# Description of two-particle transfer in superfluid systems

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Exact results of pair transfer probabilities for the Richardson model with equidistant or random level spacing are presented. The results are then compared either to particle-particle random-phase approximation (ppRPA) in the normal phase or quasiparticle random-phase approximation (QRPA) in the superfluid phase. We show that both ppRPA and QRPA are globally well reproducing the exact case although some differences are seen in the superfluid case. In particular, the QRPA overestimates the pair transfer probabilities to excited states in the vicinity of the normal-superfluid phase transition, which might explain the difficulty in detecting collective pairing phenomena as, for example, the giant pairing vibration. The shortcoming of QRPA can be traced back to the breaking of particle number that is used to incorporate pairing. A method based on direct diagonalization of the Hamiltonian in the space of two quasiparticles projected onto good particle number is shown to improve the description of pair transfer probabilities in superfluid systems.

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## I. INTRODUCTION

The importance of pairing correlations in nuclear systems has been established in several aspects: binding energies of nuclei, odd-even effects, superfluid phenomena, and pair transfer mechanisms, to mention just a few. However, despite the fact that pairing is anticipated to play a significant role in the pair transfer process, the existence of collective effects, such as giant pair vibration (GPV) [1–6], leading to an increase in the pair transfer from a superfluid nuclei, still challenges the experimental nuclear physics [7]. On the theoretical side, mean-field methods based on Hartree-Fock-Bogolyubov (HFB) sometimes augmented by quasiparticle random phase approximation (QRPA) have been used to predict pair transfer probabilities either from ground state to ground state or from ground state to excited states [8–12]. A common conclusion of most of these studies is the sensitivity of one- or two-nucleon transfer process to the internal topology of pairing in nuclei. In the present work, exact results of pair transfer probability are obtained for the Richardson model [13], consisting of a set of single-particle levels interacting through a pure pairing interaction. This model can be seen as the valence space of the last occupied level in nuclei where nucleons can be either added (pickup reactions) or removed (stripping reactions). The exact solution of the pair transfer in the Richardson model has been obtained in Ref. [14]. The possibility to perform exact calculations opens new perspectives to understand the pair transfer process and provides a benchmark for approximate treatments. In the following, we first discuss the pair transfer mechanism from a general point of view and estimate pair transfer probabilities in the pairing model.

We are interested here in a process where two particles are either added or removed in a system that is initially in its ground state formed of $N$ particles. In the following, we denote by $|\nu, A\rangle$, respectively, the eigenstates of the systems with $A$ particles associated to the set of energies $E_\nu^A$ and, by convention, $\nu = 0$ is taken for ground state. During its evolution, the system wave function can be decomposed as [15]

$$
\begin{aligned}
|\Psi(t)\rangle = e^{-itE_0^N/\hbar} \Bigg\{ & \sum_\nu c_\nu^N e^{-it(E_\nu^N - E_0^N)/\hbar} |\nu, N\rangle \\
& + \sum_\nu c_\nu^{N-2} e^{-it(E_\nu^{N-2} - E_0^N)/\hbar} |\nu, N-2\rangle \\
& + \sum_\nu c_\nu^{N+2} e^{-it(E_\nu^{N+2} - E_0^N)/\hbar} |\nu, N+2\rangle \Bigg\},
\end{aligned}
$$

where the first line describes the possibility that the system remains in its ground state or in one of its excited states without changing its particle number. The second (third) line contains the information on the removal and/or addition process. The explicit form of the coefficients $c_\nu^A$ depends on the physical process under interest, such as the stripping or pickup reactions in nuclear physics. On the theoretical side, information of these processes can be obtained by studying the small-amplitude response of the system to an external field $\hat{T}$ that changes the particle number by two units. Then, information on the transfer reduces to the knowledge of the response function, given by

$$
\begin{aligned}
S(E) &= \sum_\nu |\langle N+2, \nu|\hat{T}|N, 0\rangle|^2 \delta\left(E - \Delta E_\nu^{N+2}\right) \\
&\quad + \sum_\nu |\langle N-2, \nu|\hat{T}|N, 0\rangle|^2 \delta\left(E - \Delta E_\nu^{N-2}\right) \\
&\equiv S^{\mathrm{Add}}(E) + S^{\mathrm{Rem}}(E), \qquad (1)
\end{aligned}
$$

where $\Delta E_\nu^{N\pm2} = E_\nu^{N\pm2} - E_0^N$. In the following, $S^{\mathrm{Add}}(E)$ ($S^{\mathrm{Rem}}(E)$) will be referred to addition (removal) strength function. A common choice of $\hat{T}$ [6] to excite pairing modes is

$$
\hat{T} = \sum_i (T_{i\bar{i}} a_i^\dagger a_{\bar{i}}^\dagger + T_{i\bar{i}}^* a_{\bar{i}} a_i), \qquad (2)
$$

where $a_i^\dagger a_{\bar{i}}^\dagger$ corresponds to creation operators of a pair of time-reversed single-particle states. In the present article, we are interested in a physical process where two particles are either added or removed. In that case, it is more suitable to consider directly the non-Hermitian addition (removal)

 

transition operator, denoted by $\hat{T}^{\rm Add}$ ($\hat{T}^{\rm Rem}$) defined through

$$\hat{T}^{\rm Add} = \sum_i T_{i\bar{i}} a_i^\dagger a_{\bar{i}}^\dagger, \hat{T}^{\rm Rem} = \sum_i T_{i\bar{i}}^* a_{\bar{i}} a_i, \qquad (3)$$

respectively, associated to $S^{\rm Add}(E)$ and $S^{\rm Rem}(E)$. From the expression of the strength, we see that the understanding of the two-particle transfer passes through a good knowledge of the spectroscopy of initial and final states as well as of the capacity to provide the addition or removal probabilities defined here as

$$P_\nu^{\rm Add} = |\langle N+2, \nu|\hat{T}|N, 0\rangle|^2, \qquad (4)$$

$$P_\nu^{\rm Rem} = |\langle N-2, \nu|\hat{T}|N, 0\rangle|^2. \qquad (5)$$

If the many-body problem can be solved exactly, such quantities as well as the exact eigenvalues of the Hamiltonian can be used to have a precise estimate of the strength function (1). In most realistic situations, such treatment is impossible and an educated guess should be used. In the present work, we are interested in an initial system where pairing correlation plays a role. This case has been first considered in Refs. [1,16,17], leading to the concept of pairing vibration, that is, a coherent excitation of pairs of particles that is expected to show up in the enhancement of pair transfer probabilities. The standard way to incorporate pairing correlation is to use the BCS or HFB approach as a starting point [18,19]. This technique is indeed standardly used in nuclear physics to estimate the transfer either from ground state to ground state [9,12,20] or from ground state to excited states. In the latter case, the response is obtained using QRPA [4,5,11] or its time-dependent version [6]. For a comprehensive introduction, please refer to Ref. [21].

With the increase of computational powers, it is possible nowadays to study exactly pair transfer in schematic model that approaches realistic situations and to quantify the predictive power of mean-field based approaches. In the present work, we first present exact results of pair transfer probabilities for the Richardson model with equidistant or random level spacing. The exact results are then used to benchmark standard approaches, namely ppRPA and QRPA.

## II. EXACT DESCRIPTION IN THE RICHARDSON MODEL

A system of $\Omega$ doubly degenerated single-particle levels interacting via a pairing force with parameter $G$ is considered. The Hamiltonian is given by [13]

$$H = \sum_{i=1}^{\Omega} \epsilon_i \hat{N}_i - G \sum_{i,j=1}^{\Omega} \hat{P}_i^\dagger \hat{P}_j, \qquad (6)$$

where the particle-number operator $\hat{N}_i$ and pair creation/annihilation operators $\hat{P}_i^\dagger$, $\hat{P}_i$ are given by

$$\hat{N}_i = a_i^\dagger a_i + a_{\bar{i}}^\dagger a_{\bar{i}}, \quad \hat{P}_i^\dagger = a_i^\dagger a_{\bar{i}}^\dagger, \quad \hat{P}_i = (\hat{P}_i^\dagger)^\dagger. \qquad (7)$$

For not-too-large model space $\Omega$, exact solutions can be obtained using standard diagonalization techniques in subspace of given seniority [22]. As a test case here, a system of $N = 8$ particles with $\Omega = 10$ doubly degenerated levels is considered here. Assuming a simple transition operator (2) with $T_{i\bar{i}} = 1$ for all pairs, illustrations of addition strength function obtained for the Richardson model are shown in Fig. 1, in solid (red) lines, for equidistant level spacing, that is, $\epsilon_i = i\Delta\varepsilon$ ($i = 1, \Omega$), for different values of the pairing interaction and $\Delta\varepsilon = 1$ MeV. In the following the excitation energies are calculated with respect to the ground state of the system with $N = 10$ particles consistently in all the theories. The presented result is exact in the sense that the exact eigenvalues and eigenstates of the system with $N = 8$ and $N = 10$ have been used to compute Eq. (1). Note that in the present work, we are mainly interested in the transition from ground state to excited states of the $N + 2$ nucleus and the contribution of the ground state to ground state transfer has been omitted in the figure. Moreover, to make simpler the comparison between different results, we have folded the discrete spectra with a Lorentzian function with a width of 1 MeV. For completeness, we also show in Fig. 2 similar results obtained with a randomly spaced system whose single-particle energies are 0.551, 2.176, 4.033, 5.142, 6.444, 7.029, 7.827, 8.343, 9.226, and 9.571 in MeV units. This situation could be considered closer to realistic cases.

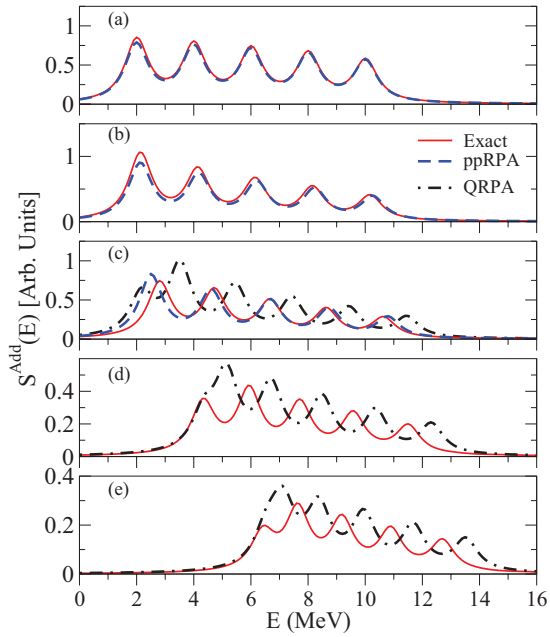


FIG. 1. (Color online) Addition strength function as a function of the excitation energy $E$ for different pairing strength G obtained using exact eigenvalues and eigenstates in Eq. (1) (red solid line) for a system of $N = 8$ particles to $N = 10$ in the case of $\Omega = 10$ equally spaced levels. From top to bottom: (a) $G/\Delta\varepsilon = 0.1$, (b) 0.3, (c) 0.5, (d) 0.7, and (e) 0.9 are shown. The results of the ppRPA (blue dashed line) and QRPA (black dot-dashed line) are also presented. Note that in panel (c), the result of ppRPA obtained at the collapse point $G = 0.48\Delta\varepsilon$ is also reported. The QRPA are shown only above the BCS threshold $G = 0.33\Delta\varepsilon$.

In the exact calculations, we observe a shift of the excitation spectrum toward higher energies as the pairing strength $G$ increases while the pair transfer probabilities of the excited states get smaller. On the contrary, as it is shown below that the pair transfer probability from ground state to ground state increases as $G$ increases.

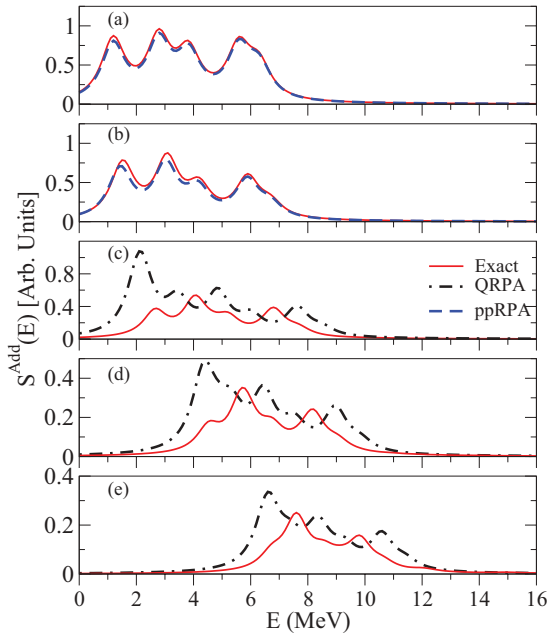


FIG. 2. (Color online) Same as Fig. 1 obtained for randomly spaced single-particle energies. See the text for more details. The QRPA are shown only above the BCS threshold $G = 0.44$ MeV.

## III. PPRPA VS QRPA APPROACHES TO PAIR TRANSFER

As already mentioned, in most cases, exact evaluation of the pair transfer probabilities cannot be performed and approximations for the many-body states are necessary. The most common strategy used in nuclear physics is to first apply the HFB or BCS theory and minimize the energy in the Hilbert space of quasiparticle vacuum imposing a mean particle number equal to $N$. This leads to an approximation for $|0, N\rangle$. Using standard notations, the quasiparticle vacuum is given by

$$|0, N\rangle \simeq |0, \mathrm{QP}\rangle = \prod_{i>0} [\alpha_i \alpha_{\bar{i}}]|-\rangle, \tag{8}$$

where $|-\rangle$ is the bare vacuum and $|0, \mathrm{QP}\rangle$ is the vacuum of the quasiparticle annihilation operators $\alpha_i$ defined through the Bogolyubov transformation:

$$\alpha_i = U_i a_i - V_i a_{\bar{i}}^\dagger, \tag{9}$$

$$\alpha_{\bar{i}} = U_i a_{\bar{i}} + V_i a_i^\dagger. \tag{10}$$

In the HFB approach, the above transformation automatically implies that the single-particle basis identifies with the canonical basis.

In the present model, the Hamiltonian (6) is already written in the canonical basis for the HFB theory. This theory is particularly suitable to provide estimates for ground state (GS) to GS transfer probabilities [9,12]. Because we want to compare with exact results, at the BCS/HFB and QRPA level the contribution to the particle-hole channel of the pairing interaction is taken into account. The result of the HFB theory for this probability is shown in Fig. 3 and compared to the exact solution. Note that below the pairing threshold (denoted by $G_{\mathrm{cr}}$ and equal to 0.33 and 0.44 MeV for the eight-particle system

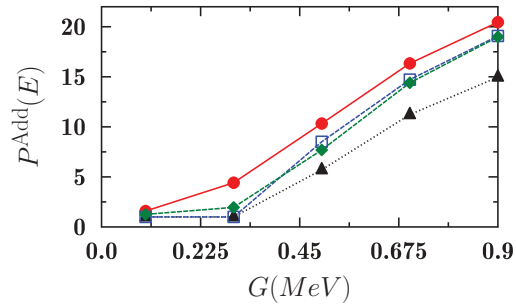


FIG. 3. (Color online) Illustration of the application of mean-field theory to provide estimate of GS to GS addition pair probability. The exact result (red circles), BCS (black triangles), and BCS projected on good particle number (blue open squares) are shown here for the equidistant single-particle level case. The result of the P-QTDA$^{(\mathrm{GS})}$ is also presented with green solid diamonds.

in the equally spaced and random spaced case, respectively), the HFB reduces to HF and probability is 1. As illustrated in this figure, while the HF theory (not shown) would have failed to reproduce the exact probabilities, the HFB framework gives estimations that are already rather close to the exact ones in the superfluid regime.

Owing to the absence of residual coupling between quasiparticle excitations, it is known that HFB alone cannot properly describe excited states. Then, linear response theory including possible particle-particle (pp), hole-hole (hh), or particle-hole (ph) excitations is applied to describe excited states, and then transfer probabilities, within the QRPA approach. In QRPA, the excited states, denoted by $|\nu\rangle$, are obtained by considering coherent superposition of two quasiparticle (2QP) excitations. This leads to

$$|\nu\rangle = Q_\nu^\dagger |0\rangle, \tag{11}$$

where $Q_\nu^\dagger$ are QRPA phonons written as

$$Q_\nu^\dagger = \sum_i \left( X_j^\nu \alpha_i^\dagger \alpha_{\bar{i}}^\dagger - Y_j^\nu \alpha_{\bar{i}} \alpha_i \right), \quad Q_\nu = (Q_\nu^\dagger)^\dagger, \tag{12}$$

while $|0\rangle$ is the phonon vacuum, defined through the conditions $Q_\nu|0\rangle = 0$. In practice, the components $X^\nu$ and $Y^\nu$, as well as the energies of the excited states $\omega_\nu$, are deduced by solving the QRPA eigenvalue problem [23]. Because these techniques are rather standard [18], we only recall here the expressions of the pair transfer probability.

In QRPA, the addition transition probability is given by

$$P_\nu^{\mathrm{Add}} = |\langle 0|\hat{T}^{\mathrm{Add}}|\nu\rangle|^2 = \left| \sum_i \left( V_i^2 X_i^{(\nu)} - U_i^2 Y_i^{(\nu)} \right) \right|^2. \tag{13}$$

It is well known that the first solution of the QRPA equations corresponds to the spurious mode, which is then not considered in the evaluation of the strength function.

In the weak coupling limit, below a certain threshold value of $G$ denoted by $G_{\mathrm{cr}}$, the minimization of the energy in HFB identifies to the Hartree-Fock approach with no pairing. Then the mean-field vacuum is a pure Slater determinant where the lowest hole states are occupied. In this case, labeling by $h$ the hole state ($V_h = 1$, $U_h = 0$) and $p$ ($V_p = 1$, $U_p = 1$) the

particle states associated to this vacuum, the excited states are described by using the particle-particle RPA (ppRPA), where the phonon creation operators (12) can be written as

$$Q_\nu^\dagger = \sum_p X_p^\nu a_p^\dagger a_{\bar{p}}^\dagger + \sum_h Y_h^\nu a_h^\dagger a_{\bar{h}}^\dagger, \tag{14}$$

while the addition probability is simply written

$$P_\nu^{\mathrm{Add}} = \left| \sum_p X_p^\nu - \sum_h Y_h^\nu \right|^2. \tag{15}$$

Note that in the ppRPA case, contrary to the QRPA case, the U(1) symmetry associated to particle number conservation is not broken. Explicit forms of the ppRPA and QRPA equations for the model considered here can be found in Refs. [24–26]

In Figs. 1 and 2, the ppRPA (dashed line) and QRPA (dot-dashed line) are compared to the exact results. Above a given threshold $G_{\mathrm{cr}}^{\mathrm{RPA}}$, ppRPA collapses and leads to imaginary energies making not possible a direct comparison with the exact and QRPA results. However, in Fig. 1 we show in the panel (c) corresponding to a pairing strength $G = 0.5$ MeV the ppRPA results obtained at the collapse point, that is, $G_{\mathrm{cr}}^{\mathrm{RPA}} = 0.48$, to show that its description is still reasonable even in the superfluid phase.

From these comparisons, the following conclusions can be drawn. (i) The ppRPA does reproduce perfectly the exact results (energies and probabilities) in the normal phase. (ii) The QRPA provides a global reproduction of the pair transfer probabilities in the superfluid phase. In particular, the threshold in energy that is directly related to pairing correlation is properly accounted for. It is worth mentioning that such a threshold can only be described in a mean-field theory by breaking particle number symmetry. Finally, it is also clearly seen that some differences exist between the exact and QRPA. In general, QRPA leads to peaks in the strength that are at slightly higher energies compared to the exact solution while probabilities are slightly overestimated. These differences are even stronger in the random space case and can stem from different origins. First, while part of the four quasiparticles (4QP) are accounted for in the QRPA GS correlations, the complete inclusion of 4QP excitations is known to modify excited-state energy spectrum [18,27]. Second, a systematic error exists owing to the breaking of the particle number. Panel (c) of Fig. 1 illustrates that when ppRPA is applicable in the superfluid phase $G \leqslant 0.48$ MeV, it gives a better agreement with the exact result compared to QRPA. Because ppRPA is a particle-number-conserving theory, this is a first indication that the breaking of U(1) symmetry might pollute the QRPA predictions.

### A. Role of particle number in the estimation of pair transfer probabilities

At this stage, it is most likely to conjecture that the failure of QRPA to reproduce two-particle transfer processes stems from the mixing of systems with different particle numbers. The QRPA approach implicitly assumes that the states $|\nu\rangle$ are relatively good approximations for the eigenstates of systems with $N+2$ (or $N-2$) particles. As an illustration of the correctness of this assumption, the mean number of particles $N_\nu = \langle\nu|\hat{N}|\nu\rangle$ is displayed in the bottom panel of Fig. 4 as a function of the excitation energy $\omega_\nu$. $N_\nu$ has been estimated using the quasiboson approximation, leading to

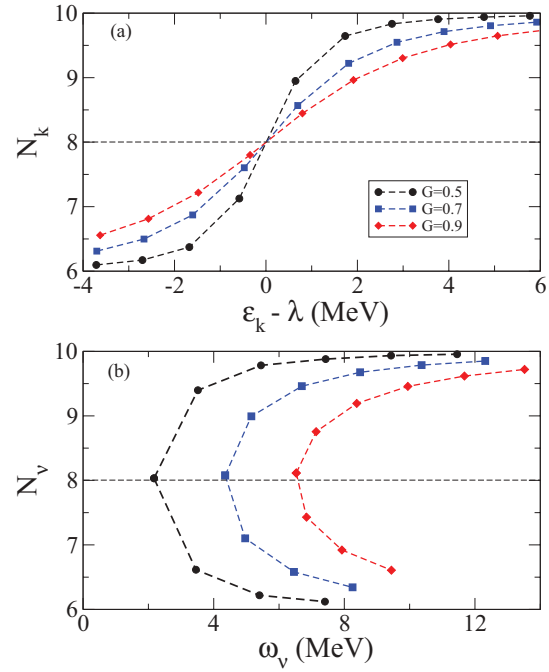


FIG. 4. (Color online) (Bottom) Mean number of particles $N_\nu$ of the QRPA states as a function of their excitation energy $\hbar\omega_\nu$. (Top) Mean number of particles $N_k$ of 2QP states as a function of $(\epsilon_k - \lambda)$, where λ is the Fermi energy.

$$N_\nu = \sum_i 2\left(U_i^2 - V_i^2\right)\left(X_i^{\nu 2} + Y_i^{\nu 2}\right) + \langle\hat{N}\rangle, \tag{16}$$

where $\langle\hat{N}\rangle$ is the number of particles in the QP vacuum.

In the same figure, the mean-particle number of the 2QP excited states $|k\rangle$ as a function of $(\epsilon_k - \lambda)$ is also shown, where λ is the Fermi energy. The 2QP states are defined through

$$|k\rangle = \alpha_k^\dagger \alpha_{\bar{k}}^\dagger |0, QP\rangle, \tag{17}$$

where the GS has $N$ particles on average. The mean particle number in $|k\rangle$ is given by

$$N_k = \langle k|\hat{N}|k\rangle = \langle\hat{N}\rangle + 2U_k^2 - 2V_k^2. \tag{18}$$

This expression as well as the illustration in Fig. 4 clearly shows that the 2QP states will be close to a state with $N+2$ ($N-2$) particles only if $U_k \to 1$, that is, well above the Fermi energy λ ($U_k \to 0$, i.e., well below the Fermi energy), but will be a bad approximation if the 2QP state involves a single-particle state in the vicinity of λ. Consequently, QRPA states will also suffer from the same problems if the state is constructed from 2QP states that are close to the Fermi energy. While the QRPA results are in a reasonable agreement with the exact case, the effect of particle number conservation on the pair transfer is largely uncontrolled within QRPA. This is anticipated to be especially crucial in exotic nuclei as the level spacing is reduced close to the drip line.

## IV. IMPROVED TREATMENT OF PAIR TRANSFER IN SUPERFLUID SYSTEMS

The effect of particle number conservation on pair transfer from GS to GS has been already studied in Ref. [12]. It has been empirically found that the breaking of U(1) symmetry has a rather small impact on the estimated probabilities. As a further illustration, we show in Fig. 3 estimations of transfer probabilities with Eq. (4) using the GS quasiparticle vacua projected either on $N$ or on $N+2$ particle numbers (see Ref. [12] for technical details). As seen in Fig. 3, the BCS approach reduces to HF below the threshold and is not able to reproduce the transfer probability at low $G$. The projection after variation (open square) obviously does not cure this problem but considerably improves the probability above the threshold, especially in the strong coupling regime. As a conclusion, the HFB and/or BCS approach augmented by an *a posteriori* projection is already very good to describe GS to GS transition in the superfluid regime. Therefore, in the following we concentrate the discussion on excited state where it is necessary to go beyond HFB.

From previous discussion, we are facing the following dilemma: To describe the physics of pairing and in particular the gap in energy between the GS and the first excited state in a superfluid system, it is necessary to break the U(1) symmetry. However, this symmetry breaking seems to be at the origin of some discrepancies between QRPA and exact pair transfer probabilities.

The most direct extension of the projection technique to estimate transition from GS to GS would be to directly estimate the transition from the BCS/HFB GS projected onto $N$ particles to the QRPA eigenstates projected onto $N+2$ particles. This has, however, two disadvantages. (i) In practice we found that the pair transfer probabilities are much smaller than the exact ones. (ii) QRPA states are no longer orthogonal after projection, leading to some difficulties in interpreting the probabilities themselves.

Alternatively, one can try to develop a RPA-like approach directly in the space of projected 2QP states. Following the Tamm-Dancoff approximation spirit, a set of states $|\Phi_k\rangle$ defined through

$$|\Phi_k\rangle = \hat{P}_{N+2}\alpha_k^\dagger\alpha_{\bar{k}}^\dagger|0, QP\rangle \tag{19}$$

are introduced, where $\hat{P}_{N+2}$ is the projector on $N+2$ particles [19]. Then, excited states of the system with $N+2$ particles decompose as

$$|\nu, N+2\rangle = \sum_k X_k^\nu|\Phi_k\rangle. \tag{20}$$

This strategy has been analyzed in Ref. [28], as well as its RPA generalization following Refs. [29] (see also Ref. [30]).

A proper description would require a full projected QRPA calculation whose practical implementation would be rather cumbersome, especially in realistic calculations. A simpler approach is to introduce what could be considered as a projected version of a two quasiparticle Tamm-Dancoff approximation. Again, it should be mentioned that, contrary to standard TDA, states $|\Phi_k\rangle$ are neither normalized nor orthogonal with each other. Therefore, special attention has to be paid when formulating the approach. In practice, this implies to diagonalize the overlap matrix $O_{kl} = \langle\Phi_k|\Phi_l\rangle$ prior to write the TDA eigenvalue problem. A practical method has been proposed in Ref. [28] to obtain the TDA equation in the projected space. Following Ref. [28], the excited states are written in terms of new states,

$$|\nu, N+2\rangle = \sum_k X_k^\nu|(\Phi_k)\rangle. \tag{21}$$

States $|(\Phi_k)\rangle$ are defined through

$$|(\Phi_k)\rangle = |\Phi_k\rangle - |0_{N+2}\rangle\langle 0_{N+2}|\Phi_k\rangle, \tag{22}$$

which guarantees the orthogonality of these states with respect to $|0_{N+2}\rangle$, corresponding here to the approximated GS with $N+2$ particles. While in the original article, this GS was anticipated to be obtained with a variation after projection (VAP) procedure, below it is simply taken as $|0_{N+2}\rangle \simeq \hat{P}_{N+2}|0, QP\rangle$. Then the TDA eigenequation is solved in the space of $|(\Phi_k)\rangle$ states. In the following, this approach is referred to projected two quasiparticle TDA (P-QTDA).

Consistent with the present approach, addition pair transfer probabilities are computed using the expectation value of the transition operator between the quasiparticle vacuum projected on particle $N$ and the excited states (21):

$$P^{\mathrm{Add}} = \left|\sum_\nu X_k^\nu\langle 0, QP|\hat{P}_N\hat{T}\hat{P}_{N+2}|(\Phi_k)\rangle\right|^2. \tag{23}$$

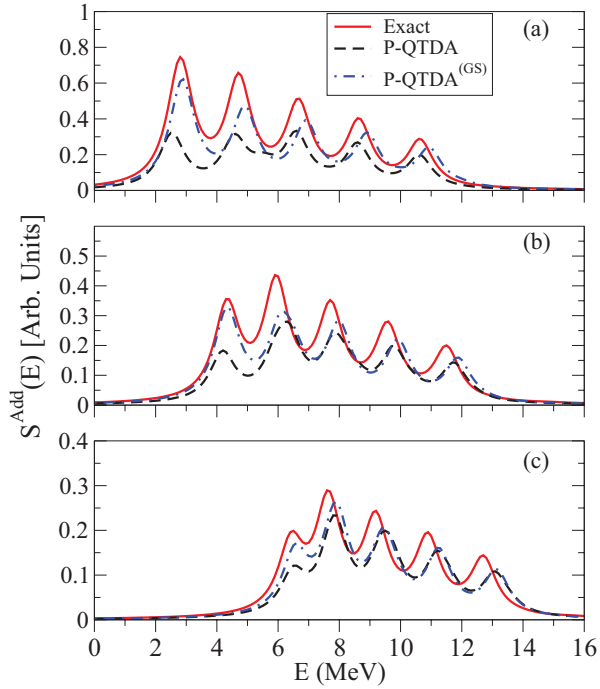


FIG. 5. (Color online) Same as Fig. 1 for (a) $G = 0.5$ MeV, (b) $G = 0.7$ MeV, and (c) $G = 0.9$ MeV. The exact result (red solid line) is compared to the P-QTDA (black dashed line) and P-QTDA$^{(GS)}$ (blue dot-dashed line).

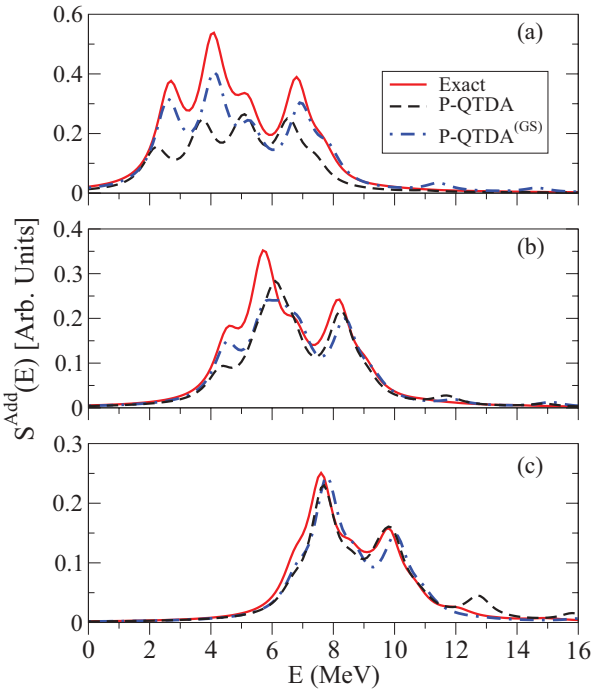


FIG. 6. (Color online) Same as Fig. 5 for the case of random level spacing.

The projection onto different particle numbers in the expression does not induce extra difficulty using the fact that

$$\langle 0, \mathrm{QP}|\hat{P}_N \hat{T} \hat{P}_{N+2}|\Phi_k\rangle = \langle 0, \mathrm{QP}|\hat{T} \hat{P}_{N+2}|\Phi_k\rangle = \langle 0, \mathrm{QP}|\hat{P}_N \hat{T}|\Phi_k\rangle.$$

Therefore, expectation values entering in expression (23) can be performed using standard projection techniques. In practice, the projection on particle number is made by discretizing the gauge angle integration using the Fomenko approach with 199 points [31,32]. A useful expression to express the overlaps as well as the Hamiltonian expectation value in a projected basis can be found in the appendix of Ref. [27].

Illustration of the method proposed in Ref. [28] applied to pair transfer are shown in Figs. 5 and 6, respectively, for the equidistant and random single-particle level spacing. These figures clearly demonstrate that the projected TDA approach provides a much better description of the energy peak positions of the excited states in the system with $N + 2$ particles. However, probabilities of transfer are underestimated especially as $G$ decreases.

Further improvements can *a priori* be made by including more correlations in the ground state $|0_{N+2}\rangle$. Here, we simply used a projection after variation (PAV) approach to approximate this state. A better treatment would be to perform a VAP as originally proposed in Ref. [28]. Further correlations, such as correlations induced by the coupling to 4QP states might be included using the projected QRPA instead of the projected TDA approach. The projected QRPA approach has also been formulated previously (Sec. III B of Ref. [28]). In that case, not only the projected 2QP states but also the projected 4QP states should be explicitly introduced.

The use of VAP and/or the introduction of 4QP, although possible in the present model [27,33,34], will considerably increase the complexity of the approach in realistic situations (see, for instance, [35–37] for the VAP). Below, we propose a simpler method inspired from P-QTDA and able to grasp part of the GS correlations without increasing the numerical complexity.

Contrary to standard linear response theory based on HF (HFB), the GS projected onto good particle number is not orthogonal to the particle-hole (2QP) excited states. Similarly, the projected 2QP states are themselves not orthogonal to the projected 4QP or higher-order excitations. At first glance, this might be seen as a disadvantage compared to RPA or QRPA because additional orthonormalization is required. On the contrary, one might take advantage of the presence of higher-order components to improve the description of the GS itself.

The aim of the P-QTDA approach was to describe excitations with respect to the projected GS. Then, the latter state has been naturally removed before solving the eigenvalues equation [Eq. (22)]. Let us assume that we restart from expression (20), where the projected mean field is also included in the sum (with the convention that $|\Phi_0\rangle = \hat{P}_{N+2}|0, QP\rangle$). Then, coefficients $X_k^\nu$ can be obtained by diagonalizing the Hamiltonian in the $\{\Phi_k\}$ (with proper treatment of the nonorthonormality of the states). This direct approach, called below P-QTDA$^{(GS)}$, not only provides a way to get excited states but might also improve the description of the GS itself. This is indeed what we observed empirically. For instance, for $G = 0.5$ MeV and equidistant level spacing, the new GS has an energy 300 keV lower than the energy of the original projected mean-field GS. The difference reduces as

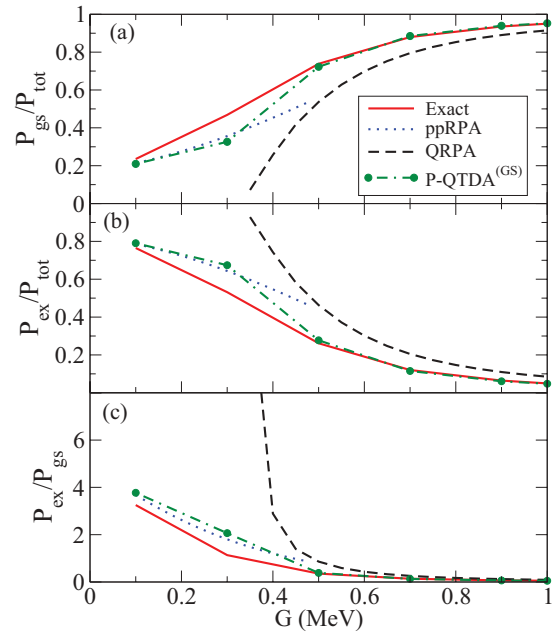


FIG. 7. (Color online) Ratios of probabilities estimated with different theories for the equidistant level case. The exact (red solid line), QRPA (black dashed line), and P-QTDA (green solid circles) are shown. The ppRPA is also shown (dotted line) up to $G = 0.48$ MeV.

$G$ increases. At $G = 0.9$ MeV the reduction is only 30 keV. For completeness, the probability to transfer from GS to GS within the P-QTDA$^{(GS)}$ is also shown in Fig. 3. This probability is slightly improved compared to the BCS case below the BCS threshold while it follows the PAV results above.

It turns out that this approach improves the description of pair transfer from GS to excited states. Results of the P-QTDA$^{(GS)}$ method are presented in Figs. 5 and 6 by dashed lines. A clear improvement is observed especially at low excitations and low $G$. The remaining difference with the exact solution is acceptable in view of the simplified approach presented here.

To further quantify the predictive power of the P-QTDA$^{(GS)}$ method, some ratios of pair transfer probabilities are shown in Fig. 7. In this figure, $P_{gs}$, $P_{ex}$, and $P_{tot}$ correspond, respectively, to the probability to transfer to the GS, the sum of probabilities to transfer to any excited states, while $P_{tot} = P_{gs} + P_{ex}$. Below the pairing threshold, the P-QTDA$^{(GS)}$ reduces to a ppTDA$^{(GS)}$, where the diagonalization is made in a reduced space of Slater determinant. In that case, the result are of the same quality as for the ppRPA. This figure clearly confirms that while QRPA is rather far from the exact results in the superfluid phase, especially in the vicinity of the BCS threshold, the projected theory provides a much better reproduction of GS to excited state pair transfer probabilities. Moreover, theoretical predictions based on the QRPA [38] might significantly overestimate the GPV cross section, as suggested also in Ref. [7], especially in the vicinity of the normal-superfluid transition.

To analyze the dependence of the results on the system size, we present in Fig. 8 the addition strength function for different $N$ to $N + 2$ systems with (a) $N = 6$, (b) $N = 12$, and (c) $N = 40$. In each case, $\Omega = N + 2$ equally spaced levels are taken while the pairing strength is set to $G = 0.5\Delta\varepsilon$. Because of the size of the matrix involved, for the $N = 40$ case, the exact calculation cannot be performed and only the QRPA and P-QTDA$^{(GS)}$ results are shown in panel (c), showing at the same time the feasibility of the P-QTDA$^{(GS)}$ for large systems. As a general feature, by comparing with the exact solution in panels (a) and (b), we can see that the P-QTDA$^{(GS)}$ is able to give a better description of the position of the energy peaks compared to the QRPA. Conjointly, while the overall strength is globally well reproduced, the transfer probability is slightly underestimated in P-QTDA$^{(GS)}$ pointing out some missing effect of more complex configurations that play a role in the exact case. The QRPA generally overestimates the collectivity of the states.

For completeness, we also show in Fig. 9 the effect of enlarging the model space size. Panels (a), (b), and (c) correspond, respectively, to $\Omega = 10$, $\Omega = 12$, and $\Omega = 14$

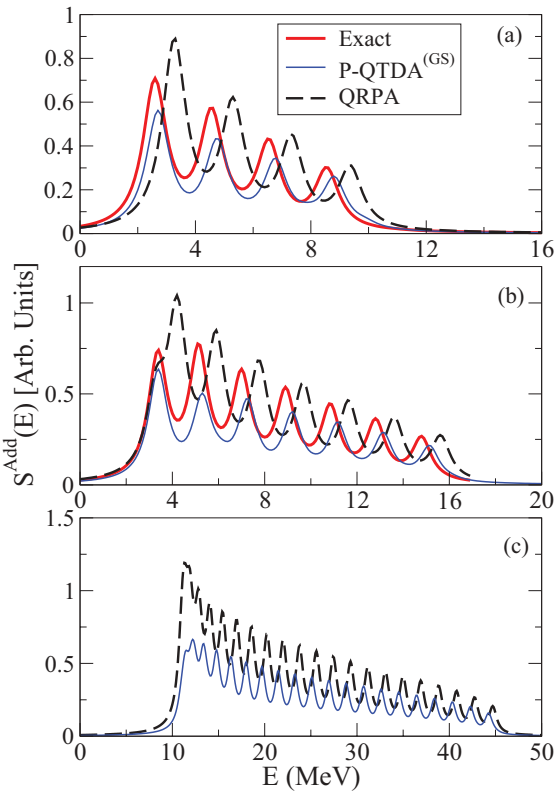


FIG. 8. (Color online) Exact (red thick solid line), QRPA (black dashed line), and P-QTDA$^{(GS)}$ (blue thin solid line) addition strength function as a function of excitation energy $E$ for different $N$ to $N + 2$ systems. Calculations are made with $\Omega = N + 2$ equally spaced levels: (a) $N = 6$, (b) $N = 12$, and (c) $N = 40$. In all cases the pairing strength $G$ used is $G/\Delta\varepsilon = 0.5$. In panel (c) only the QRPA and P-QTDA$^{(GS)}$ results are shown because in this case the full diagonalization cannot be performed.

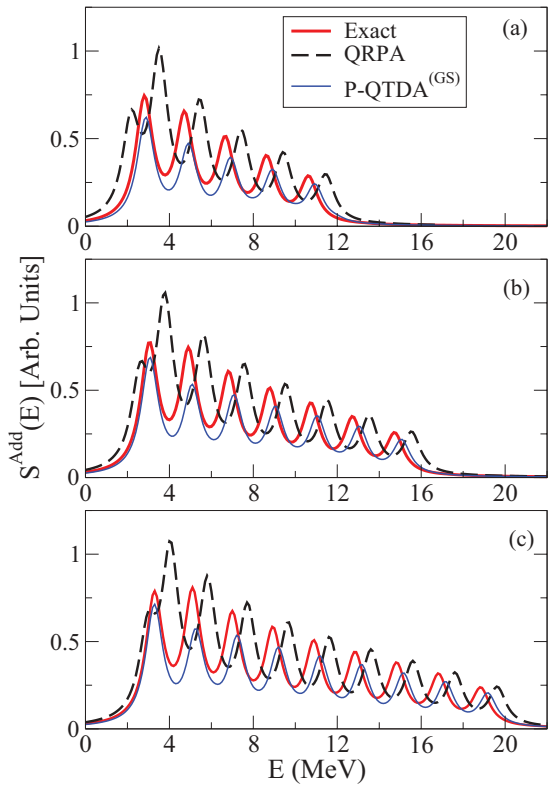


FIG. 9. (Color online) Exact (red thick solid line), QRPA (black dashed line), and P-QTDA$^{(GS)}$ (blue thin solid line) addition strength function as a function of excitation energy $E$ for the transfer from $N = 8$ to $N = 10$. Calculations are made with (a) $\Omega = 10$, (b) $\Omega = 12$, and (c) $\Omega = 14$ equally spaced levels and $G/\Delta\varepsilon = 0.5$.

doubly degenerated single-particle levels for the transfer from $N = 8$ to $N = 10$. We see that the increase of $\Omega$ does not change the conclusions drawn above and that the P-QTDA$^{(GS)}$ always leads to a significant improvement compared to the QRPA result.

## V. CONCLUSIONS

In this work, the QRPA description of the two-particle transfer mechanism is tested against the exact solution in the Richardson model for several conditions of pairing interaction strength and level spacing. It is seen that both ppRPA in the normal phase and QRPA in the superfluid region are able to grasp the gross feature of the pair transfer process. However, some differences are observed. At variance with other kinds of resonance states mainly built in terms of particle-hole excitations (see, for instance, Fig. 1 of Ref. [39]), the particle number conservation seems to play an important role when the particle number change during the physical process of interest. A method is proposed here to improve the description of pair transfer in finite superfluid systems. The new method is based on the direct diagonalization of the Hamiltonian in a reduced space of the projected GS plus two quasiparticle states. This theory improves considerably the description of the pair transfer process. On the practical side, the P-QTDA$^{(GS)}$ requires only to solve the BCS or HFB problem in the initial nucleus and, except for the additional numerical cost of projection, it does not need more effort than the original QRPA. Work is in progress to apply it to nuclear transfer reactions.

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