\mu_2}) C_5^{2k}$ in Eq. (7) will not contribute to T_1, as the indices μ and ν are "locked up" in such a way that they cannot result in a factor of $g^{\mu\nu}$ through contractions. We conclude, therefore, that the coefficient T_1 of $-g^{\mu\nu}$ in the expansion (7) is

$$T_1 = \sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \frac{(2k)!}{2^j j! (2k-2j)!} (p^2 q^2)^j (p \cdot q)^{2k-2j} \frac{2^{2k}}{(Q^2)^{2k}} C_1^{2k} A_{2k}$$

$$- \sum_{k=1}^{\infty} \sum_{j=1}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \frac{(2k-2)!}{2^{j-1}(j-1)!(2k-2j)!} (p^2)^j (q^2)^{j-1} (p \cdot q)^{2k-2j} \frac{2^{2k}}{(Q^2)^{2k}} Q^2 C_2^{2k} A_{2k}. \tag{A3}$$

Substituting $p^2 = M^2$, $q^2 = -Q^2$, and $p \cdot q = Q^2/2x$ into Eq. (A3), changing indices in each term to $l = k - j$ and j, and rearranging, gives the result

$$T_1(x, Q^2) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \binom{2l+j}{j} \mu^j \frac{1}{x^{2l}} C_1^{2l+2j} A_{2l+2j} + \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \binom{2l+j}{j} \frac{j}{(l+j)(2l+2j-1)} \mu^j \frac{1}{x^{2l}} C_2^{2l+2j} A_{2l+2j}. \tag{A4}$$

Finally, using the identity $\oint_C d\omega \, \omega^{n-m-1} = 2\pi i \, \delta_{nm}$ together with Eq. (6), we find that

$$M_1^{(n)}(Q^2) = \frac{1}{2} \frac{1}{2\pi i} \oint_C d\omega \frac{T_1(1/\omega, Q^2)}{\omega^{n+1}}$$

$$= \frac{1}{2} \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \mu^j \binom{2l+j}{j} C_1^{2l+2j} A_{2l+2j} \delta_{2l,n} + \frac{1}{2} \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \mu^j \binom{2l+j}{j} \frac{j}{(l+j)(2l+2j-1)} C_2^{2l+2j} A_{2l+2j} \delta_{2l,n}, \tag{A5}$$

which leads to Eq. (13a). The results for the other moments (13) are derived in a similar manner.

APPENDIX B: STRUCTURE FUNCTION INVERSION

In this section we illustrate the standard moment inversion procedure by presenting a detailed derivation for the case of the F_1 structure function. The derivations for the other structure functions can be deduced straightforwardly from this example.

We begin by denoting the series in Eq. (13a) involving C_1^{n+2j} and C_2^{n+2j} by $m_1^{(n)}(Q^2)$ and $m_2^{(n)}(Q^2)$, respectively. Using Eq. (14), we find for the $m_1^{(1)}$ term

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \, x^{-n} m_1^{(n)}(Q^2) = \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} dn \int_0^1 dy \, x^{-n} y^{n-1} f_1(y) \sum_{j=0}^{\infty} \binom{n+j}{j} (\mu y^2)^j. \tag{B1}$$

From the (generalized) binomial theorem, it follows then that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, x^{-n} m_1^{(n)}(Q^2) = \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} dn \int_0^1 dy\, x^{-n} y^{n-1} f_1(y) \frac{1}{(1-\mu y^2)^{n+1}} = \frac{1}{2} \int_0^1 dy\, \frac{f_1(y)}{y(1-\mu y^2)} \delta\left(\ln \frac{y}{x(1-\mu y^2)} \right),$$
(B2)

where we have used the δ-function representation

$$\delta(\ln u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dn\, e^{in(\ln u)} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, u^n.$$
(B3)

Using the relation

$$\delta(u(y)) = \sum_{a\,=\,\text{root of } u} \frac{1}{|u'(a)|} \delta(y-a)$$
(B4)

with $u(y) = \ln(y/[x(1-\mu y^2)])$ and $u'(y) = (1+\mu y^2)/[y(1-\mu y^2)]$, the only root of u on $[0, 1]$ corresponds to $\xi = 2x/(1+\rho)$, which leads to

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, x^{-n} m_1^{(n)}(Q^2) = \frac{1}{2} \int_0^1 dy\, \frac{f_1(y)}{1+\mu y^2} \delta(y-\xi).$$
(B5)

For the $m_2^{(n)}$ term, using integration by parts with Eqs. (19) and (27), we can write

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, x^{-n} m_2^{(n)}(Q^2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \int_0^1 dy\, x^{-n} y^{n-2} g_2(y) \sum_{j=1}^{\infty} j \binom{n+j}{j} (\mu y^2)^j.$$
(B6)

Next, from the relations

$$\sum_{j=1}^{\infty} j \binom{n+j}{j} (\mu y^2)^j = \sum_{j=1}^{\infty} (n+1) \binom{n+j}{j-1} (\mu y^2)^j = (n+1) \frac{\mu y^2}{(1-\mu y^2)^{n+2}},$$
(B7)

we obtain

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, x^{-n} m_2^{(n)}(Q^2) = \frac{1}{2\pi i}\, \mu \int_{-i\infty}^{i\infty} dn \int_0^1 dy\, (n+1)\, x^{-n} y^n \frac{g_2(y)}{(1-\mu y^2)^{n+2}}.$$
(B8)

Finally, since $(n+1)x^{-n} = -x^2(\partial/\partial x)\, x^{-n-1}$, we arrive at the result for the $m_2^{(n)}$ moment,

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn\, x^{-n} m_2^{(n)}(Q^2) = -\frac{1}{2\pi i}\, \mu x^2 \frac{\partial}{\partial x} \int_{-i\infty}^{i\infty} dn \int_0^1 dy\, \frac{x^{-n-1} y^n g_2(y)}{(1-\mu y^2)^{n+2}}$$

$$= -\mu x^2 \frac{\partial}{\partial x} \int_0^1 dy\, \frac{g_2(y)}{x(1-\mu y^2)^2} \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \left(\frac{y}{x(1-\mu y^2)} \right)^n \right].$$
(B9)

Combining Eqs. (B5) and (B9) then gives the final result for the inverted F_1 structure function,

$$F_1(x, Q^2) = \frac{1}{2(1+\mu\xi^2)} f_1(\xi) - \mu x^2 \frac{\partial}{\partial x} \left(\frac{g_2(\xi)}{1+\mu\xi^2} \right).$$
(B10)

The results for the other structure functions $F_2, \ldots, F_5$ in Eqs. (17) follow analogous derivations.

[1] M. E. Christy and W. Melnitchouk, J. Phys. Conf. Ser. **299**, 012004 (2011).

[2] H. Georgi and H. D. Politzer, Phys. Rev. D **14**, 1829 (1976).

[3] A. De Rújula, H. Georgi, and H. D. Politzer, Phys. Rev. D **15**, 2495 (1977).

[4] A. De Rújula, H. Georgi, and H. D. Politzer, Ann. Phys. (NY) **103**, 315 (1977).

[5] I. Schienbein *et al.*, J. Phys. G **35**, 053101 (2008).

[6] L. T. Brady, A. Accardi, T. J. Hobbs, and W. Melnitchouk, Phys. Rev. D **84**, 074008 (2011).

[7] S. Alekhin, J. Blümlein, S. Klein, and S. Moch, Phys. Rev. D **81**, 014032 (2010).

[8] A. Accardi, M. E. Christy, C. E. Keppel, W. Melnitchouk, P. Monaghan, J. G. Morfin, and J. F. Owens, Phys. Rev. D **81**, 034016 (2010).

[9] A. Accardi, W. Melnitchouk, J. F. Owens, M. E. Christy, C. E. Keppel, L. Zhu, and J. G. Morfin, Phys. Rev. D **84**, 014008 (2011).

[10] S. Alekhin, J. Blümlein, and S. Moch, Phys. Rev. D **86**, 054009 (2012).

[11] R. D. Ball *et al.*, Nucl. Phys. B **809**, 1 (2009); **816**, 293(E) (2009).

[12] E. D. Bloom and F. J. Gilman, Phys. Rev. Lett. **25**, 1140 (1970).

[13] W. Melnitchouk, R. Ent, and C. Keppel, Phys. Rep. **406**, 127 (2005).

[14] O. Nachtmann, Nucl. Phys. B **63**, 237 (1973).

[15] O. W. Greenberg and D. Bhaumik, Phys. Rev. D **4**, 2048 (1971).

[16] R. K. Ellis, R. Petronzio, and G. Parisi, Phys. Lett. B **64**, 97 (1976).

[17] R. Barbieri, J. R. Ellis, M. K. Gaillard, and G. G. Ross, Phys. Lett. B **64**, 171 (1976); Nucl. Phys. B **117**, 50 (1976).

[18] D. J. Gross, S. B. Treiman, and F. A. Wilczek, Phys. Rev. D **15**, 2486 (1977).

[19] K. Bitar, P. W. Johnson, and W. K. Tung, Phys. Lett. B **83**, 114 (1979).

[20] P. W. Johnson and W. K. Tung, contribution to Neutrino '79, Bergen, Norway, 1979, Illinois Institute of Technology Report No. Print-79-1018.

[21] W. R. Frazer and J. F. Gunion, Phys. Rev. Lett. **45**, 1138 (1980).

[22] W. Melnitchouk, Phys. Rev. Lett. **86**, 35 (2001).

[23] F. M. Steffens and W. Melnitchouk, Phys. Rev. C **73**, 055202 (2006).

[24] R. K. Ellis, W. Furmanski, and R. Petronzio, Nucl. Phys. B **212**, 29 (1983).

[25] M. A. G. Aivazis, F. I. Olness, and W. K. Tung, Phys. Rev. D **50**, 3085 (1994).

[26] S. Kretzer and M. H. Reno, Phys. Rev. D **66**, 113007 (2002).

[27] S. Kretzer and M. H. Reno, Phys. Rev. D **69**, 034002 (2004).

[28] A. Accardi and J. W. Qiu, J. High Energy Phys. 07 (2008) 090.

[29] N. Baillie *et al.*, Phys. Rev. Lett. **108**, 199902 (2012).

[30] S. Bültmann, S. Kuhn, H. Fenker, W. Melnitchouk, M. E. Christy, C. E. Keppel, V. Tvaskis, and K. Griffioen (spokespersons), Jefferson Lab Experiment E12-06-113.

[31] G. G. Petratos, J. Gomez, R. J. Holt, and R. D. Ransome (spokespersons), Jefferson Lab Experiment E12-10-103.

[32] W. Melnitchouk and A. W. Thomas, Phys. Lett. B **377**, 11 (1996).

[33] R. J. Holt and C. D. Roberts, Rev. Mod. Phys. **82**, 2991 (2010).

[34] S. Kuhlmann *et al.*, Phys. Lett. B **476**, 291 (2000).

[35] L. T. Brady, A. Accardi, W. Melnitchouk, and J. F. Owens, J. High Energy Phys. 06 (2012) 019.

[36] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).

[37] A. J. Buras, Rev. Mod. Phys. **52**, 199 (1980).

[38] J. Blümlein, arXiv:1208.6087 [Prog. Part. Nucl. Phys. (to be published)].

[39] J. Blümlein (private communication).

[40] M. D. Brown *et al.* (unpublished).