

Reaction cross sections in heavy-ion collisions

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Previously a compact formula for total reaction cross section for heavy-ion collisions as a function of energy was obtained by treating the angular momentum l as a continuous variable. The accuracy of the continuum approximation is assessed and corrections are evaluated. The accuracy of the compact equation can be improved by a simple modification, if a higher accuracy is required. Simple rules to determine the barrier heights and the penetration probability for the l partial wave from experimental data are presented for the collision of identical or nonidentical light nuclei.

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I. INTRODUCTION

Nuclear fusion in heavy-ion collisions is an important process in many phenomena. The sub-barrier fusion of light nuclei plays an important role in the evolution of massive stars, the dynamics of white dwarf Type Ia supernovae, and explosions on the surface of neutron stars [1]. The fusion of heavy nuclei just above the barrier is an important tool in the production of superheavy nuclei [2]. The fusion of identical light nuclei at high energies reveals interesting effects in isolated high-angular-momentum states of the fused system [3]. Nuclear fusion of light nuclei is utilized in accelerator-based heavy-ion inertial fusion for fusion energy production [4].

Previously, a simple analytical expression was obtained for the total reaction cross section σ_r in the collision of nuclei A_1 and A_2 as a function of the collision energy E [5]:

$$\sigma_r(E) = \pi R_0^2 \frac{\Gamma}{E} \ln \left\{ 1 + \exp \left(\frac{E - E_0}{\Gamma} \right) \right\}, \quad (1)$$

where E_0 is the barrier height for the $l = 0$ partial wave, $\Gamma = \hbar\omega/2\pi$ is the energy width parameter in the potential barrier penetration probability, ω is the frequency of the inverted parabola representing the potential barrier, and $R_0 = r_0(A_1^{1/3} + A_2^{1/3})$ is the spatial separation between the two nuclei at the potential barrier. By assuming that the fusion process is equivalent to the strong-absorption of ingoing waves passing through the potential barrier, the total reaction cross section (1) can be interpreted as the heavy-ion fusion cross section.

Even though the fusion process involves complicated dynamics of channel coupling, dynamical distortions, polarizations, deformations, particle transfers, proximity interactions, and barrier penetrations [6], the simple expression of Eq. (1) provides an efficient way to represent experimental data in terms of important physical parameters, whose systematics give valuable insights into the dynamics of the process. The relationship between the barrier penetration model used in Ref. [5] and the coupled-channels calculations for heavy-ion fusion was discussed previously [7]. Equation (1) has been successfully applied to describe fusion cross sections in the collision of many projectile and target combinations [6].

As the range of fusion energy in astrophysical processes and fusion-energy production extends to the region beyond

the sensitivity of present-day measurements, theoretical extrapolations are needed to access many relevant rates of fusion reactions [1]. It is desirable to examine the accuracy of the simple formula (1) in the sub-barrier region. In the other extreme in higher energy fusion, the recent interesting observation of the stepwise increase of the fusion cross section for two identical spin-0 nuclei [3] also calls for an analysis of its accuracy in the high-energy fusion regime.

The simple result of Eq. (1) was obtained by treating the discrete angular momentum l as a continuous variable. Such a treatment incurs errors. We therefore wish to assess the accuracy of Eq. (1) over the whole energy range and to find how it may be improved if a higher accuracy is desired.

The simple result of Eq. (1) relies also on the assumption that the fusion barrier height E_l for the l partial wave is a linear function of $l(l + 1)$. While such an assumption is a reasonable concept for many reactions, there are nuclear collisions for which such an assumption is not valid, as is evidenced by the disagreement of the $^{12}\text{C} + ^{13}\text{C}$ data [8,9] with the main features of Eq. (1). To diagnose such a pathological case, it will be useful to develop tools that will enable us to determine the fusion barrier heights E_l and the penetration probabilities $P_l(E)$, as a function of l , from experimental data. Furthermore, the direct determination of these physical quantities provides useful insight into the fusion process.

For those cases where the basic assumption of the linear dependence of the fusion barrier E_l on $l(l + 1)$ does not hold, we would like to propose alternative methods to describe the reaction cross section. We wish to design a framework to partition the reaction cross section such that contributions to different regions of l partial waves can be singled out for special scrutiny.

This paper is organized as follows. In Sec. II, we evaluate the reaction cross section and develop the rules for fusion barrier analysis. In Sec. III, we present the formulation of the reaction cross section, the continuum approximation, and its corrections. In Sec. IV, we give the numerical results and the comparison with experimental data. In Sec. V, we carry out a barrier analysis for $^{12}\text{C} + ^{13}\text{C}$ and show that the barrier E_l for that reaction is not a linear function of $l(l + 1)$. We show how the fusion cross section of such a pathological case can be described by an alternative method of partitioning the reaction cross section. We present the rules for barrier analysis

in Sec. VI and the rules for the penetration probability analysis in Sec. VII, for the collision of identical or nonidentical light nuclei. In Sec. VIII, we present our conclusions and discussion.

II. REACTION CROSS SECTION AND FUSION BARRIER ANALYSIS

We approximate various barriers for different partial l waves by inverted harmonic-oscillator potentials of height E_l and frequency ω_l in the ingoing-wave strong-absorption model. For a collision energy E , the probability for the absorption of the l partial wave is then given by the Hill-Wheeler penetration probability formula [10],

$$P_l(E) = \frac{1}{1 + \exp\{2\pi(E_l - E)/\hbar\omega_l\}}. \quad (2)$$

As a consequence, the total reaction cross section as a function of E for the collision of unequal nuclei is [5]

$$\sigma_r(E) = \frac{\pi}{k^2} \sum_{l=0,1,2,\dots} \frac{2l+1}{1 + \exp\{2\pi(E_l - E)/\hbar\omega_l\}}. \quad (3)$$

The above expression can be cast in a more illuminating form in terms of the de Broglie wavelength $\lambda = 1/k$. The natural unit of cross sectional area in de Broglie wavelength scales is $\pi\lambda^2$ [11], which can be conveniently called the de Broglie cross section. Using Eq. (3), we can construct the dimensionless measure of the reaction cross section σ_r in units of $\pi\lambda^2$, $(\sigma_r/\pi\lambda^2)$, at the collision energy E , as given by

$$(\sigma_r/\pi\lambda^2)(E) = \sum_{l=0,1,2,\dots} f(l) = f_0 + f_1 + f_2 + f_3 + \dots, \quad (4)$$

where

$$f_l = f(l) = \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma_l\}}, \quad (5)$$

and $\Gamma_l = \hbar\omega_l/2\pi$. We can evaluate $(\sigma_r/\pi\lambda^2)(E)$ at the collision energy E . We find from Eq. (4) that for the energy E such that

$$l^2 \leq (\sigma_r/\pi\lambda^2)(E) \leq (l+1)^2, \quad (6)$$

the reaction cross section at energy E is

$$(\sigma_r/\pi\lambda^2)(E) = l^2 + \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma_l\}} + C(l, E), \quad (7)$$

where the correction term $C(l, E)$ takes into account the width Γ_l for the barrier penetration. It is given explicitly by

$$C(l, E) = - \sum_{l'=0}^{l-1} \frac{2l'+1}{1 + \exp\{(E - E_{l'})/\Gamma_{l'}\}} \Theta(l-1) + \sum_{l'=l+1}^{\infty} \frac{2l'+1}{1 + \exp\{(E_{l'} - E)/\Gamma_{l'}\}}, \quad (8)$$

where $\Theta(x) = 1$ for $x \geq 0$.

For the evaluation of the reaction cross section and the correction term $C(l, E)$, we shall study a simple model in

which we assume that the barriers E_l and the frequencies $\hbar\omega_l$ (or Γ_l) are related to l by

$$E_l = E_0 + \frac{\hbar^2 l(l+1)}{2\mu R_0^2}, \quad (9)$$

$$\hbar\omega_l \sim \hbar\omega \text{ (or } \Gamma_l = \Gamma), \quad (10)$$

where $\mu = A_1 A_2 m_{\text{nucleon}}/(A_1 + A_2)$ is the reduced mass. We shall further convert the summations in Eq. (8) as integrals in the continuum approximation, then the correction term is given explicitly by

$$C(l=0, E) = \frac{2\mu R_0^2 \Gamma}{\hbar^2} \ln[1 + \exp\{(E - E_1)/\Gamma\}], \quad (11)$$

$$C(l=1, E) = \frac{2\mu R_0^2 \Gamma}{\hbar^2} \ln[1 + \exp\{(E - E_2)/\Gamma\}] - \frac{1}{1 + \exp\{(E - E_0)/\Gamma\}}, \quad (12)$$

$$C(l \geq 2, E) = \frac{2\mu R_0^2 \Gamma}{\hbar^2} \ln[1 + \exp\{(E - E_{l+1})/\Gamma\}] - \frac{2\mu R_0^2 \Gamma}{\hbar^2} \ln \left[\frac{1 + \exp\{(E_{l-1} - E)/\Gamma\}}{1 + \exp\{(E_0 - E)/\Gamma\}} \right]. \quad (13)$$

Thus, Eq. (7), with supplementary equations (6), (11), (12), and (13), gives the reaction cross section as a function of energy E .

We can evaluate the reaction cross section $(\sigma_r/\pi\lambda^2)(E_l)$ at the barrier E_l . It is given by

$$(\sigma_r/\pi\lambda^2)|_{E_l} = l(l+1) + \frac{1}{2} + C(l, E_l), \quad (14)$$

where the correction term $C(l, E_l)$ is

$$C(l=0, E_l) = \frac{2\Gamma}{E_1 - E_0} \ln[1 + \exp\{(E_l - E_{l+1})/\Gamma\}],$$

$$C(l=1, E_l) = \frac{2\Gamma}{E_1 - E_0} \ln[1 + \exp\{(E_l - E_{l+1})/\Gamma\}] - \frac{1}{1 + \exp\{(E_l - E_0)/\Gamma\}}, \quad (15)$$

$$C(l \geq 2, E_l) = \frac{2\Gamma}{E_1 - E_0} \ln[1 + \exp\{(E_l - E_{l+1})/\Gamma\}] - \frac{2\Gamma}{E_1 - E_0} \ln \left[\frac{1 + \exp\{(E_{l-1} - E_l)/\Gamma\}}{1 + \exp\{(E_0 - E_l)/\Gamma\}} \right].$$

Equation (14) has a simple physical interpretation. As illustrated in Fig. 2.1 of Blatt and Weisskopf [11], the partial wave l' contributes $2l' + 1$ units to the dimensionless measure of the reaction cross section. The total contribution is the integral of $\int dl'(2l' + 1)$. Therefore, the dimensionless measure of the reaction cross section, up to the fusion barrier of the l partial wave, is given by $l(l+1)$ on the right-hand side. The additional constant $1/2$ is purely quantum mechanical in origin and it depends on the symmetry of the colliding system, as will be discussed in Sec. VI. The correction term $C(l, E_l)$ in Eq. (14) takes into account the finite energy width Γ_l for barrier penetration probability relative to the spacing between adjacent fusion barriers.

TABLE I. The value of the dimensionless measure $\sigma_r/\pi\lambda^2$ at which the fusion barriers E_l for the l partial wave is located, for the collision of unequal nuclei, when the correction term $C(l, E_l)$ can be neglected.

l	0	1	2	3	4	5
$\sigma_r/\pi\lambda^2$	0.5	2.5	6.5	12.5	20.5	30.5

Equations (14) and (15) can be inverted to provide the rule for the ‘‘barrier analysis’’ for unequal nuclei as follows. The fusion barrier E_l for the l partial wave is located at the value of energy E at which the dimensionless reaction cross section measure, $\sigma_r/\pi\lambda^2$, is equal to $l(l+1) + 1/2 + C(l, E_l)$. If the dimensionless measure ($\sigma_r/\pi\lambda^2$) can be obtained experimentally as a function of E , the heights of various fusion barriers E_l can be determined iteratively, within the present model of fusion barrier penetration.

In the beginning of the iteration, one neglects the correction $C(l, E_l)$, and the barriers E_l can be determined from the $\sigma_r/\pi\lambda^2$ values. With the knowledge of the barriers heights E_l the correction terms $C(l, E_l)$ can be evaluated for different l partial waves, and the barrier quantities E_l can be corrected. In these iterations, it is necessary to know the width parameter Γ , which can be obtained either from a fit of the experimental fusion cross section with Eq. (1) or from the penetration probability analysis, as will be discussed in Sec. VII.

We list the values of $\sigma_r/\pi\lambda^2$ at which the fusion barriers E_l are located for the collision of unequal nuclei in Table I, when the correction term $C(l, E_l)$ can be neglected.

The ratio $\Gamma/(E_l - E_{l-1})$ in the correction term $C(l, E_l)$ varies with the colliding nuclei mass number as $A^{5/3}\Gamma/2l$. Thus, a decrease in the mass number or an increase in the angular momentum l will lead to a smaller $\Gamma/(E_l - E_{l-1})$ and a smaller correction term $C(l, E_l)$. Our investigations in subsequent sections (Tables II and III) indicate that, for light nuclei collisions, the condition of $\Gamma \ll |E_l - E_{l\pm 1}|$ is approximately fulfilled so that $C(l, E_l)$ is small. It becomes appropriate to neglect the correction term in the barrier analysis for light nuclei collisions. For heavy nuclei collisions, while the neglect of the correction $C(l, E_l)$ may be appropriate in the barrier analysis for large angular momentum l , the correction term must be properly taken into account for partial waves with small values of l .

III. THE CONTINUUM APPROXIMATION AND ITS CORRECTIONS

Our model assumption of a linear dependence of E_l on $l(l+1)$ in Eq. (9) is a reasonable concept for cases when the effective separation of the two colliding nuclei, R_0 , at the fusion barrier is insensitive to the change of the angular momentum l . While such an assumption is reasonable for most reactions, there are however cases, such as $^{12}\text{C} + ^{13}\text{C}$, in which such an assumption may not be valid. The method of barrier analysis we have just developed in Sec. II may be used to diagnose the pathological case. We shall discuss the collision of $^{12}\text{C} + ^{13}\text{C}$ in Sec. V.

By replacing the sum in Eq. (3) by an integral in the continuum approximation, the reaction cross section can be integrated to yield the analytical formula of Eq. (1) [5]. Such a replacement of the discrete l variable by a continuous variable incurs errors. It is desirable to find the magnitude of the errors and ways to correct for these errors, if a higher accuracy is required. For brevity of notation, we introduce a and g to rewrite f_l as

$$f_l = f(l) = \frac{2l+1}{1 + \exp\{[al(l+1) - \epsilon]/\Gamma\}} \equiv \frac{2l+1}{1+g}, \quad (16)$$

where $g = \exp\{[al(l+1) - \epsilon]/\Gamma\}$, $\epsilon = E - E_0$, and $a = \hbar^2/2\mu R_0^2$.

Our effort to examine the errors brings us to partition the contributions in Eq. (4) into two groups: (i) one group of l states for which the continuum approximation is a reasonable concept and analytical results can be readily obtained and (ii) another group of discrete l states which remain as they are, without applying the continuum approximation, and their contributions to the total reaction cross section can be subsequently singled out for scrutiny.

The $l=0$ state is important in sub-barrier fusion and it is not suitable for the continuum approximation. We shall keep f_0 to remain as it is in Eq. (4). We can express f_l with $l \geq 1$ as a continuous integral with a correction Δf_l :

$$f_l = \int_{l-1/2}^{l+1/2} dl f(l) + \Delta f_l. \quad (17)$$

The function $f(l)$ has an indefinite integral

$$\int dl f(l) = \frac{-\Gamma}{a} \ln \left\{ 1 + \exp \left[\frac{\epsilon - al(l+1)}{\Gamma} \right] \right\} \equiv F(l). \quad (18)$$

In terms of the function $F(l)$, we have

$$f_l = F(l+1/2) - F(l-1/2) + \Delta f_l. \quad (19)$$

By definition, the correction term Δf_l is then given by

$$\Delta f_l = f(l) - [F(l+1/2) - F(l-1/2)]. \quad (20)$$

Treating l as a continuous variable in the above equation and expanding the function $F(l \pm 1/2)$ about l in a Taylor series with $\Delta l = 1/2$, we obtain explicitly

$$\Delta f_l = -2 \sum_{n=2,4,\dots} \frac{(\Delta l)^{n+1}}{(n+1)!} \frac{d^n}{dl^n} f(l). \quad (21)$$

We thus obtain the central result that, in the continuum approximation, any term f_l with $l \geq 1$ in the series of Eq. (4) can be replaced by Eq. (19) of the known function $F(l)$, with Δf_l given by Eq. (21). For example, if we wish to partition the partial wave into those with $[0, l_L]$ as a discrete sum, with those in $[l_L + 1, \infty]$ in the continuum approximation, then we obtain for such a partition

$$\frac{\sigma_r}{\pi\lambda^2} = \sum_{l=0}^{l_L} f(l) - F(l_L + 1/2) - 2 \sum_{l=l_L+1, l_L+2, \dots} \left\{ \sum_{n=2,4,\dots} \frac{(\Delta l)^{n+1}}{(n+1)!} \frac{d^n}{dl^n} f(l) \right\}. \quad (22)$$

We can use the relation

$$\pi \lambda^2 \frac{\Gamma}{a} = \pi R_0^2 \frac{\Gamma}{E} \quad (23)$$

to write the reaction cross section as

$$\begin{aligned} \sigma_r = & \pi \lambda^2 (f_0 + f_1 + \dots + f_L) \\ & + \pi R_0^2 \frac{\Gamma}{E} \ln \left\{ 1 + \exp \left[\frac{\epsilon}{\Gamma} - \frac{(l_L + 1/2)(l_L + 3/2)a}{\Gamma} \right] \right\} \\ & - 2\pi \lambda^2 \sum_{l=l_L+1, l_L+2, \dots} \left\{ \sum_{n=2,4,\dots} \frac{(\Delta l)^{n+1}}{(n+1)!} \frac{d^n}{dl^n} f(l) \right\}. \end{aligned} \quad (24)$$

In the special partition by singling out only the lowest $l_L = 0$ wave for special consideration, then up to the third order $(\Delta l)^3$, we obtain

$$\begin{aligned} \sigma_r = & \frac{\pi \lambda^2}{1 + \exp\{\epsilon/\Gamma\}} + \pi R_0^2 \frac{\Gamma}{E} \ln \left\{ 1 + \exp \left(\frac{\epsilon}{\Gamma} - \frac{3a}{4\Gamma} \right) \right\} \\ & - \frac{\pi \lambda^2}{24} \sum_{l=1,2,3,\dots} \frac{d^2}{dl^2} f(l), \end{aligned} \quad (25)$$

where the derivative in the correction term is

$$\begin{aligned} \frac{d^2}{dl^2} f(l) = & 6(2l+1) \left(\frac{a}{\Gamma} \right) \left[- \frac{g}{(1+g)^2} \right] \\ & + (2l+1)^3 \left(\frac{a}{\Gamma} \right)^2 \left[- \frac{g}{(1+g)^2} + \frac{2g^2}{(1+g)^3} \right]. \end{aligned} \quad (26)$$

Terms on the right-hand side of Eq. (25) have direct physical meanings. The first term corresponds to the contribution from the lowest $l = 0$ partial wave, and the second term corresponds to the contribution from $l \geq 1$ partial waves in the continuum approximation, and the last term is the correction due to the continuum approximation up to the third order in $\Delta l = 1/2$.

The above considerations can be generalized. For the most general case [12],

$$\frac{\sigma_r}{\pi \lambda^2} = \sum_{l_\nu} [1 + \eta(-1)^{l_\nu}] f(l_\nu), \quad (27)$$

where (i) $\eta = 0$ and $l_\nu = \nu = 0, 1, 2, 3, \dots$ for unequal nuclei, (ii) $\eta = 1$ and $l_\nu = 2\nu = 0, 2, 4, \dots$ for identical spin-0 nuclei or for identical spin-1/2 nuclei with symmetric spatial and antisymmetric spin wave functions, and (iii) $\eta = -1$ and $l_\nu = 2\nu + 1 = 1, 3, 5, \dots$ for identical spin-1/2 nuclei with antisymmetric spatial and symmetric spin wave functions. The sum over l_ν can be converted into a sum over ν with $\nu = 0, 1, 2, \dots$. We obtain, up to the third order $(\Delta \nu)^3$ with $\Delta \nu = 1/2$,

$$\begin{aligned} \sigma_r = & \pi \lambda^2 \frac{dl_\nu}{d\nu} \frac{2l_0 + 1}{1 + \exp\{[a l_0(l_0 + 1) - \epsilon]/\Gamma\}} \\ & + \pi R_0^2 \frac{\Gamma}{E} \ln \left\{ 1 + \exp \left[\frac{\epsilon}{\Gamma} - \frac{l_{1/2}(l_{1/2} + 1)a}{\Gamma} \right] \right\} \\ & - 2\pi \lambda^2 \left(\frac{dl_\nu}{d\nu} \right)^3 \left(\sum_{\nu=1,2,3,\dots} \frac{(\Delta \nu)^3}{3!} \frac{d^2}{dl_\nu^2} f(l_\nu) \right). \end{aligned} \quad (28)$$

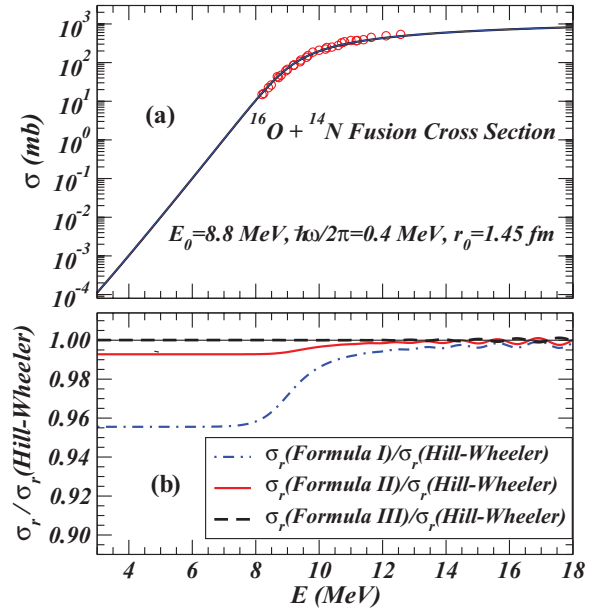


FIG. 1. (Color online) (a) The comparison of experimental fusion cross section for $^{16}\text{O} + ^{14}\text{N}$ [13] with theoretical results. (b) The ratio of the cross sections from Formulas I, II, and III relative to the Hill-Wheeler cross section.

IV. NUMERICAL RESULTS AND COMPARISON WITH DATA

In presenting our numerical results, we shall label Eq. (1) as Formula I, the sum of the first two terms in Eq. (25) [or (28)] as Formula II, and the sum of all three terms in Eq. (25) [or (28)] as Formula III. In simple physical terms, Formula I corresponds to the earlier result of Ref. [5] using the continuum approximation for all partial waves. Formula II is obtained by writing out the contribution from the lowest $l = 0$ partial wave explicitly and treating the higher $l \geq 1$ partial wave contributions in the continuum approximation. Formula III involves Formula II with the inclusion of corrections up to the third order in $\Delta l = 1/2$. Following Esbensen [3], we shall label the cross section obtained in the sum of Eq. (3) over the Hill-Wheeler penetration probability, under the assumption of Eqs. (9) and (10), as the Hill-Wheeler cross section.

We examine the sample case for the collision of $^{16}\text{O} + ^{14}\text{N}$, where the experimental data [13] are shown in Fig. 1(a). We show the fit to the fusion cross section obtained with $E_0 = 8.8$ MeV, $\Gamma = 0.40$ MeV ($\hbar\omega = 2.51$ MeV), and $r_0 = 1.45$ fm as curves in Fig. 1(a). The differences among the three formulas cannot be distinguished in the logarithmic plot. In order to see the differences, we plot the corresponding ratios of the cross sections relative to the Hill-Wheeler cross section in Fig. 1(b). In the sub-barrier region, we find that Formula I gives an error of about 4.5%, Formula II gives an error of less than 1%, and Formula III gives an error of less than 0.01%. In the high-energy region, all three formulas give small errors, of the order of at most 0.3%.

We conclude from the results of Fig. 1 that for unequal nuclei Formula I is adequate for the sub-barrier region if errors of 5% are permitted, Formula II gives a more accurate result

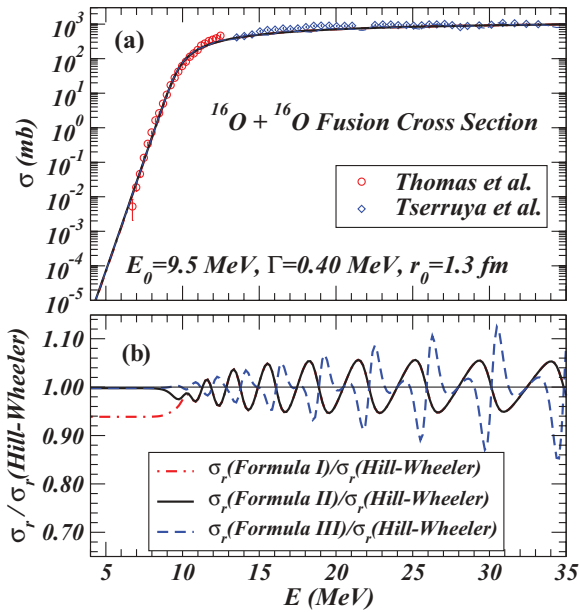


FIG. 2. (Color online) (a) Comparison of fusion cross section data for the collision of identical spin-0 nuclei $^{16}\text{O} + ^{16}\text{O}$ [14,15] with theoretical curves. (b) The ratio of cross sections from Formulas I, II, and III relative to the Hill-Wheeler cross section.

in all regions with less than 1% error, and Formula III gives even smaller errors in all regions.

In Fig. 2, we examine next the fusion cross section for the collision of $^{16}\text{O} + ^{16}\text{O}$, where the experimental data [14,15] are shown in Fig. 2(a) and the theoretical results from the three different formulas calculated with the parameters $E_0 = 9.5$ MeV, $\Gamma = 0.4$ MeV, and $r_0 = 1.3$ fm are shown as curves. In this case with identical spin-0 nuclei with spatially symmetric wave functions, only the even- l partial waves contribute to the reaction cross section. Formula II consists of the first two terms on the right-hand side of Eq. (28) and Formula III consists of all three terms in Eq. (28).

On the logarithmic scale of Fig. 2(a), the results from all three formulas cannot be well distinguished. The agreement of the experimental data with the theoretical curves may appear reasonable. However, in Fig. 2(b) we examine the ratio of the cross section obtained with the three different formulas relative to the Hill-Wheeler cross section. In the sub-barrier region, Formula I gives errors of order 6%, Formulas II and III give errors of less than 0.3%.

In the high-energy region, all three formulas give errors oscillating regularly about zero as a function of E . The magnitude of the oscillation is nearly constant for Formula II at high energies, but it increases as the energy increases for Formula III. These results are in agreement with the earlier observation of Esbensen [3], who noted that, as a result of the spatial symmetry of the wave function such that only even- l states contribute, the energy separation between the contributing l state and the $l + 2$ state increases as energy increases, and the total reaction cross section exhibits a stepwise increase when a high- l state enters into the formation of a fused system. As a consequence, the continuum approximation contains large and oscillating errors. In mathematical terms, the large error

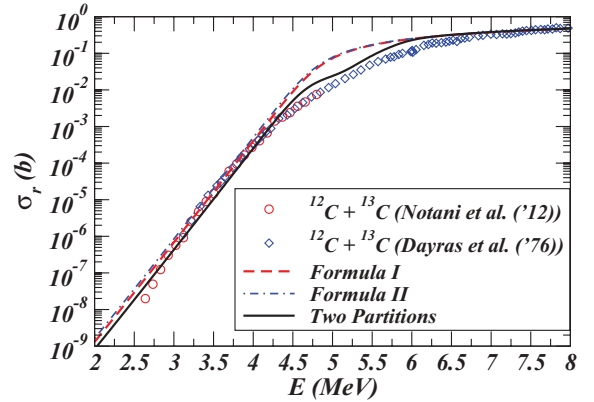


FIG. 3. (Color online) Comparison of fusion cross section data for $^{12}\text{C} + ^{13}\text{C}$ [8,9] with theoretical curves.

arises from the fact that even though the expansion parameter $\Delta\nu = 1/2$ is less than unity in Eq. (28), it is multiplied by the factor $dl_v/d\nu$ with $dl_v/d\nu = 2$. Thus the effective expansion parameter is $(\Delta\nu)(dl_v/d\nu) = 1$ and the expansion in Formula III does not properly converge.

We conclude from Fig. 2 that, for the collision of identical spin-0 nuclei at high energies, the continuum approximation incurs large errors. Formulas I and II give errors of about 5% while Formula III gives even greater errors up to 15%. On the other hand, near the sub-barrier region Formula II gives very small errors.

V. BARRIER ANALYSIS FOR $^{12}\text{C} + ^{13}\text{C}$

The results in the last few sections pertain to the collisions of both light and heavy nuclei. In the collision of heavy nuclei, however, there is the complication that the correction term $C(l, E_l)$ for low- l partial waves for the barrier analysis must be properly taken into account in an iterative procedure, as specified by Eqs. (14) and (15). In contrast, for light nuclei collisions, the width parameter Γ is found to be substantially smaller than the separation between adjacent barriers so that these correction terms can be neglected in the barrier analysis, leading to a great simplification of the problem. For simplicity, we shall therefore specialize to light nuclei collisions in subsequent sections.

Our ability to reach the simple results in the last sections relies on the assumption that the fusion barrier E_l for the l partial wave is a linear function of $l(l + 1)$, as given by Eq. (9). There may be nuclear collisions in which such an assumption may not be valid.

We examine $^{12}\text{C} + ^{13}\text{C}$, where the data are shown in Fig. 3. The data can be explained well by a coupled-channels calculation with the ingoing-wave boundary condition (IWBC) and the M3Y+repulsive potential [8]. Nevertheless, it is useful to examine these data from a complementary perspective in barrier-penetration points of view. One then finds that the $^{12}\text{C} + ^{13}\text{C}$ data [8,9] cannot be described by Formula I, II, or III. Any fit to the data near the threshold will miss the data at some other energy region. We show in Fig. 3 the results of Formulas I and II obtained with the parameters $E_0 = 4.7$ MeV,

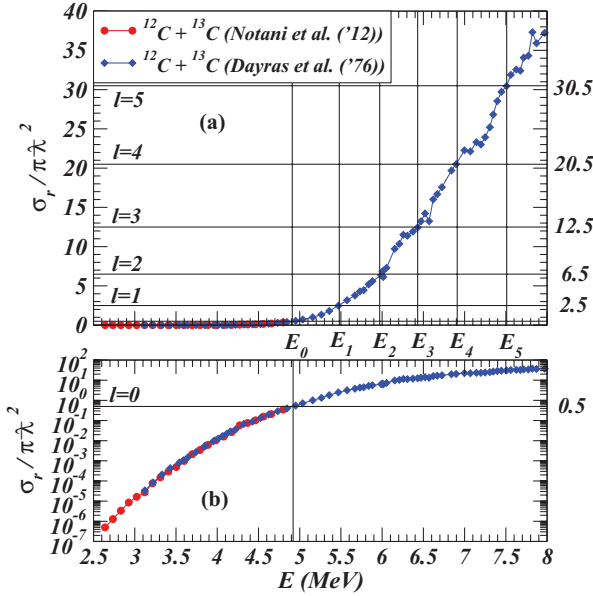


FIG. 4. (Color online) The dimensionless quantity $(\sigma_r/\pi\lambda^2)$ as a function of E , on a linear scale (a) and on a logarithmic scale (b), for the collision of $^{12}\text{C} + ^{13}\text{C}$. In the barrier analysis, the value of E at which $(\sigma_r/\pi\lambda^2)$ is $l(l+1) + 1/2$ is the fusion barrier E_l . The horizontal lines indicate the $l(l+1) + 1/2$ values shown as numbers given along the right vertical axis. The vertical lines indicate the positions of the fusion barriers E_l at which $\sigma_r/\pi\lambda^2 = l(l+1) + 1/2$. Data points are from [8,9].

$\Gamma = 0.15$ MeV, and $r_0 = 1.3$ fm as the dashed curve and the dash-dotted curve, respectively. The region around $E \sim 5$ MeV is not well reproduced. From the viewpoint of barrier penetration and the simple model with the assumption of $E_l = E_0 + al(l+1)$ in Eq. (9), the shape of the fusion cross section in the collision of $^{12}\text{C} + ^{13}\text{C}$ poses a problem.

To check whether the assumption of Eq. (9) is valid for $^{12}\text{C} + ^{13}\text{C}$, we can carry out a ‘‘barrier analysis’’ by plotting $\sigma_r/\pi\lambda^2$ as a function of E , as shown in Fig. 4. The plots in Fig. 4(a) are on a linear scale and those in Fig. 4(b) are on a logarithm scale. The rule in Eq. (14) stipulates that the barrier E_l is the value of energy E at which $\sigma_r/\pi\lambda^2$ is $l(l+1) + 1/2 + C(l, E_l)$. For light nuclei collision for which the width Γ is substantially smaller than the spacing between adjacent barriers, the correction term $C(l, E_l)$ is small and can be neglected. We plot $\sigma_r/\pi\lambda^2 = l(l+1) + 1/2$ as horizontal lines in Fig. 4. The energy values E where the horizontal lines meet the data points give the locations of the fusion barriers E_l in Fig. 4.

On plotting the barrier E_l obtained in the analysis of Fig. 4 as a function of $l(l+1)$, one observes in Fig. 5 that E_l is not a linear function of $l(l+1)$, as assumed in Eq. (9). While the linear relationship is reasonable for $l \gtrsim 2$, the systematics of E_l for $l \leq 2$ appears to be different from those with $l \gtrsim 2$.

To examine the problem of barrier penetration, we partition the partial waves into two parts in the intervals $[0, l_L]$ and $[l_L+1, \infty]$. We assume the Hill-Wheeler penetration probability for the partition in $[0, l_L]$ and describe the cross section from the partition $[l_L+1, \infty]$ by the continuum approximation.

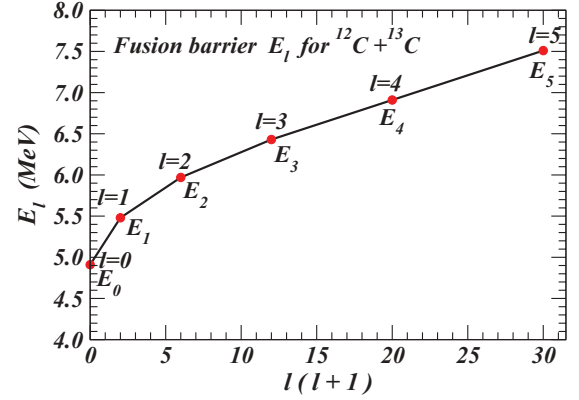


FIG. 5. (Color online) The fusion barrier E_l as a function of $l(l+1)$ in the collision of $^{12}\text{C} + ^{13}\text{C}$.

As given by Eq. (24) the reaction cross section with the neglect of the correction term and the assumption of $\Gamma_l = \Gamma$ is

$$\sigma_r = \pi\lambda^2 \sum_{l=0,1,\dots,l_L} \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma\}} + \pi R_0^2 \frac{\Gamma}{E} \ln \left\{ 1 + \exp \left[\frac{\epsilon}{\Gamma} - \frac{(l_L + 1/2)(l_L + 3/2)a}{\Gamma} \right] \right\}. \quad (29)$$

Figure 5 indicates that the dependence of the fusion barriers E_l on $l(l+1)$ deviates from a linear relationship, for the first three partial waves with $l = 0, 1$, and 2 . Consequently, we partition the partial waves into two partitions of $[0, 2]$ and $[3, \infty]$ with $l_L = 2$. The values of E_0, E_1 , and E_2 can be read off from Fig. 5 as the starting point for parameter search, with minor fine tuning. The results with $E_0 = 4.7$ MeV, $E_1 = 5.4$ MeV, $E_2 = 5.85$ MeV, $\Gamma = 0.15$ MeV, and $r_0 = 1.3$ fm are shown as the solid curve in Fig. 3. We observe that, although the fit of Eq. (29) to the experimental data is not perfect, the agreement with experimental data around $E \sim 5$ MeV is substantially improved. The simple comparison indicates that a possible solution of the peculiar shape of the fusion cross section may involve fusion barriers increasing in a nonlinear way as a function of $l(l+1)$, corresponding to a fusion radial distance occurring at a much reduced separation for the lowest partial waves in $^{12}\text{C} + ^{13}\text{C}$ collisions. This may be related to the need for a repulsive core in the interaction and the mutual excitation between the colliding nuclei, as shown in [8].

VI. BARRIER ANALYSIS FOR THE COLLISION OF IDENTICAL LIGHT NUCLEI

The results in the last section illustrate the application of the rule for barrier analysis in the collision of unequal nuclei. The reaction cross section for the collision of identical nuclei will need to obey the symmetry of the total wave function with respect to the interchange of the colliding nuclei. As a consequence, the barrier height analysis rule will be modified for the collisions of identical nuclei.

For the collision of identical spin-0 nuclei, the quantity $(\sigma_r/\pi\lambda^2)$ is given by

$$(\sigma_r/\pi\lambda^2) = 2 \sum_{l=0,2,4,\dots} \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma_l\}}. \quad (30)$$

For light nuclei collisions, $|E_l - E_{l\pm 1}| \gg \Gamma_l$ and we can evaluate the above quantity $(\sigma_r/\pi\lambda^2)$ at the fusion barrier E_l analytically. For the collision of identical spin-0 nuclei, the quantity $(\sigma_r/\pi\lambda^2)$ at the fusion barrier E_l (with even l value) is

$$(\sigma_r/\pi\lambda^2)|_{E_l} = l(l+1) + 1. \quad (31)$$

For the collision of identical spin-1/2 nuclei, the total spin can be $S = 0$ or $S = 1$, with a weight of 1/4 and 3/4, respectively. As a consequence, the reaction cross section is given by

$$\begin{aligned} (\sigma_r/\pi\lambda^2) = 2 \times \frac{1}{4} \sum_{l=0,2,4,\dots} \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma_l\}} \\ + 2 \times \frac{3}{4} \sum_{l=1,3,5,7,\dots} \frac{2l+1}{1 + \exp\{(E_l - E)/\Gamma_l\}}. \end{aligned} \quad (32)$$

The barrier analysis rule is different for E_l with even- l or odd- l partial waves. Assuming $|E_l - E_j| \gg \Gamma_l$ for $l \neq j$ for light nuclei collisions, we find that, for the collision of identical spin-1/2 nuclei, the quantity $(\sigma_r/\pi\lambda^2)$ at the barrier E_l with even l is given by

$$(\sigma_r/\pi\lambda^2)|_{E_l} = l(l+1) + \frac{1}{4}, \quad (33)$$

and the quantity $\sigma_r/\pi\lambda^2$ at the barrier E_l with odd l is given by

$$(\sigma_r/\pi\lambda^2)|_{E_l} = l(l+1) + \frac{3}{4}. \quad (34)$$

These equations can be utilized to determine the fusion barriers from experimental $(\sigma_r/\pi\lambda^2)$ data for the collision of identical or nonidentical light nuclei.

We can summarize the rules for the barrier analysis as follows. The dimensionless measure of the reaction cross section $(\sigma_r/\pi\lambda^2)$ at the fusion barrier E_l is

$$(\sigma_r/\pi\lambda^2)|_{E_l} = l(l+1) + K, \quad (35)$$

where K is given by

$$K = \begin{cases} \frac{1}{2}, & \text{nonidentical nuclei,} \\ 1, & \text{identical spin-0 nuclei,} \\ \frac{1}{4}, & \text{even-}l, \text{ identical spin-}\frac{1}{2} \text{ nuclei,} \\ \frac{3}{4}, & \text{odd-}l, \text{ identical spin-}\frac{1}{2} \text{ nuclei.} \end{cases} \quad (36)$$

The differences of $(\sigma_r/\pi\lambda^2)|_{E_l}$ in the different cases are large for the lowest $l = 0$ partial wave. The differences of $(\sigma_r/\pi\lambda^2)|_{E_l}$ in the different cases are small, in comparison with the first term $l(l+1)$, when l is large.

VII. PENETRATION PROBABILITY ANALYSIS AND RESONANCES FOR LIGHT NUCLEI COLLISIONS

The penetration probability $P_l(E)$ for the l partial wave is a physical quantity that reveals important information on the

dynamics of the fusion process. The energy E at which $P_l(E)$ is 1/2 is at the top of the fusion barrier, and the shape of the potential barrier is governed by the shape and the energy dependence of $P_l(E)$. It is desirable to extract such a quantity from experimental data for light nuclei collisions for which the width Γ is substantially smaller than the separation between adjacent barriers.

A. Collision of unequal light nuclei

We shall consider first the collision of unequal light nuclei and express the dimensionless cross section $(\sigma_r/\pi\lambda^2)$ in terms of the penetration probability $P_l(E)$ as

$$\frac{\sigma_r}{\pi\lambda^2} = \sum_{l=0,1,2,3,\dots} (2l+1)P_l(E). \quad (37)$$

The dimensionless cross section $(\sigma_r/\pi\lambda^2)$ appears so frequently that it is appropriate to abbreviate it by $\Sigma(E)$, which is explicitly a function of the energy E .

The penetration probability $P_l(E)$ can be extracted from the dimensionless cross section $\Sigma(E) = (\sigma_r/\pi\lambda^2)$ if we assume that the contributions of different partial waves to the dimensionless cross section are well separated in energy as in light nuclei collisions. Under such an assumption, we can consider the contributions to $P_l(E)$ from different partial waves. In the domain of E in which $P_l(E)$ is significant, the contribution from each of the lower $l' < l$ partial waves is saturated to $P_{l'}(E) = 1$ while the contribution from each of the higher $l' > l$ partial waves is negligible. We can decompose the sum over l in Eq. (37) into individual contributions. For the l partial wave in the collision of unequal nuclei, the penetration probability is then given by

$$P_l(E) = \frac{\Sigma(E) - B(l)}{2l+1} \Theta[T(l) - \Sigma(E)] \Theta[\Sigma(E) - B(l)], \quad (38)$$

where Θ is the step function, $T(l)$ is the top delimiter of $\Sigma(E)$, and $B(l)$ is the bottom delimiter of $\Sigma(E)$. For unequal nuclei collisions, the sum of l is over $l = 0, 1, 2, 3, \dots$, and the delimiters can be shown to be

$$T(l) = (l+1)^2, \quad (39)$$

$$B(0) = 0, \quad \text{and} \quad B(l) = T(l-1) \quad \text{for} \quad l \geq 1. \quad (40)$$

If $\Sigma(E) = (\sigma_r/\pi\lambda^2)$ is measured experimentally as a function of E , the penetration probability $P_l(E)$ for different partial waves can be determined.

In Fig. 6 we show the penetration probability $P_l(E)$ as a function of E for various l partial waves in the collision of $^{12}\text{C} + ^{13}\text{C}$, obtained by using Eq. (38) and data from [8,9]. For a given l , it is possible to determine the fusion barrier E_l as the energy at which $P_l(E) = 0.5$, as discussed in an equivalent procedure in Secs. II and V. We can also extract an empirical width Γ_l , where $2\Gamma_l$ is defined as the separation of E between $P_l(E) = 1/(1+e^{-1}) = 0.731$ and $P_l(E) = 1/(1+e) = 0.269$. This empirical Γ_l would be the same as the Γ_l in the Hill-Wheeler formula, if the penetration probability follows the Hill-Wheeler formula.

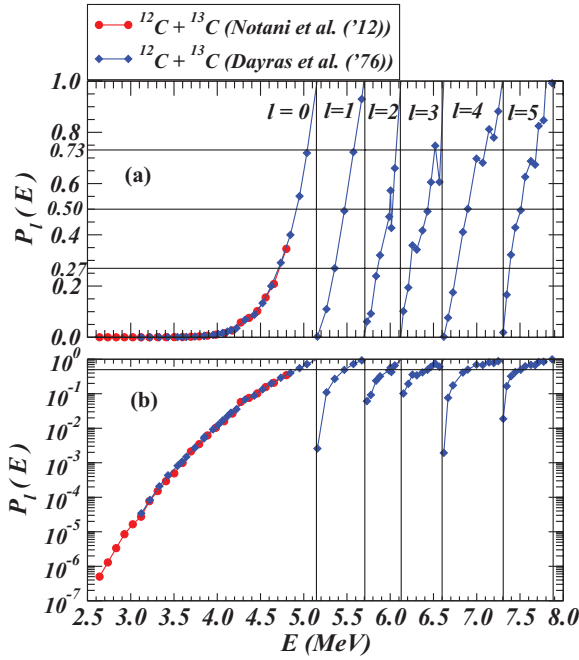


FIG. 6. (Color online) The penetration probability $P_l(E)$ as a function of E on a linear scale (a) and on a logarithmic scale (b) for different partial waves l extracted from the data of [8,9] for the collision of $^{12}\text{C} + ^{13}\text{C}$.

In Table II, we list the fusion barrier E_l and the width Γ_l extracted from $P_l(E)$ in such a procedure for $^{12}\text{C} + ^{13}\text{C}$. As one observes, the widths for most of the partial waves are about equal to 0.15 MeV except for the $l = 1$ and 2 partial waves, which are about 0.11–0.12 MeV.

B. Collision of identical spin-0 nuclei

We shall consider next the collision of two identical spin-0 nuclei. The dimensionless cross section ($\sigma_r/\pi\lambda^2$) in terms of the penetration probability $P_l(E)$ is

$$\frac{\sigma_r}{\pi\lambda^2} = 2 \sum_{l=0,2,4,6,\dots} (2l+1)P_l(E). \quad (41)$$

Under the assumption that $|E_l - E_j| \gg \Gamma_l$ for $l \neq j$ for light nuclei collisions, the contributions of different partial waves to the dimensionless cross section are well separated in energy. For the even- l partial wave in the collision of equal spin-0 light nuclei, the penetration probability is given by

$$P_l(E) = \frac{\Sigma(E) - B(l)}{2(2l+1)} \Theta[T(l) - \Sigma(E)] \Theta[\Sigma(E) - B(l)], \quad (42)$$

TABLE II. The empirical values of E_l and Γ_l from $P_l(E)$, as extracted from the data of [16] for the collision of $^{12}\text{C} + ^{13}\text{C}$.

l	0	1	2	3	4	5
E_l (MeV)	4.91	5.48	5.97	6.43	6.91	7.51
Γ_l (MeV)	0.16	0.11	0.12	~0.16	~0.15	~0.16

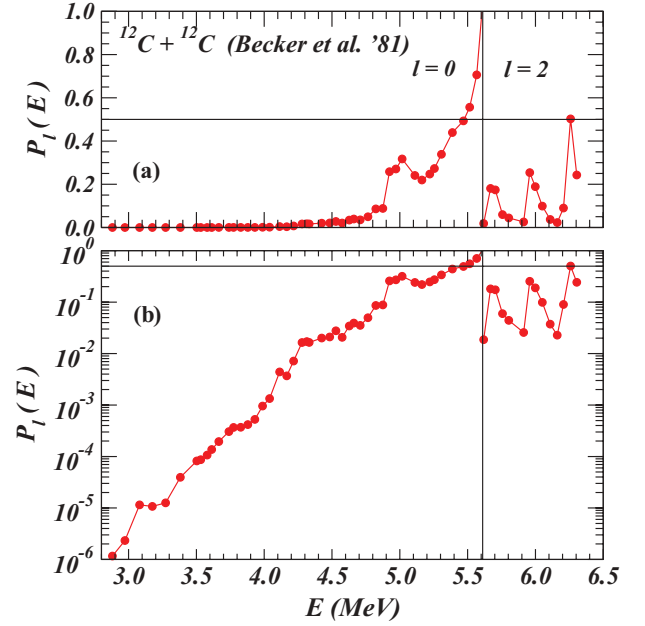


FIG. 7. (Color online) The penetration probability $P_l(E)$ as a function of E on a linear scale (a) and a logarithmic scale (b) extracted from the $^{12}\text{C} + ^{12}\text{C}$ data of [16].

where we find

$$T(l) = l(l+1) + 2l + 2, \quad (43)$$

$$B(0) = 0, \quad \text{and} \quad B(l) = T(l-2) \quad \text{for} \quad l \geq 2. \quad (44)$$

The penetration probability $P_l(E)$ extracted from the experimental $^{12}\text{C} + ^{12}\text{C}$ data [16] and Eq. (42) is shown on a linear scale in Fig. 7(a) and on a logarithmic scale in Fig. 7(b). One finds the fusion barrier for the $l = 0$ partial wave, E_0 , at 5.46 MeV, at which $P_{l=0}(E) = 0.5$. The boundary between the $l = 0$ and $l = 2$ partial waves is approximately at $E = 5.6$ MeV. One observes that $P_l(E)$ exhibits resonances. The resonances below $E = 5.6$ MeV are most likely $l = 0$ resonances whereas those resonances above $E = 5.6$ MeV are most likely $l = 2$ resonances.

C. Collision of identical spin-1/2 nuclei

We shall consider next the collision of two identical spin-1/2 nuclei. The dimensionless cross section ($\sigma_r/\pi\lambda^2$) written in terms of the penetration probability $P_l(E)$ is

$$\frac{\sigma_r}{\pi\lambda^2} = 2 \left[\frac{1}{4} \sum_{l=0,2,4,6,\dots} + \frac{3}{4} \sum_{l=1,3,5,7,\dots} \right] (2l+1)P_l(E). \quad (45)$$

Under the assumption that $|E_l - E_j| \gg \Gamma_l$ for $l \neq j$ for light nuclei collisions, the contributions of different partial waves to the dimensionless cross section are well separated in energy. In the collision of identical spin-1/2 nuclei, the penetration probability for the even- l partial wave is given by

$$P_l(E) = \frac{\Sigma(E) - B(l)}{(2l+1)/2} \Theta[T(l) - \Sigma(E)] \Theta[\Sigma(E) - B(l)], \quad (46)$$

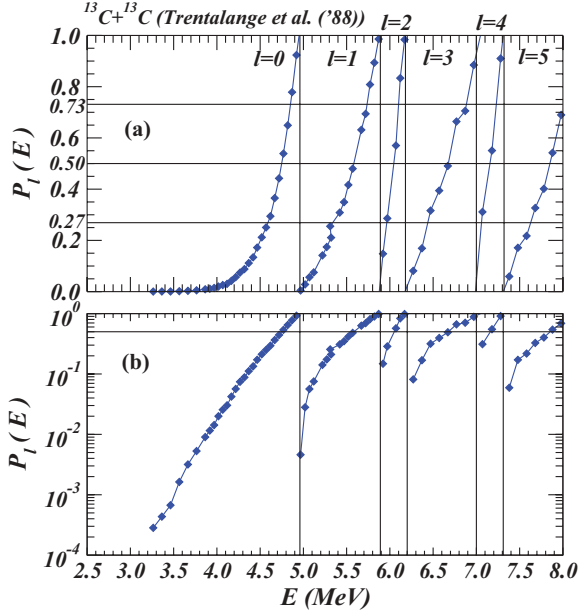


FIG. 8. (Color online) The penetration probability $P_l(E)$ as a function of E on a linear scale (a) and a logarithmic scale (b) extracted from the $^{13}\text{C} + ^{13}\text{C}$ data of [17].

where $T(l)$ with even l is given by

$$T(l) = l(l+1) + (l+1)/2, \quad (47)$$

and the function $B(l)$ is given by

$$B(0) = 0, \quad \text{and} \quad B(l) = T(l-1) \quad \text{for} \quad l \geq 1. \quad (48)$$

In the collision of identical spin-1/2 nuclei, the penetration probability for the odd- l partial wave is

$$P_l(E) = \frac{\Sigma(E) - B(l)}{3(2l+1)/2} \Theta[T(l) - \Sigma(E)] \Theta[\Sigma(E) - B(l)], \quad (49)$$

where $T(l)$ with odd l is

$$T(l) = l(l+1) + (3l+3)/2, \quad (50)$$

and the function $B(l)$ is given again by

$$B(0) = 0, \quad \text{and} \quad B(l) = T(l-1) \quad \text{for} \quad l \geq 1. \quad (51)$$

Using Eq. (46) or (49) and the $^{13}\text{C} + ^{13}\text{C}$ data from [17], we extract the penetration probability $P_l(E)$ as a function of E for various l partial waves. The results are shown on a linear scale in Fig. 8(a) and on a logarithmic scale in Fig. 8(b). In Table III, we list the fusion barrier E_l and the width Γ_l extracted from $P_l(E)$ in such a procedure for $^{13}\text{C} + ^{13}\text{C}$. As one observes, the widths for the even- l states and the widths from the odd- l states appear to fall into two different groups, with Γ for the even- l

TABLE III. The empirical values of E_l and Γ_l from $P_l(E)$, as extracted from the data of [17] for the collision of $^{13}\text{C} + ^{13}\text{C}$.

l	0	1	2	3	4	5
E_l (MeV)	4.70	5.59	6.04	6.68	7.16	7.86
Γ_l (MeV)	0.12	0.19	0.07	0.22	0.10	0.20

states in the 0.7–0.12 MeV range, while the width parameters Γ for the odd- l states lie in the 0.19–0.22 MeV range. There seems to be strong dependence on the even or odd property of the angular momentum l of the fused system.

It should be noted that, in deriving the barrier and penetration probability rules, we have made the assumption that $|E_l - E_j| \gg \Gamma_l$ for $l \neq j$. The results in Table II and III for E_l and Γ_l indicate that such an assumption is substantially valid and is a reasonable and approximate idealization for light nuclei collisions. The extracted barrier height and penetration probabilities are approximate quantities that reveal the gross features of the fusion process.

VIII. CONCLUSIONS AND DISCUSSION

By treating the angular momentum as a continuous variable, the reaction cross section can be evaluated in a simple analytical form. The continuum approximation of the discrete angular momentum variable carries errors, and these errors can be evaluated and amended to previous results.

Three different formulas have been presented in the present formulations. Formula I corresponds to the earlier result of Ref. [5] using the continuum approximation for all partial waves. Formula II is obtained by writing out the contribution from the lowest $l=0$ partial wave explicitly and treating the higher $l \geq 1$ partial wave contributions in the continuum approximation. Formula III involves Formula II with the inclusion of corrections up to the second order in $\Delta l = 1/2$.

For the collision of unequal nuclei, the better formula is Formula II, which incurs errors of order 0.7% in the sub-barrier regions and errors of order 0.2% at high energies. The simpler Formula I incurs errors of about 4.4% in the sub-barrier region and errors of about 0.4% at high energies. Higher order corrections in Formula III can be used if high accuracy is desired, with errors of about 0.005% in the sub-barrier region and errors of about 0.12% at high energies.

For the collision of identical spin-0 nuclei, the application of these formulas incurs substantial errors. The best formula for identical spin-0 nuclei is Formula II, which incurs errors of about 0.2% in the sub-barrier regions and errors of about 5.0% at high energies. On the other hand, the simpler Formula I incurs errors of about 6.0% in the sub-barrier region and errors of about 5.5% at high energies.

Simple rules have been presented to determine the barriers E_l and the penetration probabilities $P_l(E)$ for different l partial waves from experimental data for the collision of identical or nonidentical light nuclei. The direct determination of the physical quantities as a function of l gives new insight into the fusion process. The barrier analysis rule has been successfully applied to examine the relation between the fusion barrier and l for the pathological case of $^{12}\text{C} + ^{13}\text{C}$. The application of the penetration probability analysis reveals quantitatively the resonance structure in $^{12}\text{C} + ^{12}\text{C}$ collisions.

We note that the partitioning of the partial waves into the lowest l region and the higher l region has some advantages in phenomenology. There are situations in which the properties of the potential barriers for the lowest l states may deviate from the systematics of those for the higher l states. These

lowest l states may need to be specially handled. One may provide a different description of the penetration probabilities for the lowest partial waves, with contributions from higher l partial waves represented analytically in the continuum approximation. By this partition, the new degrees of freedom, if any, can be incorporated into the penetration probability to provide a clearer picture of the dynamics of the fusion process.

For simplicity, we have carried out the barrier analysis and the penetration probability analysis for light nuclei collisions. For collision with heavy nuclei, however, Γ is not small compared to adjacent barrier separations $|E_l - E_{l\pm 1}|$. The barrier analysis for low- l partial waves needs to be carried out iteratively. While analytical expressions have been obtained to

carry out such an iterative procedure, whether such a barrier analysis for heavy-nuclei collisions may be practical remains to be investigated.

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