

Relativistic quantum mechanics and the quark-pair creation model

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In the context of a 3P_0 quark-pair creation model for the process $\rho \rightarrow \pi\pi$, a method is developed for taking relativity into account in the calculation of hadron decays. Following a brief review of relativistic quantum mechanics, an expression is derived for the general relation between a momentum-space, two-particle, instant form wave function in an arbitrary frame and the wave function associated with the c.m. frame. This relation is used to develop relativistic wave functions for the π and ρ mesons. Second quantized state vectors for $\pi\pi$ and ρ states are constructed with the help of these relativistic wave functions. The $\rho \rightarrow \pi\pi$ transition amplitude is obtained by using these state vectors to calculate matrix elements of a second quantized 3P_0 quark-pair creation operator derived from a scalar Lagrangian density. The amplitude differs from the one obtained using nonrelativistic wave functions in the appearance of Wigner rotations. In spite of the complications arising from these rotations the calculation of the relativistic amplitude is reduced to carrying out a two-dimensional integral. The amplitude is of the same form as one derived from an effective $\rho\pi\pi$ Lagrangian except for the presence of a form factor that depends on the magnitude of the three-momentum of a final-state pion. The shape of the form factor is determined by the relativistic π and ρ wave functions. Using the $\rho \rightarrow \pi\pi$ transition amplitude as a vertex interaction in a relativistic model of $\pi\pi$ scattering, the p -wave, $\pi\pi$ scattering amplitude is calculated and fit to data by adjusting the interaction strength and the ρ bare mass. This leads to a mass shift and decay width for the ρ meson. Using nonrelativistic wave functions to calculate the form factor leads to a negligible mass shift, whereas using the relativistic wave functions leads to a bare ρ mass of 855.7 MeV, corresponding to a physical ρ mass of 775.5 MeV. The quark-pair creation operator strength parameter for the relativistic case is roughly a factor of 2 larger than that for the nonrelativistic case.

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I. INTRODUCTION

Here relativistic quantum mechanics refers to the framework for developing relativistic models that grew out of an important paper by Dirac [1] in which he discussed the various ways for incorporating interactions in the generators of the Poincaré group. Dirac called these various possibilities the instant form, the point form, and the front form. Each form is associated with a three-dimensional hypersurface in Minkowski space that is invariant under a subgroup of the Poincaré transformation, $x' = ax + b$, and intersects every world line just once. For the instant, point, and front forms the hypersurfaces can be taken to be $t = \text{const.}$, $c^2t^2 - \mathbf{x}^2 = a^2$ with $t > 0$, and $ct + z = 0$, respectively. In Dirac's approach the generators associated with these hypersurfaces are taken to be noninteracting, and interactions are put into the remaining generators. In the instant form, which is the one we adopt here, the three-momentum \mathbf{P} and the angular momentum \mathbf{J} are noninteracting, while the Hamiltonian H and the generator of rotationless boosts \mathbf{K} contain interactions. Since \mathbf{P} and \mathbf{J} generate translations and rotations in ordinary three-dimensional space, it is clear that these transformation do not move a space-time point off of the hypersurface $t = \text{const.}$

A practical method for constructing models in the various forms of relativistic quantum mechanics is due to Bakamjian and Thomas [2,3]. In their approach for constructing instant form generators the 10 Poincaré generators are expressed in terms of the 10 operators $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$, where M is the mass operator, \mathbf{S} is a spin operator, and \mathbf{X} is the so-called Newton-Wigner position operator [3,4]. This set of operators satisfies

much simpler commutation rules than the Poincaré generators, which facilitates the construction of models. In the Bakamjian-Thomas scheme \mathbf{P} , \mathbf{S} , and \mathbf{X} are taken to be noninteracting, and an interaction is put only in the mass operator M .

Here we show that the Bakamjian-Thomas scheme provides a natural framework for constructing relativistic quark models in which the strong coupling to decay channels is taken into account by means of the 3P_0 quark-pair creation model. In this model it is assumed that the $q\bar{q}$ pair that is created has the quantum numbers of the vacuum. This implies that the $q\bar{q}$ pair must be a color singlet and a flavor singlet and have zero total linear momentum and zero total angular momentum. For a fermion-antifermion pair $P = (-1)^{l+1}$ and $C = (-1)^{l+s}$, where P is parity, C is charge conjugation, l is orbital angular momentum, and s is spin angular momentum. For the vacuum we have $j^{PC} = 0^{++}$; therefore the $q\bar{q}$ pair must have $l = 1$ and $s = 1$ combined to give $j = 0$; i.e., the $q\bar{q}$ pair is created in a 3P_0 state.

The 3P_0 model grew out of some work by Micu [5] in which he used the model to calculate the decay rates of meson resonances described by the quark model. The model was subsequently applied to both meson and baryon strong decays by Le Yaouanc *et al.* [6,7]. A very thorough early application of the 3P_0 model to three-meson vertices was carried out by Chaichian and Kögerler [8]. All of these analyses used nonrelativistic $q\bar{q}$ and qqq wave functions to describe the hadrons. Here we propose a remedy for dealing with this shortcoming.

The 3P_0 mechanism is somewhat similar to the mechanism for strong decays that arises in the Isgur-Paton [9] flux-tube

model, a model derived from the strong-coupling Hamiltonian lattice formulation of QCD. The flux tubes provide the interactions between quarks as well as a decay mechanism. A decay can occur when a flux tube breaks, thereby leading to the appearance of a quark and an antiquark on the new ends [10]. Ultimately the flux-tube decay model turns out to be a fairly straightforward extension [11,12] of the usual 3P_0 model. In the usual model the q and \bar{q} are assumed to be pointlike and to occur with equal amplitude everywhere in space. Geiger and Isgur [11,12] introduced three modifications. In their model the pair is created in a finite region defined by the overlap of the initial and final flux-tube wave functions, a form factor is introduced to take account of the constituent quark size, and another factor is introduced to suppress pair creation at short distances. Here we do not deal explicitly with the Geiger-Isgur extension of the 3P_0 model, but it will become clear that the relativistic approach developed here is capable of doing so.

One of the most interesting questions that arises when strong coupling to decay channels is included in quark models is the effect on the hadron mass spectrum. These couplings can provide self-energy contributions to the masses of mesons (m) through the process $m \rightarrow m' + m'' \rightarrow m$, as well as to baryons (b) through the process $b \rightarrow b' + m \rightarrow b$. Furthermore, mixing between hadrons with the same quantum numbers can arise from processes such as $m \rightarrow m' + m'' \rightarrow m'''$ and $b \rightarrow b' + m \rightarrow b''$. These processes, of course, are associated with loop diagrams.

The effect of loop diagrams on the meson spectrum was investigated in detail by Törnqvist [13–16] and on the baryon spectrum by Törnqvist and Zenczykowski [17–19]. In their approach the loop diagrams are generated by unitarity so the model is referred to as the unitarized quark model (UQM). With this model the mass shift of hadron A due to various open and closed decay channels can be written as the dispersion relation

$$m_A^2 - (m_A^0)^2 = \sum_i w_i^A \int_{s_{\text{threshold}}}^{\infty} \frac{\phi(s, m_B, m_C)}{m_A^2 - s} ds, \quad i = (B, C), \quad (1.1)$$

where the sum runs over the decay channels. Here m_A^0 and m_A are the bare mass and physical or dressed mass of hadron A , respectively, ϕ is a spectral function, w_i^A is a numerical weight, and m_B and m_C are the masses of the hadrons that comprise channel i . If A is a meson then B and C are mesons, while if A is a baryon B is a baryon and C is a meson. The weights and the spectral function can be determined within a specific decay model such as the 3P_0 pair-creation model. With the UQM it is possible to explain many mass shifts, mass widths, and mixing effects among the light mesons and among the heavy $c\bar{c}$ and $b\bar{b}$ states [13–16], as well as among the ground state and P -wave baryons [17–19].

In a variation of the UQM Silvestre-Brac and Gignoux [20] used the dispersion relation

$$E_i - E_i^0 = \lim_{\varepsilon \rightarrow 0} \sum_c \int_0^{\infty} \frac{k^2 dk |V_{ic}(k)|^2}{E_i - E_c(k) + i\varepsilon}, \quad (1.2)$$

where E_i is the perturbed energy of a state, E_i^0 is the corresponding unperturbed energy, and the sum on c is over the various decay channels. In their calculations the matrix elements, $V_{ic}(k)$, that determine the coupling of state i to the decay channels c are taken from the 3P_0 model. Their analysis helps to explain the order and importance of spin-orbit splitting in $L = 1$ baryons, as well as partial and total decay widths.

Roberts and Silvestre-Brac [21] developed a very general formalism for applying the 3P_0 model to any hadron decay of the form $A \rightarrow B + C$. Capstick and Roberts [22–24] used this formalism to calculate the decay amplitudes of nonstrange baryons to a large number of baryon-meson decay channels. They employed hadron wave functions derived from the relativized quark model of Capstick and Isgur [25]. This model uses the relativistic expression $E = \sqrt{\mathbf{p}^2 + m^2}$ for the quark energies rather than the nonrelativistic form $E = p^2/2m + m$, and in it the quark masses m that appear in the potentials with $E = \sqrt{\mathbf{p}^2 + m^2}$ are also replaced. A very useful review paper on quark models of baryon masses and decays is by Capstick and Roberts [26].

It should be noted that introducing relativistic expressions into nonrelativistic quark models does not guarantee Poincaré invariance. In principle to do this it is necessary to establish the existence of 10 Poincaré generators with which the unitary operators that map quantum mechanical state vectors from one inertial frame to another can be constructed. The Bakamjian-Thomas scheme [2,3] described above is probably the simplest way to satisfy this requirement for Poincaré invariance. In the instant form, for example, it is only necessary to construct a mass operator M that commutes with \mathbf{P} , \mathbf{S} , and \mathbf{X} . A number of authors have used the Bakamjian-Thomas scheme to construct Poincaré invariant quark models for mesons and baryons.

Coester and Riska [27] used the point form to construct a Poincaré invariant quark model for the baryons. The interaction in their mass operator involves harmonic oscillator confinement, a hyperfine interaction whose spin-flavor structure is motivated by the spin-flavor part of the interaction mediated by the exchange of the octet of light pseudoscalar bosons [28], and an angular-momentum-dependent term. An interesting feature of their model is that the quark masses do not appear in their mass operator. Their model reproduces the empirical baryon masses up to ~ 1700 MeV to an accuracy of $\sim 6\%$. A more sophisticated version of the model [29] reproduces the empirical spectra of the baryons in all flavor sectors to an accuracy of a few percent. This model has a confining interaction that is a function of a hyper-radial variable that is the radius of a six-dimensional hypersphere. It also includes an operator that depends on the flavor quantum numbers of strangeness, charm, and beauty, as well as a phenomenological hyperfine interaction.

The semirelativistic quark model for baryons due to Glazman *et al.* [30] consists of a linear confinement interaction and a Goldstone boson exchange potential. Wagenbrunn *et al.* [31] made this model Poincaré invariant by reinterpreting its Hamiltonian as a point form mass operator.

Krassnigg *et al.* [32] used the point form to construct a Poincaré invariant quark model for vector mesons. Their model

employs harmonic oscillator confinement and a hyperfine interaction generated by Goldstone boson exchange.

Here we develop a method for incorporating a Poincaré invariant version of the 3P_0 quark-pair creation model into Poincaré invariant quark models such as those presented in Refs. [27,29,31,32]. This is done in the context of a specific decay process, i.e., $\rho \rightarrow \pi\pi$. Here the ρ and π mesons are each described as bound states of a quark and an antiquark.

In Sec. II we briefly review relativistic quantum mechanics in the instant form, with emphasis on the Lorentz transformation of instant form state vectors from one inertial frame to another. The relation between the momentum-space wave function of a two-particle system in an arbitrary frame and in a c.m. or rest frame is derived.

Section III presents a derivation of the 3P_0 interaction from the scalar Lagrangian density $L_I(x) = g\bar{\psi}(x)\psi(x)$, where $\psi(x)$ is a Dirac quark field [33]. This shows that there is nothing inherently nonrelativistic about the 3P_0 interaction.

By using the momentum-space wave function relation derived in Sec. II, the relativistic wave functions for the π and ρ mesons, including the color and flavor variables, are presented in Sec. IV. The properly normalized $\pi\pi$ and ρ second quantized state vectors are also given, which makes it possible to evaluate the $\rho \rightarrow \pi\pi$ transition amplitude by taking matrix elements of the second quantized 3P_0 interaction.

The $\rho \rightarrow \pi\pi$ transition amplitude is developed in Sec. V. Here we see the distinctive feature that occurs in the relativistic treatment of the π and ρ wave functions, but not in a nonrelativistic treatment—namely, the appearance of Wigner rotations [3]. By using standard SU(2) relations we find that it is possible, even with the complication of the Wigner rotations, to reduce the calculation of the transition amplitude to evaluating a two-dimensional integral. The final result is identical to the form obtained from the effective hadronic, Lagrangian density $L_{\rho\pi\pi}(x) = -g_{\rho\pi\pi}[\boldsymbol{\pi}(x) \times \partial_\mu \boldsymbol{\pi}(x)] \cdot \boldsymbol{\rho}^\mu(x)$, except for the appearance of a form factor that depends on the magnitude of the three-momentum of a pion in the final state c.m. frame. The momentum dependence of the form factor is determined by the quark wave functions of the mesons in their rest frames.

In Sec. VI a simple mass operator is given whose eigenstates are the $q\bar{q}$ wave functions for the π and ρ mesons. The operator consists of a harmonic oscillator confinement interaction and a hyperfine interaction. Both mesons have the same $q\bar{q}$ rest frame radial wave function, which is a simple Gaussian.

The nonrelativistic $\rho \rightarrow \pi\pi$ transition amplitude is developed in Sec. VII. We find that the relativistic result found in Sec. V goes over to the nonrelativistic one if a couple of relativistic relations for energy and momentum are replaced by the nonrelativistic ones and the Wigner rotations are “turned off.”

In Sec. VIII we analyze p -wave $\pi\pi$ scattering using a mass operator that acts in the space spanned by p -wave $\pi\pi$ and ρ meson states. The only interaction is the vertex interaction that follows from the $\rho \rightarrow \pi\pi$ transition amplitude. The p -wave $\pi\pi$ scattering amplitude is calculated and the two adjustable parameters of the model are fit to the data. This leads to results for the mass shift and width of the ρ meson. The nonrelativistic

and relativistic results for these properties of the ρ meson are compared.

Section IX gives a brief discussion of future extensions and applications of the method developed here. Necessary results for Wigner rotations are given in the Appendix.

Throughout we work in units in which $\hbar = c = 1$.

II. RELATIVISTIC QUANTUM MECHANICS

Here we analyze the construction of states that have well-defined transformation properties under Lorentz transformations and indicate how they are used in a Bakamjian-Thomas construction of an instant form model of a two-particle system.

Single-particle states are denoted by $|\mathbf{p}m\rangle$, where \mathbf{p} is the particle’s three-momentum and m is the z component of its spin s . In the rest frame of the particle the state rotates according to

$$U(r)|\mathbf{0}m\rangle = \sum_{m'} |\mathbf{0}m'\rangle D_{m'm}^{(s)}(r), \quad (2.1)$$

where $U(r)$ is a unitary operator corresponding to the rotation r , and $D^{(s)}(r)$ is a standard SU(2) matrix representative of the rotation r . We boost the rest frame state to one of three-momentum \mathbf{p} by applying the unitary operator corresponding to a canonical Lorentz boost $l_c(p)$, i.e.,

$$|\mathbf{p}m\rangle = U[l_c(p)]|\mathbf{0}m\rangle[\mu/\varepsilon(\mathbf{p})]^{1/2}, \quad (2.2a)$$

$$\varepsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}, \quad p = (\varepsilon(\mathbf{p}), \mathbf{p}), \quad (2.2b)$$

$$\langle \mathbf{p}m|\mathbf{p}'m'\rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{mm'}. \quad (2.2c)$$

Here μ is the mass of the particle. The square root factor in (2.2a) is consistent with the inner product (2.2c) [34]. The canonical boost is given by

$$x = l_c(p)x_{\text{c.m.}}, \quad p = (p^0, \mathbf{p}), \quad (2.3a)$$

$$x^0 = (p^0 x_{\text{c.m.}}^0 + \mathbf{p} \cdot \mathbf{x}_{\text{c.m.}})/W,$$

$$\mathbf{x} = \mathbf{x}_{\text{c.m.}} + \left(x_{\text{c.m.}}^0 + \frac{\mathbf{p} \cdot \mathbf{x}_{\text{c.m.}}}{p^0 + W} \right) \frac{\mathbf{p}}{W},$$

$$W = (p \cdot p)^{1/2}. \quad (2.3b)$$

Under a general Lorentz transformation $x' = ax$ the state (2.2a) transforms according to [3,34]

$$U(a)|\mathbf{p}m\rangle = \sum_{m'} |\mathbf{p}'m'\rangle D_{m'm}^{(s)}[r_c(a, p)][\varepsilon(\mathbf{p}')/\varepsilon(\mathbf{p})]^{1/2},$$

$$\mathbf{p}' = a\mathbf{p}, \quad (2.4)$$

where $r_c(a, p)$ is a so-called Wigner rotation [3,4] defined by

$$r_c(a, p) = l_c^{-1}(ap)al_c(p). \quad (2.5)$$

If a is a rotation r this simplifies to [3,34]

$$r_c(r, p) = r. \quad (2.6)$$

Two-particle states can be obtained by boosting from the c.m. frame of the two particles according to [34]

$$|\mathbf{p}\mathbf{k}m_1m_2\rangle = U[l_c(p)]|\mathbf{k}m_1\rangle \otimes |-\mathbf{k}, m_2\rangle \\ \times [W(\mathbf{k})/E(\mathbf{p}, \mathbf{k})]^{1/2}, \quad (2.7a)$$

$$\varepsilon_1(\mathbf{k}, \mathbf{k}) = l_c^{-1}(p)\varepsilon(\mathbf{p}_1, \mathbf{p}_1), \\ \varepsilon_2(-\mathbf{k}, -\mathbf{k}) = l_c^{-1}(p)\varepsilon_2(\mathbf{p}_2, \mathbf{p}_2), \quad (2.7b)$$

$$\langle \mathbf{p}\mathbf{k}m_1m_2 | \mathbf{p}'\mathbf{k}'m'_1m'_2 \rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta^3(\mathbf{k} - \mathbf{k}')\delta_{m_1m'_1}\delta_{m_2m'_2}. \quad (2.7c)$$

Here \mathbf{k} is the three-momentum of particle 1 in the c.m. frame, \mathbf{p}_1 and \mathbf{p}_2 are the three-momenta of particles 1 and 2 in a general frame, while the other quantities that appear are given by

$$W(\mathbf{k}) = \varepsilon_1(\mathbf{k}) + \varepsilon_2(-\mathbf{k}), \quad (2.8a)$$

$$p = (\varepsilon(\mathbf{p}_1) + \varepsilon_2(\mathbf{p}_2), \mathbf{p}_1 + \mathbf{p}_2) = (E(\mathbf{p}, \mathbf{k}), \mathbf{p}), \quad (2.8b)$$

$$E(\mathbf{p}, \mathbf{k}) = \sqrt{\mathbf{p}^2 + W^2(\mathbf{k})}. \quad (2.8c)$$

Under a general Lorentz transformation a , the state (2.7a) transforms according to [3,34]

$$U(a)|\mathbf{p}\mathbf{k}m_1m_2\rangle = \sum_{m'_1m'_2} |\mathbf{p}', r_c(a, p)\mathbf{k}, m'_1m'_2\rangle D_{m'_1m_1}^{(s_1)}[r_c(a, p)] \\ \times D_{m'_2m_2}^{(s_2)}[r_c(a, p)][E(\mathbf{p}', \mathbf{k})/E(\mathbf{p}, \mathbf{k})]^{1/2}, \\ p' = ap. \quad (2.9)$$

Using (2.4) in (2.7a) we can show that the direct product of two single-particle states is related to the state (2.7) by the relation

$$|\mathbf{p}_1m_1, \mathbf{p}_2m_2\rangle = |\mathbf{p}_1m_1\rangle \otimes |\mathbf{p}_2m_2\rangle \\ = \left[\frac{\varepsilon_1(\mathbf{k})\varepsilon_2(-\mathbf{k})}{W(\mathbf{k})} \frac{E(\mathbf{p}, \mathbf{k})}{\varepsilon(\mathbf{p}_1)\varepsilon_2(\mathbf{p}_2)} \right]^{1/2} \\ \times \sum_{m'_1m'_2} |\mathbf{p}\mathbf{k}, m'_1m'_2\rangle D_{m'_1m_1}^{(s_1)} \{r_c^{-1}[l_c(p), k_1]\} \\ \times D_{m'_2m_2}^{(s_2)} \{r_c^{-1}[l_c(p), k_2]\}, \quad (2.10a)$$

$$k_1 = (\varepsilon_1(\mathbf{k}), \mathbf{k}), \quad k_2 = (\varepsilon_2(-\mathbf{k}), -\mathbf{k}). \quad (2.10b)$$

With the help of this relation we can derive the Jacobian that relates the $\mathbf{p}_1, \mathbf{p}_2$ variables to the \mathbf{p}, \mathbf{k} variables. The completeness relations for the states that appear in (2.10) are given by

$$1 = \sum_{m_1m_2} \int |\mathbf{p}_1m_1, \mathbf{p}_2m_2\rangle d^3p_1 d^3p_2 \langle \mathbf{p}_1m_1, \mathbf{p}_2m_2| \\ = \sum_{m_1m_2} \int |\mathbf{p}\mathbf{k}m_1m_2\rangle d^3p d^3k \langle \mathbf{p}\mathbf{k}m_1m_2|. \quad (2.11)$$

Inserting (2.10a) into this relation immediately leads to the result

$$d^3p_1 d^3p_2 = \frac{W(\mathbf{k})}{\varepsilon_1(\mathbf{k})\varepsilon_2(-\mathbf{k})} \frac{\varepsilon(\mathbf{p}_1)\varepsilon_2(\mathbf{p}_2)}{E(\mathbf{p}, \mathbf{k})} d^3p d^3k \\ = \frac{W(\mathbf{k})}{\varepsilon_1(\mathbf{k})\varepsilon_2(-\mathbf{k})} \frac{\varepsilon(\mathbf{p}_1)\varepsilon_2(\mathbf{p}_2)}{\varepsilon(\mathbf{p}_1) + \varepsilon_2(\mathbf{p}_2)} d^3p d^3k. \quad (2.12)$$

By coupling angular momenta we can construct from the state (2.7a) a state that transforms irreducibly under a Lorentz transformation [3]. It is given by

$$|\mathbf{p}klsjm\rangle \\ = \sum_{m_1m_2} \sum_{m'_1m'_2} \int d\Omega_{\mathbf{k}} |\mathbf{p}\mathbf{k}m_1m_2\rangle Y_l^{m_l}(\hat{\mathbf{k}}) \langle s_1s_2m_1m_2 | sm_s \rangle \\ \times \langle lsm_1m_s | jm \rangle, \quad (2.13a)$$

$$\langle \mathbf{p}klsjm | \mathbf{p}'k'l's'j'm' \rangle \\ = \delta^3(\mathbf{p} - \mathbf{p}')k^{-2}\delta(k - k')\delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{mm'}. \quad (2.13b)$$

Using the identities

$$\delta_{JJ'}D_{MM'}^{(J)} = \sum_{m_1m_2} \sum_{m'_1m'_2} \langle j_1j_2m_1m_2 | JM \rangle D_{m_1m'_1}^{(j_1)} D_{m_2m'_2}^{(j_2)} \\ \times \langle j_1j_2m'_1m'_2 | J'M' \rangle, \quad (2.14)$$

$$Y_l^{m_l}(r^{-1}\hat{\mathbf{k}}) = \sum_{m'_l} Y_l^{m'_l}(\hat{\mathbf{k}}) D_{m'_lm_l}^{(l)}(r), \quad (2.15)$$

it can be verified that under the Lorentz transformation $x' = ax$ the state (2.13a) transforms according to

$$U(a)|\mathbf{p}klsjm\rangle = \sum_{m'} |\mathbf{p}'k'l's'j'm'\rangle D_{m'm}^{(j)}[r_c(a, p)] \\ \times [E(\mathbf{p}', k)/E(\mathbf{p}, k)]^{1/2}, \\ p' = ap. \quad (2.16)$$

We note that the transformation is very similar to the transformation of a single-particle state as given by (2.4). By solving (2.13a) for $|\mathbf{p}\mathbf{k}m_1m_2\rangle$ and putting the result in (2.10a) we find the relation

$$|\mathbf{p}_1m_1, \mathbf{p}_2m_2\rangle \\ = \left[\frac{\varepsilon_1(\mathbf{k})\varepsilon_2(-\mathbf{k})}{W(\mathbf{k})} \frac{E(\mathbf{p}, \mathbf{k})}{\varepsilon(\mathbf{p}_1)\varepsilon_2(\mathbf{p}_2)} \right]^{1/2} \\ \times \sum_{m'_1m'_2} \sum_{lm_1} \sum_{sm_s} \sum_{jm} |\mathbf{p}klsjm\rangle \langle lsm_1m_s | jm \rangle \\ \times Y_l^{m_l}(\hat{\mathbf{k}}) \langle s_1s_2m'_1m'_2 | sm_s \rangle D_{m'_1m_1}^{(s_1)} \{r_c^{-1}[l_c(p), k_1]\} \\ \times D_{m'_2m_2}^{(s_2)} \{r_c^{-1}[l_c(p), k_2]\}. \quad (2.17)$$

The result (2.17) will allow us to incorporate relativistic wave functions for the bound states of quarks into the quark-pair creation model. The framework we will use to construct such wave functions is based on the Bakamjian-Thomas [2,3] instant form scheme for relativistic quantum mechanics. In a satisfactory relativistic quantum mechanics there exist unitary operators $U(a, b)$ that correspond to the Poincaré transformation $x' = ax + b$ and map quantum mechanical state vectors from the x frame to the x' frame. For proper transformations these unitary operators can be expressed in terms of the 10 Poincaré generators, four of which are the components of the four-momentum operator $P = (H, \mathbf{P})$, while the other six are the components of the three-vector operators \mathbf{J} and \mathbf{K} . Here H is the Hamiltonian operator, \mathbf{P} is the three-momentum operator, \mathbf{J} is the angular momentum operator, and \mathbf{K} is the generator of rotationless boosts.

In a Bakamjian-Thomas scheme the 10 generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ are expressed in terms of another set of ten Hermitian operators $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$ by means of the relations

$$H = (\mathbf{P}^2 + M^2)^{1/2}, \quad (2.18a)$$

$$\mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S}, \quad (2.18b)$$

$$\mathbf{K} = -\frac{1}{2}(\mathbf{X}H + H\mathbf{X}) - \frac{\mathbf{P} \times \mathbf{S}}{M + H}. \quad (2.18c)$$

Here M is the mass operator, \mathbf{S} is the spin operator, and \mathbf{X} is the Newton-Wigner [3,4] position operator. The advantage of the second set of operators over the generators is that they satisfy much simpler commutation rules. In particular the only nonzero commutators are given by the well-known commutation rules

$$[P_j, X_k] = -i\delta_{jk}, \quad [S_j, S_k] = i\varepsilon_{jkl}S_l. \quad (2.19)$$

The operators \mathbf{P} , \mathbf{S} , and \mathbf{X} are chosen to be the same as those for the system of particles without interactions, while the mass operator M contains interactions. The commutation rules for \mathbf{P} , \mathbf{S} , and \mathbf{X} are then automatically satisfied, and in order to guarantee Poincaré invariance it is only necessary to ensure that

$$[M, \mathbf{P}] = \mathbf{0}, \quad [M, \mathbf{S}] = \mathbf{0}, \quad [M, \mathbf{X}] = \mathbf{0}. \quad (2.20)$$

With this scheme the generators \mathbf{P} and \mathbf{J} are noninteracting, while H and \mathbf{K} contain interactions. This defines an instant form of relativistic quantum mechanics, since the Poincaré transformations constructed from the noninteracting generators map a Minkowski subspace $t = \text{const.}$ into itself.

The representatives of the noninteracting operators in the space spanned by the states (2.13) are given by

$$\begin{aligned} \langle \mathbf{p}klsjm | \mathbf{P} = \mathbf{p} \langle \mathbf{p}klsjm |, \quad \langle \mathbf{p}klsjm | \mathbf{X} = i\nabla_{\mathbf{p}} \langle \mathbf{p}klsjm |, \\ \langle \mathbf{p}klsjm | \mathbf{S}^2 = j(j+1) \langle \mathbf{p}klsjm |, \\ \langle \mathbf{p}klsjm | S_3 = m \langle \mathbf{p}klsjm |, \\ \langle \mathbf{p}klsjm | S_{\pm} = \langle \mathbf{p}klsjm | (S_1 \pm iS_2) \\ = [j(j+1) - m(m \mp 1)]^{1/2} \langle \mathbf{p}klsj, m \mp 1|. \end{aligned} \quad (2.21)$$

Using (2.21) and the conditions imposed by (2.20) we can show that the matrix elements of the mass operator must be of the form

$$\langle \mathbf{p}klsjm | M | \mathbf{p}'k'l's'j'm' \rangle = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{jj'} \delta_{mm'} M_{ls, l's'}^j(k, k'). \quad (2.22)$$

The operators M , \mathbf{P} , \mathbf{S}^2 , and S_3 commute with each other; therefore we can construct simultaneous eigenstates of them according to

$$M | \mathbf{q}Wjm \rangle = W | \mathbf{q}Wjm \rangle, \quad (2.23a)$$

$$\mathbf{P} | \mathbf{q}Wjm \rangle = \mathbf{q} | \mathbf{q}Wjm \rangle,$$

$$\mathbf{S}^2 | \mathbf{q}Wjm \rangle = j(j+1) | \mathbf{q}Wjm \rangle, \quad (2.23b)$$

$$S_3 | \mathbf{q}Wjm \rangle = m | \mathbf{q}Wjm \rangle.$$

If we write the mass eigenstates in the form

$$| \mathbf{q}Wjm \rangle = \sum_{ls} \int k^2 dk \phi_{ls}^{Wj}(k), \quad (2.24)$$

we see that they explicitly satisfy (2.23b). By putting this form into (2.23a), contracting with one of the states (2.13), and using (2.21) and (2.22), we find the integral equation

$$\sum_{l's'} \int M_{ls, l's'}^j(k, k') k'^2 dk' \phi_{l's'}^{Wj}(k') = W \phi_{ls}^{Wj}(k). \quad (2.25)$$

We note that this equation verifies that the wave function $\phi_{ls}^{Wj}(k)$ that appears in (2.24) does not depend on \mathbf{q} or m . Assuming that

$$\sum_{ls} \int_0^{\infty} \phi_{ls}^{Wj*}(k) \phi_{ls}^{Wj}(k) k^2 dk = \delta_{WW'}, \quad (2.26)$$

and using (2.13b), we find that the mass eigenstates are normalized according to

$$\langle \mathbf{q}Wjm | \mathbf{q}'W'j'm' \rangle = \delta^3(\mathbf{q} - \mathbf{q}') \delta_{WW'} \delta_{jj'} \delta_{mm'}. \quad (2.27)$$

Using (2.10), (2.24), (2.13a), and (2.12) we find that in the $| \mathbf{p}_1 m_1, \mathbf{p}_2 m_2 \rangle$ basis the mass eigenstates are given by

$$\begin{aligned} \Psi_{m_1 m_2}^{Wjm}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) \\ = \langle \mathbf{p}_1 m_1, \mathbf{p}_2 m_2 | \mathbf{q}Wjm \rangle \\ = \delta^3(\mathbf{p} - \mathbf{q}) \sum_{m'_1 m'_2} A_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(\mathbf{p}_1, \mathbf{p}_2) \psi_{m'_1 m'_2}^{Wjm}(\mathbf{k}), \end{aligned} \quad (2.28a)$$

$$\begin{aligned} A_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(\mathbf{p}_1, \mathbf{p}_2) \\ = \left[\frac{\varepsilon_1(\mathbf{k}) \varepsilon_2(-\mathbf{k})}{W(\mathbf{k})} \frac{E(\mathbf{p}, \mathbf{k})}{\varepsilon(\mathbf{p}_1) \varepsilon_2(\mathbf{p}_2)} \right]^{1/2} \\ \times D_{m_1 m'_1}^{(s_1)} \{r_c[l_c(p), k_1]\} D_{m_2 m'_2}^{(s_2)} \{r_c[l_c(p), k_2]\}, \end{aligned} \quad (2.28b)$$

$$\begin{aligned} \psi_{m_1 m_2}^{Wjm}(\mathbf{k}) \\ = \sum_{ls} \sum_{m_l m_s} \langle s_1 s_2 m_l m_s | s m_s \rangle \\ \times \langle l s m_l m_s | j m \rangle Y_l^{m_l}(\hat{\mathbf{k}}) \phi_{ls}^{Wj}(|\mathbf{k}|), \end{aligned} \quad (2.28c)$$

$$\begin{aligned} \mathbf{p} = (\varepsilon(\mathbf{p}_1) + \varepsilon_2(\mathbf{p}_2), \mathbf{p}_1 + \mathbf{p}_2) \\ = (E(\mathbf{p}, \mathbf{k}), \mathbf{p}) = (\sqrt{\mathbf{p}^2 + W^2(\mathbf{k})}, \mathbf{p}), \end{aligned} \quad (2.28d)$$

$$\begin{aligned} k_1 = l_c^{-1}(p) \varepsilon(\mathbf{p}_1, \mathbf{p}_1) = (\varepsilon_1(\mathbf{k}), \mathbf{k}), \\ k_2 = l_c^{-1}(p) \varepsilon(\mathbf{p}_2, \mathbf{p}_2) = (\varepsilon_2(-\mathbf{k}), -\mathbf{k}). \end{aligned} \quad (2.28e)$$

Equation (2.28a) gives the relation between $\Psi_{m_1 m_2}^{Wjm}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q})$, the wave function of the system in an arbitrary frame, and $\psi_{m_1 m_2}^{Wjm}(\mathbf{k})$, the rest frame wave function of the system. This equation plays an essential role in incorporating relativistic wave functions into the 3P_0 quark-pair creation model. For future reference we note that by inverting (2.10a) we can easily show that

$$\langle \mathbf{p}k m_1 m_2 | \mathbf{q}Wjm \rangle = \delta^3(\mathbf{p} - \mathbf{q}) \psi_{m_1 m_2}^{Wjm}(\mathbf{k}). \quad (2.29)$$

III. THE INTERACTION

The interaction in the quark-pair creation model can be derived from the quantum field theory Hamiltonian [33]

$$H_I = g \int d^3x \bar{\psi}(x)\psi(x), \quad t = 0, \quad (3.1)$$

where $\psi(x)$ is a Dirac field operator associated with the space-time point x . Since under a Lorentz transformation $x' = ax$, the interaction density $\bar{\psi}(x)\psi(x)$ is a Lorentz scalar function the above interaction can be said to be relativistic. The field operator can be expanded in the form

$$\begin{aligned} \psi(x) = & \sum_{r=1}^2 \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\varepsilon(\mathbf{p})}} [b_r(\mathbf{p})e^{-ip \cdot x} u_r(\mathbf{p}) \\ & + (-1)^{r+1} d_r^\dagger(\mathbf{p})e^{ip \cdot x} v_r(\mathbf{p})]. \end{aligned} \quad (3.2)$$

Here b_r^\dagger and d_r^\dagger create a quark and an antiquark, respectively. The corresponding spinors are given by

$$\begin{aligned} u_r(\mathbf{p}) &= \sqrt{\frac{\varepsilon(\mathbf{p}) + m}{2m}} \begin{bmatrix} \chi_r \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\varepsilon(\mathbf{p}) + m} \chi_r \end{bmatrix}, \\ v_r(\mathbf{p}) &= \sqrt{\frac{\varepsilon(\mathbf{p}) + m}{2m}} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\varepsilon(\mathbf{p}) + m} \chi'_r \\ \chi'_r \end{bmatrix}, \\ \chi_1 &= \chi'_2 = \chi_{1/2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \chi_2 = \chi'_1 = \chi_{-1/2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \quad (3.3a)$$

$$(3.3b)$$

The factor $(-1)^{r+1}$ that appears in (3.2) has been put there so that d_r^\dagger creates angular momentum states that follow the Condon-Shortley phase conventions [35]. With this factor the angular momentum operator is given by

$$\begin{aligned} \mathbf{J} &= \int d^3x \psi^\dagger(\mathbf{x}) \left[\mathbf{L}(\mathbf{x}) + \frac{1}{2} \boldsymbol{\sigma} \right] \psi(\mathbf{x}) \\ &= \int d^3p \sum_{r,s} \left\{ b_r^\dagger(\mathbf{p}) \left[\delta_{rs} \mathbf{L}(\mathbf{p}) + \frac{1}{2} \boldsymbol{\sigma}_{rs} \right] b_s(\mathbf{p}) \right. \\ &\quad \left. + d_r^\dagger(\mathbf{p}) \left[\delta_{rs} \mathbf{L}(\mathbf{p}) + \frac{1}{2} \boldsymbol{\sigma}_{rs} \right] d_s(\mathbf{p}) \right\}, \end{aligned} \quad (3.4a)$$

$$\mathbf{L}(\mathbf{p}) = i \nabla_{\mathbf{p}} \times \mathbf{p}. \quad (3.4b)$$

Thanks to the $(-1)^{r+1}$ factor the roles of the quark and antiquark creation and annihilation operators are completely symmetrical in their application to creating angular momentum eigenstates.

Putting (3.2) into (3.1) we find that the quark-pair creation part of the interaction Hamiltonian is given by

$$\begin{aligned} H_{\text{QPC}} &= g \sum_{r,s} \int d^3p_1 d^3p_2 \frac{m}{\sqrt{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)}} b_r^\dagger(\mathbf{p}_1) d_s^\dagger(\mathbf{p}_2) \\ &\quad \times \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \bar{u}_r(\mathbf{p}_1) v_s(\mathbf{p}_2) (-1)^{s+1}. \end{aligned} \quad (3.5)$$

By using (3.3) it is straightforward to show that

$$\begin{aligned} & \lim_{\mathbf{p}_1 + \mathbf{p}_2 \rightarrow 0} \sum_{r,s} m b_r^\dagger(\mathbf{p}_1) d_s^\dagger(\mathbf{p}_2) \bar{u}_r(\mathbf{p}_1) v_s(\mathbf{p}_2) (-1)^{s+1} \\ &= -\sqrt{\frac{8\pi}{3}} \left\{ \mathcal{Y}_1^{-1}(\mathbf{p}) b_1^\dagger(\mathbf{p}_1) d_1^\dagger(\mathbf{p}_2) - \frac{1}{\sqrt{2}} \mathcal{Y}_1^0(\mathbf{p}) \right. \\ &\quad \times [b_1^\dagger(\mathbf{p}_1) d_2^\dagger(\mathbf{p}_2) + b_2^\dagger(\mathbf{p}_1) d_1^\dagger(\mathbf{p}_2)] \\ &\quad \left. + \mathcal{Y}_1^1(\mathbf{p}) b_2^\dagger(\mathbf{p}_1) d_2^\dagger(\mathbf{p}_2) \right\}. \end{aligned} \quad (3.6a)$$

$$\mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2, \quad (3.6b)$$

where

$$\mathcal{Y}_1^m(\mathbf{p}) = \sqrt{\frac{3}{4\pi}} \boldsymbol{\varepsilon}_m \cdot \mathbf{p} = |\mathbf{p}| Y_1^m(\hat{\mathbf{p}}), \quad (3.7)$$

with $\boldsymbol{\varepsilon}_m$ a spherical unit vector defined by

$$\boldsymbol{\varepsilon}_\pm = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_1 \pm i \mathbf{e}_2), \quad \boldsymbol{\varepsilon}_0 = \mathbf{e}_3. \quad (3.8)$$

The result (3.6) can be written more compactly by introducing the Clebsch-Gordon coefficients $\langle 1, 1, m, -m | 0, 0 \rangle$ and $\langle 1/2, 1/2, m_1, m_2 | 1, -m \rangle$ and slightly changing notation. This leads to the following expression for the quark-pair creation interaction:

$$\begin{aligned} H_{\text{QPC}} &= -g \sqrt{8\pi} \sum_{r,s} \int \frac{d^3p_1 d^3p_2}{\sqrt{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)}} \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \\ &\quad \times \sum_m \sum_{m_1 m_2} \langle 1, 1, m, -m | 0, 0 \rangle \mathcal{Y}_1^m(\mathbf{p}) \\ &\quad \times \langle 1/2, 1/2, m_1, m_2 | 1, -m \rangle b_{m_1}^\dagger(\mathbf{p}_1) d_{m_2}^\dagger(\mathbf{p}_2). \end{aligned} \quad (3.9)$$

In order to apply this interaction to quarks we must introduce the flavor and color degrees of freedom. For the SU(3) flavor singlet ϕ_0 and the SU(3) color singlet ω_0 we write

$$\begin{aligned} \phi_0 &= (1/\sqrt{3})(-u\bar{u} + d\bar{d} + s\bar{s}), \\ \omega_0 &= (1/\sqrt{3})(-r\bar{r} + y\bar{y} + b\bar{b}). \end{aligned} \quad (3.10)$$

It should be noted that we are following deSwart's conventions [36] for the SU(3) Clebsch-Gordon coefficients. Numbering the quark flavor and color states 1, 2, 3, we can write

$$\begin{aligned} (\phi_0)_{i_1 i_2} &= \eta_{i_1 i_2} = \delta_{i_1 i_2} \eta_{i_1} / \sqrt{3}, \\ (\omega_0)_{j_1 j_2} &= \eta_{j_1 j_2} = \delta_{j_1 j_2} \eta_{j_1} / \sqrt{3}, \\ \eta_1 &= -1, \quad \eta_2 = \eta_3 = 1. \end{aligned} \quad (3.11)$$

We now take for our quark-pair creation operator

$$\begin{aligned} T &= -2m_q \gamma \sum_{i_1 i_2} \sum_{j_1 j_2} \int \frac{d^3p_1 d^3p_2}{\sqrt{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)}} \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \\ &\quad \times \sum_m \sum_{m_1 m_2} \langle 1, 1, m, -m | 0, 0 \rangle \mathcal{Y}_1^m \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right) \\ &\quad \times \langle 1/2, 1/2, m_1, m_2 | 1, -m \rangle \\ &\quad \times b_{m_1 i_1 j_1}^\dagger(\mathbf{p}_1) \eta_{i_1 i_2} \eta_{j_1 j_2} d_{m_2 i_2 j_2}^\dagger(\mathbf{p}_2), \end{aligned} \quad (3.12)$$

where m_q is a quark mass and γ is a dimensionless strength parameter.

IV. MESON STATE VECTORS

To specify the state vectors of the π and ρ mesons we must add color and flavor variables to the relativistic wave function (2.28). The color singlet for these mesons is the ω_0 given by (3.10) and (3.11). The flavor states for them are the same and are given by

$$f_1 = u\bar{d}, \quad f_0 = (1/\sqrt{2})(u\bar{u} + d\bar{d}), \quad f_{-1} = d\bar{u}, \quad (4.1)$$

which describe particles with isospin one. u and \bar{d} have isospin components $i = 1/2$ while d and \bar{u} have isospin $i = -1/2$. If we let i_1 and i_2 designate the isospin components of particles 1 and 2, we can write

$$(f_t)_{i_1 i_2} = \langle 1/2, 1/2, i_1, i_2 | 1t \rangle, \quad (4.2)$$

$$t = \pm 1, 0.$$

Since our meson states involve only up and down quarks and antiquarks the $s\bar{s}$ term in (3.10) makes no contribution, so in (3.11) and (3.12) we can make the replacements

$$\phi_0 \rightarrow -\sqrt{\frac{2}{3}} \left(\frac{1}{\sqrt{2}} u\bar{u} - \frac{1}{\sqrt{2}} d\bar{d} \right), \quad (4.3)$$

$$(\phi_0)_{i_1 i_2} = \eta_{i_1 i_2} \rightarrow -\sqrt{2/3} \langle 1/2, 1/2, i_1, i_2 | 0, 0 \rangle. \quad (4.4)$$

Adapting the wave function (2.28) by adding the color and flavor variables, we can specify the π and ρ state vectors by writing

$$\begin{aligned} \pi_{m_1 i_1 j_1, m_2 i_2 j_2}^t(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) \\ = \delta^3(\mathbf{p} - \mathbf{q}) \sum_{m'_1 m'_2} A_{m_1 m_2, m'_1 m'_2}^{(1/2, 1/2)}(\mathbf{p}_1, \mathbf{p}_2) \langle 1/2, 1/2, m'_1, m'_2 | 0, 0 \rangle \\ \times \langle 1/2, 1/2, i_1, i_2 | 1, t \rangle \eta_{j_1 j_2} \phi(|\mathbf{k}|) / \sqrt{4\pi}, \end{aligned} \quad (4.5a)$$

$$\begin{aligned} \rho_{m_1 i_1 j_1, m_2 i_2 j_2}^{mt}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) \\ = \delta^3(\mathbf{p} - \mathbf{q}) \sum_{m'_1 m'_2} A_{m_1 m_2, m'_1 m'_2}^{(1/2, 1/2)}(\mathbf{p}_1, \mathbf{p}_2) \langle 1/2, 1/2, m'_1, m'_2 | 1, m \rangle \\ \times \langle 1/2, 1/2, i_1, i_2 | 1, t \rangle \eta_{j_1 j_2} \chi(|\mathbf{k}|) / \sqrt{4\pi}. \end{aligned} \quad (4.5b)$$

In order to condense our notation we let α be a cover index for the indices m, i , and j , i.e., $\alpha = \{m, i, j\}$. Using this notation, along with (2.12), (2.26), (2.28b), and (3.11), we can easily show that the orthonormality relations of the π and ρ state vectors are given by

$$\begin{aligned} \langle \pi \mathbf{q} t | \pi \mathbf{q}' t' \rangle &= \sum_{\alpha_1 \alpha_2} \int d^3 p_1 d^3 p_2 \\ &\times \pi_{\alpha_1 \alpha_2}^{t_1*}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) \pi_{\alpha_1 \alpha_2}^{t_1'}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}') \\ &= \delta^3(\mathbf{q} - \mathbf{q}') \delta_{tt'}, \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \langle \rho \mathbf{q} m t | \rho \mathbf{q}' m' t' \rangle &= \sum_{\alpha_1 \alpha_2} \int d^3 p_1 d^3 p_2 \\ &\times \rho_{\alpha_1 \alpha_2}^{m t*}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) \rho_{\alpha_1 \alpha_2}^{m' t'}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}') \\ &= \delta^3(\mathbf{q} - \mathbf{q}') \delta_{mm'} \delta_{tt'}, \end{aligned} \quad (4.6b)$$

$$\langle \pi \mathbf{q} t | \rho \mathbf{q}' m' t' \rangle = 0. \quad (4.6c)$$

In order to use the π and ρ wave functions in conjunction with the quark-creation operator (3.12) we write the $\pi\pi$ state vector and the ρ state vector in second quantized form, i.e.,

$$\begin{aligned} |\pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2\rangle &= (1/\sqrt{2}) \sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \\ &\times b_{\alpha_1}^\dagger(\mathbf{p}_1) d_{\alpha_2}^\dagger(\mathbf{p}_2) b_{\alpha_3}^\dagger(\mathbf{p}_3) d_{\alpha_4}^\dagger(\mathbf{p}_4) |0\rangle \\ &\times \pi_{\alpha_1 \alpha_2}^{t_1}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1) \pi_{\alpha_3 \alpha_4}^{t_2}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_2), \end{aligned} \quad (4.7a)$$

$$\begin{aligned} |\rho \mathbf{q} m t\rangle &= \sum_{\alpha_1 \alpha_2} \int d^3 p_1 d^3 p_2 b_{\alpha_1}^\dagger(\mathbf{p}_1) d_{\alpha_2}^\dagger(\mathbf{p}_2) |0\rangle \\ &\times \rho_{\alpha_1 \alpha_2}^{m t}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}). \end{aligned} \quad (4.7b)$$

The normalization factor $(1/\sqrt{2})$ in (4.7a) requires some justification. If we attempt to evaluate the inner product $\langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | \pi \mathbf{q}_3 t_3, \pi \mathbf{q}_4 t_4 \rangle$ we encounter in the integrand the expression

$$\begin{aligned} \pi_{\alpha_1 \alpha_2}^{t_1*}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1) \pi_{\alpha_3 \alpha_4}^{t_2*}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_2) [e + (13)(24) - (13) \\ - (24)] \pi_{\alpha_1 \alpha_2}^{t_3}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_3) \pi_{\alpha_3 \alpha_4}^{t_4}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_4), \end{aligned} \quad (4.8)$$

where the permutations act on the subscripts of the α 's and \mathbf{p} 's, but not on those of the t 's and \mathbf{q} 's. The permutation (13)(24) switches the quarks and antiquarks between the two pions, while (13) switches just the quarks, and (24) switches just the antiquarks. If we envision the final-state pions in a configuration-space, time-dependent picture, we recognize that the pions are far apart and receding from each other. The issue of antisymmetrizing between two nonoverlapping systems is discussed in a number of references [21,37,38]. Here we consider a simple argument. If we ignore relativity for the moment and let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, and \mathbf{r}_4 be the position vectors corresponding to the \mathbf{p} 's in (4.7a) then the bound-state, quark-antiquark wave functions are significant only when $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and $r_{34} = |\mathbf{r}_3 - \mathbf{r}_4|$ are small. When, for example, we switch quark 1 and quark 3, then $r_{12} \rightarrow r_{32} = |\mathbf{r}_3 - \mathbf{r}_2|$ and $r_{34} \rightarrow r_{14} = |\mathbf{r}_1 - \mathbf{r}_4|$, where both r_{32} and r_{14} are large, so the quark-antiquark wave functions become vanishingly small. As a result of this the product $\pi_{\alpha_1 \alpha_2}^{t_1*}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{q}_1) \pi_{\alpha_3 \alpha_4}^{t_2*}(\mathbf{r}_3, \mathbf{r}_4; \mathbf{q}_2) \pi_{\alpha_3 \alpha_2}^{t_3}(\mathbf{r}_3, \mathbf{r}_2; \mathbf{q}_3) \pi_{\alpha_1 \alpha_4}^{t_4}(\mathbf{r}_1, \mathbf{r}_4; \mathbf{q}_4)$ essentially vanishes, and similarly for the (24) switch. The upshot of this is that in determining the normalization factor for the pion-pion final state only the first two terms in (4.8) are taken into account. Using (4.6a) we

find

$$\begin{aligned}
(1/2) & \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \pi_{\alpha_1\alpha_2}^{t_1^*}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1) \pi_{\alpha_3\alpha_4}^{t_2^*}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_2) \\
& \times [\pi_{\alpha_1\alpha_2}^{t_3}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_3) \pi_{\alpha_3\alpha_4}^{t_4}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_4) + \pi_{\alpha_1\alpha_2}^{t_4}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_4) \pi_{\alpha_3\alpha_4}^{t_3}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_3)] \\
& = (1/2) [\delta^3(\mathbf{q}_1 - \mathbf{q}_3) \delta_{t_1 t_3} \delta^3(\mathbf{q}_2 - \mathbf{q}_4) \delta_{t_2 t_4} + \delta^3(\mathbf{q}_1 - \mathbf{q}_4) \delta_{t_1 t_4} \delta^3(\mathbf{q}_2 - \mathbf{q}_3) \delta_{t_2 t_3}].
\end{aligned} \tag{4.9}$$

The factor (1/2) takes care of the fact that there are two contributions to the normalization of the state.

V. THE $\rho \rightarrow \pi\pi$ AMPLITUDE

Combining (3.12), (4.4), and (4.7) we find, for the $\rho \rightarrow \pi\pi$ amplitude,

$$\begin{aligned}
\langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{q} m t \rangle & = (2m_q \gamma / \sqrt{3}) \sum_{m_1 m_2 m_3 m_4 i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \{ [e + (13)(24) - (13) - (24)] \\
& \times \pi_{m_1 i_1 j_1, m_2 i_2 j_2}^{t_1^*}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1) \pi_{m_3 i_3 j_3, m_4 i_4 j_4}^{t_2^*}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_2) \} \frac{\delta^3(\mathbf{p}_1 + \mathbf{p}_2)}{\sqrt{\varepsilon(\mathbf{p}_1) \varepsilon(\mathbf{p}_2)}} \sum_n \langle 1, 1, n, -n | 0, 0 \rangle \\
& \times \mathcal{Y}_1^n \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right) \langle 1/2, 1/2, m_1, m_2 | 1, -n \rangle \langle 1/2, 1/2, i_1, i_2 | 0, 0 \rangle \eta_{j_1 j_2} \rho_{m_3 i_3 j_3, m_4 i_4 j_4}^{m t}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}).
\end{aligned}$$

As a result of the orthogonality relation (4.6c), the e and the (13)(24) terms drop out. We now consider the (13) term. If we switch the indices on the \mathbf{p} 's, i 's, and j 's according to (12)(34), and use (4.5) and (2.83), we find that the (13) term transforms into the (24) term, so we can replace $[e + (13)(24) - (13) - (24)]$ with $[-2(24)]$, leading to

$$\begin{aligned}
\langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{q} m t \rangle & = -(4m_q \gamma / \sqrt{3}) \sum_{m_1 m_2 m_3 m_4 i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \pi_{m_1 i_1 j_1, m_4 i_4 j_4}^{t_1^*}(\mathbf{p}_1, \mathbf{p}_4; \mathbf{q}_1) \\
& \times \pi_{m_3 i_3 j_3, m_2 i_2 j_2}^{t_2^*}(\mathbf{p}_3, \mathbf{p}_2; \mathbf{q}_2) \frac{\delta^3(\mathbf{p}_1 + \mathbf{p}_2)}{\sqrt{\varepsilon(\mathbf{p}_1) \varepsilon(\mathbf{p}_2)}} \sum_n \langle 1, 1, n, -n | 0, 0 \rangle \mathcal{Y}_1^n \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right) \\
& \times \langle 1/2, 1/2, m_1, m_2 | 1, -n \rangle \langle 1/2, 1/2, i_1, i_2 | 0, 0 \rangle \eta_{j_1 j_2} \rho_{m_3 i_3 j_3, m_4 i_4 j_4}^{m t}(\mathbf{p}_3, \mathbf{p}_4; \mathbf{q}).
\end{aligned} \tag{5.1}$$

With the help of (3.11) and (4.5), we find that the color factor in (5.1) is given by

$$\sum_{j_1 j_2 j_3 j_4} \eta_{j_1 j_4} \eta_{j_3 j_2} \eta_{j_1 j_2} \eta_{j_3 j_4} = 1/3. \tag{5.2}$$

Using (4.5) and (3.8), we find that the flavor factor is given by

$$\begin{aligned}
& \sum_{i_1 i_2 i_3 i_4} \langle 1/2, 1/2, i_1, i_4 | 1, t_1 \rangle \langle 1/2, 1/2, i_3, i_2 | 1, t_2 \rangle \\
& \times \langle 1/2, 1/2, i_1, i_2 | 0, 0 \rangle \langle 1/2, 1/2, i_3, i_4 | 1, t \rangle \\
& = \frac{i}{2} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \mathbf{e}_t.
\end{aligned} \tag{5.3}$$

According to (5.1), the total and relative momentum variables that appear in the meson wave functions (4.5a) and (4.5b), are defined by

$$p_{ij} = (\varepsilon(\mathbf{p}_i) + \varepsilon(\mathbf{p}_j), \mathbf{p}_i + \mathbf{p}_j) = (E(\mathbf{p}_{ij}, \mathbf{k}_{ij}), \mathbf{p}_{ij}), \tag{5.4a}$$

$$k_{ij} = (\varepsilon(\mathbf{k}_{ij}), \mathbf{k}_{ij}) = l_c^{-1}(p_{ij})(\varepsilon(\mathbf{p}_i), \mathbf{p}_i),$$

$$k'_{ij} = (\varepsilon(-\mathbf{k}_{ij}), -\mathbf{k}_{ij}) = l_c^{-1}(p_{ij})(\varepsilon(\mathbf{p}_j), \mathbf{p}_j). \tag{5.4b}$$

We can use (2.12) to replace the integration elements in (5.1) according to

$$\begin{aligned}
& d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \\
& \rightarrow \frac{W(\mathbf{k}_{14})}{\varepsilon(\mathbf{k}_{14}) \varepsilon(-\mathbf{k}_{14})} \frac{\varepsilon(\mathbf{p}_1) \varepsilon(\mathbf{p}_4)}{E(\mathbf{p}_{14}, \mathbf{k}_{14})} d^3 p_{14} d^3 k_{14} d^3 p_2 d^3 p_3.
\end{aligned} \tag{5.5}$$

We now assume the ρ meson is at rest, i.e., $\mathbf{q} = \mathbf{0}$. With the help of (4.5b), (2.26), and (2.5), we find that the rest frame ρ wave function is simply

$$\begin{aligned}
& \rho_{m_3 m_4}^m(\mathbf{p}_3, \mathbf{p}_4; \mathbf{0}) \\
& = \delta^3(\mathbf{p}_3 + \mathbf{p}_4) \langle 1/2, 1/2, m_3, m_4 | 1, m \rangle \chi(|-\mathbf{p}_4|) / \sqrt{4\pi},
\end{aligned} \tag{5.6}$$

where we have removed the color and flavor factors. If we integrate out \mathbf{p}_{14} , \mathbf{p}_2 , and \mathbf{p}_3 , and use the inverse of the Lorentz transformation (2.3) to express \mathbf{k}_{14} and \mathbf{k}_{32} in terms of the \mathbf{p}_i 's, we find that

$$\begin{aligned}
\mathbf{p}_2 & \rightarrow -\mathbf{p}_1, \quad \mathbf{p}_3 \rightarrow -\mathbf{p}_4, \quad p_{32} \rightarrow (p_{14}^0, -\mathbf{p}_{14}), \quad \mathbf{k}_{32} \rightarrow \mathbf{k}_{14}, \\
p_{14} & \rightarrow (E(\mathbf{q}_1, \mathbf{k}_{14}), \mathbf{q}_1), \quad p_{32} \rightarrow (E(\mathbf{q}_2, \mathbf{k}_{14}), \mathbf{q}_2).
\end{aligned} \tag{5.7}$$

We now let $\mathbf{k} = \mathbf{k}_{14}$, $\mathbf{q} = \mathbf{q}_1$, and $\mathbf{p}_1 \rightarrow \mathbf{p}$ and define

$$q_1 = (E(\mathbf{q}, \mathbf{k}), \mathbf{q}), \quad q_2 = (E(-\mathbf{q}, -\mathbf{k}), -\mathbf{q}), \quad (5.8a)$$

$$k_1 = (\varepsilon(\mathbf{k}), \mathbf{k}), \quad k_2 = (\varepsilon(-\mathbf{k}), -\mathbf{k}), \quad (5.8b)$$

$$\varepsilon(\mathbf{p}) = \frac{E(\mathbf{q}, \mathbf{k})\varepsilon(\mathbf{k}) + \mathbf{q} \cdot \mathbf{k}}{W(\mathbf{k})}, \quad \mathbf{p} = \mathbf{k} + \left[\varepsilon(\mathbf{k}) + \frac{\mathbf{q} \cdot \mathbf{k}}{E(\mathbf{q}, \mathbf{k}) + W(\mathbf{k})} \right] \frac{\mathbf{q}}{W(\mathbf{k})}. \quad (5.8c)$$

With the help of these relations, as well as (2.28b) and (4.5), we find that the amplitude (5.1) can be written in the form

$$\begin{aligned} \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \left[-\frac{i2m_q \mathcal{V}}{(12\pi)^{3/2}} \right] (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t \sum_{m_1 m_2 m_3 m_4 n_1 n_2 n_3 n_4} \int \frac{d^3 k}{\varepsilon(\mathbf{p})} \phi^2(|\mathbf{k}|) \chi(|\mathbf{p} - \mathbf{q}|) \\ &\times \langle 1/2, 1/2, n_1, n_4 | 0, 0 \rangle D_{n_1 m_1}^{(1/2)} \{ r_c^{-1}[l_c(q_1), k_1] \} D_{n_4 m_4}^{(1/2)} \{ r_c^{-1}[l_c(q_1), k_2] \} \\ &\times \langle 1/2, 1/2, n_3, n_2 | 0, 0 \rangle D_{n_3 m_3}^{(1/2)} \{ r_c^{-1}[l_c(q_2), k_1] \} D_{n_2 m_2}^{(1/2)} \{ r_c^{-1}[l_c(q_2), k_2] \} \\ &\times \sum_n \langle 1, 1, n, -n | 0, 0 \rangle \mathcal{Y}_1^n(\mathbf{p}) \langle 1/2, 1/2, m_1, m_2 | 1, -n \rangle \langle 1/2, 1/2, m_3, m_4 | 1, m \rangle. \end{aligned} \quad (5.9)$$

Using the results from the Appendix, we can write for the Wigner rotations that appear in (5.9)

$$D^{(1/2)} \{ r_c^{-1}[l_c(q_1), k_1] \} = D^{(1/2)} \{ r_c^{-1}[l_c(q_2), k_2] \} = \exp(-i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2) = D^{(1/2)}(\boldsymbol{\theta}), \quad \boldsymbol{\theta} = \zeta(\mathbf{q}, \mathbf{k}) \mathbf{u}(\mathbf{q}, \mathbf{k}), \quad (5.10a)$$

$$D^{(1/2)} \{ r_c^{-1}[l_c(q_1), k_2] \} = D^{(1/2)} \{ r_c^{-1}[l_c(q_2), k_1] \} = \exp(-i\boldsymbol{\xi} \cdot \boldsymbol{\sigma}/2) = D^{(1/2)}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} = -\zeta(\mathbf{q}, -\mathbf{k}) \mathbf{u}(\mathbf{q}, \mathbf{k}), \quad (5.10b)$$

where ζ and \mathbf{u} are given by (A9) and (A8c), respectively. Clearly the D 's in (5.10) are the spin-1/2 irreducible representations for the Wigner rotations. We also need the spin-1, irreducible representations for these rotations. The spin-1 D 's can be expressed in terms of the spin-1 matrix given by

$$\mathbf{S} = \begin{bmatrix} \boldsymbol{\varepsilon}_0 & \boldsymbol{\varepsilon}_{-1} & \mathbf{0} \\ -\boldsymbol{\varepsilon}_1 & \mathbf{0} & \boldsymbol{\varepsilon}_{-1} \\ \mathbf{0} & -\boldsymbol{\varepsilon}_1 & -\boldsymbol{\varepsilon}_0 \end{bmatrix}, \quad (5.11)$$

which has the convenient property

$$(\mathbf{S} \cdot \mathbf{u})^3 = \mathbf{S} \cdot \mathbf{u}. \quad (5.12)$$

By using this property, it is straightforward to show that the spin-1 D is given by

$$\begin{aligned} D^{(1)}(\boldsymbol{\psi}) &= D^{(1)}(\boldsymbol{\psi} \mathbf{u}) \\ &= \exp(-i\boldsymbol{\psi} \mathbf{S} \cdot \mathbf{u}) \\ &= 1 - i(\mathbf{S} \cdot \mathbf{u}) \sin(\psi) - (\mathbf{S} \cdot \mathbf{u})^2 [1 - \cos(\psi)]. \end{aligned} \quad (5.13)$$

We now turn our attention to carrying out the sums that appear in (5.9). Putting (5.10) into (5.9) we are led to define

$$\begin{aligned} B_{nm} &= \sum_{n_1 n_2 n_3 n_4} \langle 1/2, 1/2, n_1, n_4 | 0, 0 \rangle \langle 1/2, 1/2, n_3, n_2 | 0, 0 \rangle \\ &\times \sum_{m_1 m_2} D_{n_1 m_1}^{(1/2)}(\boldsymbol{\theta}) D_{n_2 m_2}^{(1/2)}(\boldsymbol{\theta}) \langle 1/2, 1/2, m_1, m_2 | 1, -n \rangle \\ &\times \sum_{m_3 m_4} D_{n_3 m_3}^{(1/2)}(\boldsymbol{\xi}) D_{n_4 m_4}^{(1/2)}(\boldsymbol{\xi}) \langle 1/2, 1/2, m_3, m_4 | 1, m \rangle. \end{aligned} \quad (5.14)$$

Using the well-known identity [see (2.14)]

$$\begin{aligned} D_{m_1 m_1'}^{(j_1)} D_{m_2 m_2'}^{(j_2)} &= \sum_{J M M'} \langle j_1, j_2, m_1, m_2 | J, M \rangle D_{M M'}^{(J)} \\ &\times \langle j_1, j_2, m_1', m_2' | J, M' \rangle, \end{aligned} \quad (5.15)$$

we can rewrite (5.14) in the form

$$\begin{aligned} B_{nm} &= \sum_{n_1 n_2 n_3 n_4 n' m'} \langle 1/2, 1/2, n_1, n_4 | 0, 0 \rangle \\ &\times \langle 1/2, 1/2, n_3, n_2 | 0, 0 \rangle \langle 1/2, 1/2, n_1, n_2 | 1, -n' \rangle \\ &\times \langle 1/2, 1/2, n_3, n_4 | 1, m' \rangle D_{-n', -n}^{(1)}(\boldsymbol{\theta}) D_{m' m}^{(1)}(\boldsymbol{\xi}). \end{aligned} \quad (5.16)$$

A straightforward evaluation of the Clebsch-Gordon coefficients in (5.16) leads to the result

$$B_{nm} = (1/2) \sum_a (-1)^a D_{-a, -n}^{(1)}(\boldsymbol{\theta}) D_{am}^{(1)}(\boldsymbol{\xi}). \quad (5.17)$$

With the help of the identity [37]

$$D_{M M'}^{(J)*} = (-1)^{M-M'} D_{-M, -M'}^{(J)}, \quad (5.18)$$

we can further simplify (5.17) to the result

$$\begin{aligned} B_{nm} &= (1/2) (-1)^n [D^{(1)\dagger}(\boldsymbol{\theta}) D^{(1)}(\boldsymbol{\xi})]_{nm} \\ &= (1/2) (-1)^n \{ \exp[i(\theta + \xi) \mathbf{S} \cdot \mathbf{u}] \}_{nm} \\ &= (1/2) (-1)^n \{ I + i(\mathbf{S} \cdot \mathbf{u}) \sin(\theta + \xi) - (\mathbf{S} \cdot \mathbf{u})^2 \\ &\quad \times [1 - \cos(\theta + \xi)] \}_{nm}, \end{aligned} \quad (5.19a)$$

$$\boldsymbol{\theta} = \zeta(\mathbf{q}, \mathbf{k}), \quad \boldsymbol{\xi} = \zeta(\mathbf{q}, -\mathbf{k}), \quad \mathbf{u} = \mathbf{u}(\mathbf{q}, \mathbf{k}). \quad (5.19b)$$

According to (5.9) we now need to calculate

$$\begin{aligned} C_m &= \sum_n \langle 1, 1, n, -n | 0, 0 \rangle \mathcal{Y}_1^n(\mathbf{p}) B_{nm} \\ &= (1/\sqrt{4\pi}) \sum_n (-1)^{n+1} (\boldsymbol{\varepsilon}_n \cdot \mathbf{p}) B_{nm}, \end{aligned} \quad (5.20)$$

where we have used (3.7). Using the identity

$$\mathbf{a} \times \mathbf{b} = i \det \begin{bmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_0 & \boldsymbol{\varepsilon}_{-1} \\ \boldsymbol{\varepsilon}_1 \cdot \mathbf{a} & \boldsymbol{\varepsilon}_0 \cdot \mathbf{a} & \boldsymbol{\varepsilon}_{-1} \cdot \mathbf{a} \\ \boldsymbol{\varepsilon}_1 \cdot \mathbf{b} & \boldsymbol{\varepsilon}_0 \cdot \mathbf{b} & \boldsymbol{\varepsilon}_{-1} \cdot \mathbf{b} \end{bmatrix}, \quad (5.21)$$

we can derive the results

$$\sum_n (\boldsymbol{\varepsilon}_n \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{u})_{nm} = i \boldsymbol{\varepsilon}_m \cdot (\mathbf{p} \times \mathbf{u}), \quad (5.22a)$$

$$\sum_n (\boldsymbol{\varepsilon}_n \cdot \mathbf{p})[(\mathbf{S} \cdot \mathbf{u})^2]_{nm} = \boldsymbol{\varepsilon}_m \cdot [\mathbf{u} \times (\mathbf{p} \times \mathbf{u})] = \boldsymbol{\varepsilon}_m \cdot \mathbf{p}. \quad (5.22b)$$

In deriving (5.22b) we used the fact that $\mathbf{u} \cdot \mathbf{p} = 0$, which follows from (A8c) and (5.8c). Using these results in (5.20) we obtain

$$C_m = -(1/\sqrt{16\pi})\boldsymbol{\varepsilon}_m \cdot [\mathbf{p} \cos(\theta + \xi) + (\mathbf{u} \times \mathbf{p}) \sin(\theta + \xi)]. \quad (5.23)$$

Combining the above results, we find that (5.9) becomes

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{im_q \gamma}{48\sqrt{3}\pi^2} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t \\ & \times \int \frac{d^3 k}{\varepsilon(\mathbf{p})} \phi^2(|\mathbf{k}|) \chi(|\mathbf{p} - \mathbf{q}|) \\ & \times \boldsymbol{\varepsilon}_m \cdot \{ \mathbf{p} \cos[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})] + (\mathbf{u} \times \mathbf{p}) \\ & \times \sin[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})] \}, \\ & \mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (5.24)$$

With the help of (A9) and standard trigonometric identities, we can easily show that

$$\cos[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})] = 2 \frac{[E(\mathbf{q}, \mathbf{k})m + W(\mathbf{k})\varepsilon(\mathbf{k})]^2}{[E(\mathbf{q}, \mathbf{k})\varepsilon(\mathbf{k}) + W(\mathbf{k})m]^2 - (\mathbf{q} \cdot \mathbf{k})^2} - 1, \quad (5.25a)$$

$$\sin[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})] = 2 \frac{[\mathbf{q}^2 \mathbf{k}^2 - (\mathbf{q} \cdot \mathbf{k})^2]^{1/2} [E(\mathbf{q}, \mathbf{k})m + W(\mathbf{k})\varepsilon(\mathbf{k})]}{[E(\mathbf{q}, \mathbf{k})\varepsilon(\mathbf{k}) + W(\mathbf{k})m]^2 - (\mathbf{q} \cdot \mathbf{k})^2}. \quad (5.25b)$$

Putting these results, along with (A8c) and (5.8c), into (5.24), we are led to define

$$F(\mathbf{q}, \mathbf{k}) = \frac{\phi^2(|\mathbf{k}|)\chi(|\mathbf{p} - \mathbf{q}|)}{\varepsilon(\mathbf{p})} \left\{ \cos[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})] + \frac{(\mathbf{p} \cdot \mathbf{q}) \sin[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})]}{[\mathbf{q}^2 \mathbf{k}^2 - (\mathbf{q} \cdot \mathbf{k})^2]^{1/2}} \right\}, \quad (5.26a)$$

$$G(\mathbf{q}, \mathbf{k}) = \frac{\phi^2(|\mathbf{k}|)\chi(|\mathbf{p} - \mathbf{q}|)}{\varepsilon(\mathbf{p})} \left\{ \left[\varepsilon(\mathbf{k}) + \frac{\mathbf{q} \cdot \mathbf{k}}{E(\mathbf{q}, \mathbf{k}) + W(\mathbf{k})} \right] \times \frac{\cos[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})]}{W(\mathbf{k})} - \frac{(\mathbf{p} \cdot \mathbf{k}) \sin[\zeta(\mathbf{q}, \mathbf{k}) + \zeta(\mathbf{q}, -\mathbf{k})]}{[\mathbf{q}^2 \mathbf{k}^2 - (\mathbf{q} \cdot \mathbf{k})^2]^{1/2}} \right\}. \quad (5.26b)$$

Now we can rewrite (5.24) as

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{im_q \gamma}{48\sqrt{3}\pi^2} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t \\ & \times \int d^3 k \boldsymbol{\varepsilon}_m \cdot [F(\mathbf{q}, \mathbf{k})\mathbf{k} + G(\mathbf{q}, \mathbf{k})\mathbf{q}], \quad \mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (5.27)$$

With the help of (5.8c) we see that $F(\mathbf{q}, \mathbf{k})$ and $G(\mathbf{q}, \mathbf{k})$ are actually only functions of $q = |\mathbf{q}|$, $k = |\mathbf{k}|$, and $x = \hat{\mathbf{q}} \cdot \hat{\mathbf{k}}$. Accordingly, we can write

$$F(\mathbf{q}, \mathbf{k}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l F_l(q, k) Y_l^{m*}(\hat{\mathbf{k}}) \frac{4\pi}{2l+1} Y_l^m(\hat{\mathbf{q}}), \quad (5.28a)$$

$$F_l(q, k) = \frac{2l+1}{2} \int_{-1}^1 dx P_l(x) F(\mathbf{q}, \mathbf{k}),$$

$$G(\mathbf{q}, \mathbf{k}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l G_l(q, k) Y_l^{m*}(\hat{\mathbf{k}}) \frac{4\pi}{2l+1} Y_l^m(\hat{\mathbf{q}}),$$

$$G_l(q, k) = \frac{2l+1}{2} \int_{-1}^1 dx P_l(x) G(\mathbf{q}, \mathbf{k}). \quad (5.28b)$$

Using these expressions in (5.27), along with (3.7), and integrating over the direction of \mathbf{k} , we find that

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{im_q \gamma}{48\sqrt{3}\pi^2} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t (\boldsymbol{\varepsilon}_m \cdot \mathbf{q}) E(q), \end{aligned} \quad (5.29a)$$

$$\mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2, \quad q = |\mathbf{q}|, \quad x = \hat{\mathbf{q}} \cdot \hat{\mathbf{k}},$$

$$E(q) = \int_0^{\infty} dk k^2 \int_{-1}^1 2\pi dx [xF(\mathbf{q}, \mathbf{k})(k/q) + G(\mathbf{q}, \mathbf{k})]. \quad (5.29b)$$

It is worth noting that $\int_{-1}^1 dx x F(\mathbf{q}, \mathbf{k})$ vanishes as q goes to zero, so the integral in (5.29b) is well defined in this limit. For the physical process $\rho \rightarrow \pi\pi$ the magnitude of the final momentum of each pion is given by

$$q_\rho = [(m_\rho/2)^2 - m_\pi^2]^{1/2}. \quad (5.30)$$

We define a form factor normalized to one at $q = q_\rho$ by

$$\Lambda(q) = E(q)/E(q_\rho). \quad (5.31)$$

With this definition (5.29) becomes

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{im_q \gamma E(q_\rho)}{48\sqrt{3}\pi^2} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t (\boldsymbol{\varepsilon}_m \cdot \mathbf{q}) \Lambda(q), \\ & \mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (5.32)$$

It is instructive to compare this result with the one obtained from the effective $\rho\pi\pi$ interaction Lagrangian density and the corresponding Hamilton given by

$$\begin{aligned} L_{\rho\pi\pi}(x) &= -g_{\rho\pi\pi} [\boldsymbol{\pi}(x) \times \partial_\mu \boldsymbol{\pi}(x)] \cdot \boldsymbol{\rho}^\mu(x), \\ H_{\rho\pi\pi} &= - \int d^3 x L_{\rho\pi\pi}(x)|_{t=0}. \end{aligned} \quad (5.33)$$

A straightforward calculation leads to the amplitude

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | H_{\rho\pi\pi} | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{i g_{\rho\pi\pi}}{(2\pi)^3 2\omega_\pi(\mathbf{q}) \sqrt{m_\rho}} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \cdot \boldsymbol{\varepsilon}_t (\boldsymbol{\varepsilon}_m \cdot \mathbf{q}), \end{aligned} \quad (5.34)$$

Comparing this result with (5.32) at $q = q_\rho$, we see that we can rewrite (5.32) as

$$\begin{aligned} \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T | \rho \mathbf{0} m t \rangle &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{i g_{\rho\pi\pi}}{(2\pi m_\rho)^{3/2}} (\boldsymbol{\varepsilon}_{t_1}^* \times \boldsymbol{\varepsilon}_{t_2}^*) \\ &\quad \cdot \boldsymbol{\varepsilon}_t (\boldsymbol{\varepsilon}_m \cdot \mathbf{q}) \Lambda(q), \\ \mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (5.35)$$

VI. MESON MASS OPERATOR

In order to construct a mass operator for the π and ρ mesons we need representatives of the various angular momentum operators. From (2.9) it follows that

$$\langle \mathbf{p} k m_1 m_2 | U^\dagger(r) = \sum_{m'_1 m'_2} D_{m_1 m'_1}^{(1/2)}(r^{-1}) D_{m_2 m'_2}^{(1/2)}(r^{-1}) \langle r \mathbf{p}, r \mathbf{k}, m'_1 m'_2 |, \quad (6.1)$$

where the relevant representations for the rotations are given by

$$\begin{aligned} r(\boldsymbol{\psi}) &= \exp(-i \boldsymbol{\psi} \cdot \mathbf{j}), \quad (j_i)_{mn} = -i \varepsilon_{lmn}, \\ U(r) &= \exp(-i \boldsymbol{\psi} \cdot \mathbf{J}), \quad D^{(1/2)}(r) = \exp(-i \boldsymbol{\psi} \cdot \boldsymbol{\sigma} / 2). \end{aligned} \quad (6.2)$$

Expanding both sides of (6.1) to first order in $\boldsymbol{\psi}$, comparing, and using (2.21), we find

$$\mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{L} + \boldsymbol{\Sigma}, \quad (6.3a)$$

$$\langle \mathbf{p} k m_1 m_2 | \mathbf{L} = \mathbf{L}(\mathbf{k}) \langle \mathbf{p} k m_1 m_2 |, \quad \mathbf{L}(\mathbf{k}) = i \nabla_{\mathbf{k}} \times \mathbf{k}, \quad (6.3b)$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(2)},$$

$$\langle \mathbf{p} k m_1 m_2 | \boldsymbol{\Sigma}^{(1)} = \sum_{m'_1 m'_2} \langle \mathbf{p} k m'_1 m'_2 | (1/2) \boldsymbol{\sigma}_{m'_1 m_1}^{(1)} \delta_{m'_2 m_2},$$

$$\langle \mathbf{p} k m_1 m_2 | \boldsymbol{\Sigma}^{(2)} = \sum_{m'_1 m'_2} \langle \mathbf{p} k m'_1 m'_2 | \delta_{m'_1 m_1} (1/2) \boldsymbol{\sigma}_{m'_2 m_2}^{(2)}. \quad (6.3c)$$

According to (2.18b) the spin operator is given by

$$\mathbf{S} = \mathbf{L} + \boldsymbol{\Sigma}. \quad (6.4)$$

For our mass operator we choose

$$M = M_0 + M_1, \quad (6.5a)$$

$$\langle \mathbf{p} k m_1 m_2 | M_0^2 = [W^2(\mathbf{k}) + 4\omega^4 (i \nabla_{\mathbf{k}})^2] \langle \mathbf{p} k m_1 m_2 |, \quad (6.5b)$$

$$M_1 = C \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{m_u^2}, \quad \boldsymbol{\sigma}^{(i)} = 2\boldsymbol{\Sigma}^{(i)}, \quad (6.5c)$$

where M_0 is a harmonic oscillator interaction and M_1 is a hyperfine interaction. We suppress the color and flavor variables since they do not play a role here. We choose the masses of the up and down quarks to be the same.

We note that the operators \mathbf{L} and $\boldsymbol{\Sigma}$ satisfy the commutation relations

$$[\mathbf{L}, M_0] = [\boldsymbol{\Sigma}, M_0] = [\mathbf{L}, M_1] = [\boldsymbol{\Sigma}, M_1] = \mathbf{0}; \quad (6.6)$$

therefore we can construct simultaneous eigenstates of M , \mathbf{P} , \mathbf{L}^2 , $\boldsymbol{\Sigma}^2$, \mathbf{S}^2 , and S_3 . Extending (2.23), we can write

$$\begin{aligned} M | \mathbf{q} W l s j m \rangle &= W | \mathbf{q} W l s j m \rangle, \\ \mathbf{P} | \mathbf{q} W l s j m \rangle &= \mathbf{q} | \mathbf{q} W l s j m \rangle, \\ \mathbf{L}^2 | \mathbf{q} W l s j m \rangle &= l(l+1) | \mathbf{q} W l s j m \rangle, \\ \boldsymbol{\Sigma}^2 | \mathbf{q} W l s j m \rangle &= s(s+1) | \mathbf{q} W l s j m \rangle, \\ \mathbf{S}^2 | \mathbf{q} W l s j m \rangle &= j(j+1) | \mathbf{q} W l s j m \rangle, \\ S_3 | \mathbf{q} W l s j m \rangle &= m | \mathbf{q} W l s j m \rangle. \end{aligned} \quad (6.7)$$

We see from (2.13a) and (2.21) that except for the mass eigenvalue equation all of these equations are satisfied by the state

$$| \mathbf{q} W l s j m \rangle = \int_0^\infty | \mathbf{q} k l s j m \rangle k^2 dk \phi_{l s}^{W j}(k), \quad (6.8)$$

for any choice of the function $\phi_{l s}^{W j}(k)$. Of course the mass eigenvalue equation determines this function. By using the elementary identity $\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} = 2\boldsymbol{\Sigma}^2 - 3$ we can replace the mass eigenvalue equation with

$$M_0^2 | \mathbf{q} W l s j m \rangle = \left[W - C \frac{2s(s+1) - 3}{m_u^2} \right]^2 | \mathbf{q} W l s j m \rangle. \quad (6.9)$$

With the help of (2.13) we can show that

$$\begin{aligned} \langle \mathbf{p} k l s j m | M_0^2 &= \left[W^2(k) - \frac{4\omega^4}{k} \frac{\partial^2}{\partial k^2} k + 4\omega^4 \frac{l(l+1)}{k^2} \right] \\ &\quad \times \langle \mathbf{p} k l s j m |. \end{aligned} \quad (6.10)$$

Letting this identity act on (6.9), and using (6.8) and (2.13b), we find the differential equation

$$\begin{aligned} \left\{ \frac{1}{k} \frac{\partial^2}{\partial k^2} k - \frac{l(l+1)}{k^2} - \frac{k^2 + m_u^2}{\omega^4} \right. \\ \left. + \frac{1}{4\omega^4} \left[W - C \frac{2s(s+1) - 3}{m_u^2} \right]^2 \right\} \phi_{l s}^{W j}(k) = 0. \end{aligned} \quad (6.11)$$

The solution of this harmonic oscillator differential equation is well known and is given by

$$\begin{aligned} W_{nl s} &= [4m_u^2 + 8\omega^2(2n + l + 3/2)]^{1/2} + C \frac{2s(s+1) - 3}{m_u^2}, \\ n &= 0, 1, 2, \dots, \quad l = 0, 1, 2, \dots, \end{aligned} \quad (6.12)$$

$$\begin{aligned} \phi_{l s}^{W j}(k) &\rightarrow \phi_{nl}(k) \\ &= N_{nl} \left(\frac{k}{\omega} \right)^l \exp\left(-\frac{k^2}{2\omega^2}\right) M\left(-n, l + \frac{3}{2}, \frac{k^2}{\omega^2}\right) \\ &= N_{nl} \left(\frac{k}{\omega} \right)^l \exp\left(-\frac{k^2}{2\omega^2}\right) \frac{n!}{(l+3/2)_n} L_n^{(l+1/2)}\left(\frac{k^2}{\omega^2}\right), \end{aligned} \quad (6.13a)$$

$$N_{nl} = \left[\frac{2(l+3/2)_n}{\omega^{3n} \Gamma(l+3/2)} \right]^{1/2}. \quad (6.13b)$$

Here $M(a, b, z)$ is a Kummer function, $L_n^{(\alpha)}(z)$ is a generalized Laguerre polynomial, and $(z)_n$ is a Pochhammer symbol.

In order to verify the Poincaré invariance of our model we must check to see whether we have agreement with (2.22). Our mass eigenstates have the normalization

$$\langle \mathbf{q}nl s j m | \mathbf{q}' n' l' s' j' m' \rangle = \delta^3(\mathbf{q} - \mathbf{q}') \delta_{nn'} \delta_{ll'} \delta_{ss'} \delta_{jj'} \delta_{mm'}, \quad (6.14)$$

which leads to the following representation for the mass operator:

$$M = \sum_{nlsjm} \int d^3q W_{nls}(\mathbf{q}nl s j m). \quad (6.15)$$

Using (6.8) and (2.13b) we find

$$\langle \mathbf{p}k l s j m | \mathbf{p}' k' l' s' j' m' \rangle = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ll'} \delta_{ss'} \delta_{jj'} \delta_{mm'} \phi_{nl}(k), \quad (6.16)$$

which in turn leads to

$$\langle \mathbf{p}k l s j m | M | \mathbf{p}' k' l' s' j' m' \rangle = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{jj'} \delta_{mm'} M_{ls, l' s'}^j(k, k'), \quad (6.17a)$$

$$M_{ls, l' s'}^j(k, k') = \delta_{ll'} \delta_{ss'} \sum_n \phi_{nl}(k) W_{nls} \phi_{nl}(k'). \quad (6.17b)$$

This validates (2.22), so we have Poincaré invariance.

For both the π and ρ we have $n = l = 0$, while their spins are given by $s = 0$ and $s = 1$, respectively, so it follows from (6.13) that their common spatial wave function is given by

$$\phi(k) = \chi(k) = \phi_{00}(k) = \frac{2}{\omega^{3/2} \pi^{1/4}} \exp\left(-\frac{k^2}{2\omega^2}\right), \quad (6.18)$$

and from (6.12) their masses are given by

$$m_\pi = \sqrt{4m_u^2 + 12\omega^2} - \frac{3C}{m_u^2},$$

$$m_\rho^{(0)} = \sqrt{4m_u^2 + 12\omega^2} + \frac{C}{m_u^2}. \quad (6.19)$$

The superscript on $m_\rho^{(0)}$ has been introduced to indicate that this is not the physical mass of the ρ since it has been determined by a model in which the decay channel $\rho \rightarrow \pi\pi$ has been ignored. We will rectify this omission in Sec. VIII. Determining C from the difference of the masses we have $C/m_u^2 = (m_\rho^{(0)} - m_\pi)/4$, which in turn leads to

$$\omega^2 = \frac{1}{3} \left[\left(\frac{m_\pi + 3m_\rho^{(0)}}{8} \right)^2 - m_u^2 \right]. \quad (6.20)$$

VII. COMPARISONS

Here we compare the relativistic result for the $\rho \rightarrow \pi\pi$ amplitude, given by (5.29), with the nonrelativistic result. In order to obtain the nonrelativistic result we make the replacement

$$A_{m_1 m_2, m'_1 m'_2}^{(1/2, 1/2)}(\mathbf{p}_1, \mathbf{p}_2) \rightarrow \delta_{m_1 m'_1} \delta_{m_2 m'_2} \quad (7.1)$$

in (4.5). By putting the resulting π and ρ wave functions in (5.1), and using (5.2), (5.3), and (5.14) and (5.16) with $\theta = \xi = 0$, it is straightforward to show that the nonrelativistic amplitude is given by

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T_{nr} | \rho \mathbf{q} m t \rangle \\ &= \frac{i\gamma}{\sqrt{3}(12\pi)^{3/2}} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \mathbf{e}_t \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 \\ & \quad \times \delta^3(\mathbf{p}_1 + \mathbf{p}_4 - \mathbf{q}_1) \delta^3(\mathbf{p}_3 + \mathbf{p}_2 - \mathbf{q}_2) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \\ & \quad \times \delta^3(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{q}) \phi(|\mathbf{k}_{14}|) \phi(|\mathbf{k}_{32}|) \chi(|\mathbf{k}_{34}|) \mathcal{Y}_1^m \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right), \\ & \quad \mathbf{k}_{ab} = (\mathbf{p}_a - \mathbf{p}_b)/2. \end{aligned} \quad (7.2)$$

Here \mathbf{k}_{ab} is the nonrelativistic rest frame momentum for particle a . If we replace the integration variables \mathbf{p}_1 and \mathbf{p}_4 with $\mathbf{p}_{14} = \mathbf{p}_1 + \mathbf{p}_4$ and $\mathbf{k} = \mathbf{k}_{14}$, integrate out \mathbf{p}_2 and \mathbf{p}_3 using the delta functions, and set $\mathbf{q} = \mathbf{0}$, we find

$$\begin{aligned} & \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T_{nr} | \rho \mathbf{0} m t \rangle \\ &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{i\gamma}{48\sqrt{3}\pi^2} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \mathbf{e}_t \\ & \quad \times \int d^3 k \phi(|\mathbf{k}|)^2 \chi(|\mathbf{p} - \mathbf{q}|) \mathbf{e}_m \cdot \mathbf{p}, \\ & \quad \mathbf{p} = \mathbf{k} + \mathbf{q}/2, \quad \mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (7.3)$$

We note that the \mathbf{p} defined here is given by a nonrelativistic expression, while previously it was given by the relativistic result (5.8c). Comparing (7.3) with (5.4), we see that (7.3) is transformed into (5.4) if we let $\varepsilon(\mathbf{p}) \rightarrow m_\rho$, replace the relativistic expression for \mathbf{p} with the nonrelativistic one, and turn off the Wigner rotations by letting $\zeta \rightarrow 0$.

For the π and ρ wave functions we use (6.20). The integral in (7.3) can be done analytically, which leads to

$$\begin{aligned} \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T_{nr} | \rho \mathbf{0} m t \rangle &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{i\gamma 2\sqrt{2}}{81\pi^{5/4} \omega^{3/2}} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \\ & \quad \cdot \mathbf{e}_t (\mathbf{e}_m \cdot \mathbf{q}) \exp\left(-\frac{\mathbf{q}^2}{12\omega^2}\right). \end{aligned} \quad (7.4)$$

If we compare (7.4) with (5.34) at $q = q_\rho$ we see that we can replace (7.4) with

$$\begin{aligned} \langle \pi \mathbf{q}_1 t_1, \pi \mathbf{q}_2 t_2 | T_{nr} | \rho \mathbf{0} m t \rangle &= \delta^3(\mathbf{q}_1 + \mathbf{q}_2) \frac{ig_{\rho\pi\pi}}{(2\pi m_\rho)^{3/2}} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \\ & \quad \cdot \mathbf{e}_t (\mathbf{e}_m \cdot \mathbf{q}) \Lambda_{nr}(q), \end{aligned} \quad (7.5a)$$

$$\begin{aligned} \Lambda_{nr}(q) &= \exp\left(-\frac{q^2 - q_\rho^2}{12\omega^2}\right), \\ \mathbf{q} &= \mathbf{q}_1 = -\mathbf{q}_2. \end{aligned} \quad (7.5b)$$

VIII. p -WAVE PION-PION SCATTERING

Here we construct a simple, Poincaré invariant model of p -wave $\pi\pi$ scattering in a model space spanned by p -wave $\pi\pi$ states and ρ states. The $\pi\pi$ states are given by (2.7) and

(2.8) with isospin indices added, i.e.,

$$|\mathbf{p}\mathbf{q}t_1t_2\rangle = U[l_c(p)]|\mathbf{q}t_1\rangle \otimes |-\mathbf{q}, t_2\rangle [W(\mathbf{q})/E(\mathbf{p}, \mathbf{q})]^{1/2}, \quad (8.1a)$$

$$W(\mathbf{q}) = 2\omega_\pi(\mathbf{q}), \quad E(\mathbf{p}, \mathbf{q}) = \sqrt{\mathbf{p}^2 + W^2(\mathbf{q})}, \quad (8.1b)$$

$$\langle \mathbf{p}\mathbf{q}t_1t_2 | \mathbf{p}'\mathbf{q}'t'_1t'_2 \rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta^3(\mathbf{q} - \mathbf{q}')\delta_{t_1t'_1}\delta_{t_2t'_2}. \quad (8.1c)$$

The ρ states are given by (2.2) with an isospin index added, i.e.,

$$|\mathbf{p}m t\rangle = U[l_c(p)]|\mathbf{0}m t\rangle [m_\rho/\omega_\rho(\mathbf{p})]^{1/2}, \quad (8.2a)$$

$$\langle \mathbf{p}m t | \mathbf{p}'m' t' \rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{mm'}\delta_{tt'}. \quad (8.2b)$$

We assume that the $\pi\pi$ states are orthogonal to the ρ states. The noninteracting mass operator is defined by its action on the above states. We have

$$M_0|\mathbf{p}\mathbf{q}t_1t_2\rangle = W(\mathbf{q})|\mathbf{p}\mathbf{q}t_1t_2\rangle, \quad M_0|\mathbf{p}m t\rangle = m_\rho^{(0)}|\mathbf{p}m t\rangle. \quad (8.3)$$

The only interaction is a vertex interaction, which according to (5.35) can be taken to be

$$\langle \mathbf{p}\mathbf{q}t_1t_2 | V | \mathbf{p}'\mathbf{q}'t'_1t'_2 \rangle = \delta^3(\mathbf{p} - \mathbf{p}')V(\mathbf{q}t_1t_2; mt), \quad (8.4a)$$

$$V(\mathbf{q}t_1t_2; mt) = \frac{ig_{\rho\pi\pi}^{(0)}}{(2\pi m_\rho)^{3/2}} (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \mathbf{e}_t (\mathbf{e}_m \cdot \mathbf{q}) \Lambda(q). \quad (8.4b)$$

The superscript (0) has been added to the coupling constant to indicate that it will be renormalized, as we shall soon see. By using the techniques described in Sec. V of Ref. [34] it is relatively straightforward to verify that the interaction V commutes with \mathbf{P} , \mathbf{X} , and \mathbf{S} , the three-momentum, position, and spin operators for the noninteracting $\pi\pi$ - ρ system. This ensures that the $\pi\pi$ - ρ model presented here is Poincaré invariant.

Using the time-independent theory of scattering we can take for our scattering states the solutions of the equation

$$|\Psi(\mathbf{p}'\mathbf{q}'t'_1t'_2)\rangle = |\mathbf{p}'\mathbf{q}'t'_1t'_2\rangle + [W(\mathbf{q}') + i\varepsilon - M_0]^{-1}V \times |\Psi(\mathbf{p}'\mathbf{q}'t'_1t'_2)\rangle, \quad (8.5)$$

where we have taken for our “initial state” the $\pi\pi$ state $|\mathbf{p}'\mathbf{q}'t'_1t'_2\rangle$. The projections of our state vector onto our basis states are given by

$$\langle \mathbf{p}m t | \Psi(\mathbf{p}'\mathbf{q}'t'_1t'_2)\rangle = \delta^3(\mathbf{p} - \mathbf{p}')\psi(mt; \mathbf{q}'t'_1t'_2), \quad (8.6a)$$

$$\langle \mathbf{p}\mathbf{q}t_1t_2 | \Psi(\mathbf{p}'\mathbf{q}'t'_1t'_2)\rangle = \delta^3(\mathbf{p} - \mathbf{p}')\psi(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2). \quad (8.6b)$$

Using (8.3)–(8.6) along with the completeness relation for the states given by (8.1) and (8.2), we can readily derive the coupled equations

$$\psi(mt; \mathbf{q}'t'_1t'_2) = [W(\mathbf{q}') - m_\rho^{(0)}]^{-1} \sum_{t''_1t''_2} \int V^*(\mathbf{q}''t''_1t''_2; mt) \times d^3q'' \psi(\mathbf{q}''t''_1t''_2; \mathbf{q}'t'_1t'_2), \quad (8.7)$$

$$\psi(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2) = \delta^3(\mathbf{q} - \mathbf{q}')\delta_{t_1t'_1}\delta_{t_2t'_2} + [W(\mathbf{q}') + i\varepsilon - W(\mathbf{q})]^{-1} \times \sum_{m''t''} V(\mathbf{q}t_1t_2; m''t'')\psi(m''t''; \mathbf{q}'t'_1t'_2). \quad (8.8)$$

Inserting (8.7) into (8.8) we find that the $\pi\pi$ component of the wave function can be obtained by solving the equation

$$\psi(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2) = \delta^3(\mathbf{q} - \mathbf{q}')\delta_{t_1t'_1}\delta_{t_2t'_2} + \sum_{t''_1t''_2} \int \frac{B[\mathbf{q}t_1t_2; \mathbf{q}''t''_1t''_2; W(\mathbf{q}')] }{W(\mathbf{q}') + i\varepsilon - W(\mathbf{q})} \times d^3q'' \psi(\mathbf{q}''t''_1t''_2; \mathbf{q}'t'_1t'_2), \quad (8.9)$$

where B is an effective $\pi\pi$ potential given by

$$B(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; z) = \sum_{m''t''} V(\mathbf{q}t_1t_2; m''t'')(z - m_\rho^{(0)})^{-1} \times V^*(\mathbf{q}'t'_1t'_2; m''t''). \quad (8.10)$$

It is straightforward to verify that the solution to (8.9) can be expressed in the form

$$\psi(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2) = \delta^3(\mathbf{q} - \mathbf{q}')\delta_{t_1t'_1}\delta_{t_2t'_2} + \frac{X[\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; W(\mathbf{q}') + i\varepsilon]}{W(\mathbf{q}') + i\varepsilon - W(\mathbf{q})}, \quad (8.11)$$

where X is the solution of the equation

$$X(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; z) = B(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; z) + \sum_{t''_1t''_2} \int \frac{B(\mathbf{q}t_1t_2; \mathbf{q}''t''_1t''_2; z)}{z - W(\mathbf{q}'')} \times d^3q'' X(\mathbf{q}''t''_1t''_2; \mathbf{q}'t'_1t'_2; z). \quad (8.12)$$

This equation can be simplified by coupling the isospins according to the relations

$$\sum_{t_1t_2} \sum_{t'_1t'_2} \langle 1, 1, t_1, t_2 | T, M \rangle X(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; z) \langle 1, 1, t'_1, t'_2 | T', M' \rangle = \delta_{TT'}\delta_{MM'} X_T(\mathbf{q}, \mathbf{q}'; z), \quad (8.13a)$$

$$\sum_{t_1t_2} \sum_{t'_1t'_2} \langle 1, 1, t_1, t_2 | T, M \rangle B(\mathbf{q}t_1t_2; \mathbf{q}'t'_1t'_2; z) \langle 1, 1, t'_1, t'_2 | T', M' \rangle = \delta_{TT'}\delta_{MM'} B_T(\mathbf{q}, \mathbf{q}'; z). \quad (8.13b)$$

Instead of (8.12) we now have

$$X_T(\mathbf{q}, \mathbf{q}'; z) = B_T(\mathbf{q}, \mathbf{q}'; z) + \int B_T(\mathbf{q}, \mathbf{q}''; z) \frac{d^3q''}{z - W(\mathbf{q}'')} \times X_T(\mathbf{q}'', \mathbf{q}'; z). \quad (8.14)$$

Combining (8.10), (8.4b), (8.13b), and the identity

$$\sum_{t_1t_2} \sum_{t'_1t'_2} \sum_t \langle 1, 1, t_1, t_2 | T, M \rangle (\mathbf{e}_{t_1}^* \times \mathbf{e}_{t_2}^*) \cdot \mathbf{e}_t \mathbf{e}_t^* (\mathbf{e}_{t_1} \times \mathbf{e}_{t_2}) \times \langle 1, 1, t'_1, t'_2 | T', M' \rangle = 2\delta_{TT'}\delta_{MM'}\delta_{T1}, \quad (8.15)$$

we find that

$$B_T(\mathbf{q}, \mathbf{q}'; z) = \delta_{T1} \frac{g_{\rho\pi\pi}^{(0)2}}{4\pi^3 m_\rho^3} \sum_m (\mathbf{e}_m \cdot \mathbf{q}) \frac{\Lambda(q)\Lambda(q')}{z - m_\rho^{(0)}} (\mathbf{e}_m^* \cdot \mathbf{q}'). \quad (8.16)$$

As a result of the separable nature of the potential (8.16) it is straightforward to solve (8.14). With the help of (3.7) we find

$$X_1(\mathbf{q}, \mathbf{q}'; z) = s_{\pi\rho}^{(0)} \sum_{m=-1}^1 Y_1^m(\hat{\mathbf{q}}) \frac{[q\Lambda(q)][\Lambda(q')q']}{d(z)} Y_1^{m*}(\hat{\mathbf{q}}'), \quad (8.17)$$

$$s_{\pi\rho}^{(0)} = \frac{g_{\rho\pi\pi}^{(0)2}}{3\pi^2 m_\rho^3}, \quad (8.18)$$

$$d(z) = z - m_\rho^{(0)} - s_{\pi\rho}^{(0)} \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{z - W(q)}. \quad (8.19)$$

We take for the on-shell, p -wave elastic scattering amplitude

$$X(k) = \frac{s_{\pi\rho}^{(0)} [k\Lambda(k)]^2}{d[W(k+i\varepsilon)]}. \quad (8.20)$$

It follows from (8.19) that

$$\begin{aligned} \text{Re } d[W(k+i\varepsilon)] &= W(k) - m_\rho^{(0)} - s_{\pi\rho}^{(0)} \mathcal{P} \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{W(k) - W(q)}, \quad (8.21a) \\ \frac{d[W(k+i\varepsilon)] - d[W(k-i\varepsilon)]}{2i} &= \text{Im } d[W(k+i\varepsilon)] = \frac{\pi}{4} s_{\pi\rho}^{(0)} k^3 W(k) \Lambda^2(k). \quad (8.21b) \end{aligned}$$

Writing $d = |d| \exp(-\delta_1^1)$, where δ_1^1 is the p -wave, $\pi\pi$ phase shift, and using (8.21) we can easily derive the results

$$X(k) = -\frac{4}{\pi W(k)k} \exp[i\delta_1^1(k)] \sin[\delta_1^1(k)], \quad (8.22a)$$

$$\cot[\delta_1^1(k)] = -\frac{\text{Re } d[W(k+i\varepsilon)]}{\text{Im } d[W(k+i\varepsilon)]}. \quad (8.22b)$$

We determine the position of the ρ resonance by setting $\text{Re } d = 0$, which gives the relation between the physical ρ mass m_ρ and the bare mass $m_\rho^{(0)}$, i.e.,

$$m_\rho - m_\rho^{(0)} - s_{\pi\rho}^{(0)} \mathcal{P} \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{m_\rho - W(q)} = 0. \quad (8.23)$$

Using this equation to eliminate $m_\rho^{(0)}$ from (8.19) we find

$$\begin{aligned} \text{Re } d[W(k+i\varepsilon)] &= [W(k) - m_\rho] \\ &\times \left\{ 1 + s_{\pi\rho}^{(0)} \mathcal{P} \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{[W(k) - W(q)][m_\rho - W(q)]} \right\}. \quad (8.24) \end{aligned}$$

As in the renormalization of the Lee model [39,40], we define a renormalization parameter Z_ρ by requiring that

$$\lim_{W(k) \rightarrow m_\rho} \text{Re } d[W(k+i\varepsilon)]/[W(k) - m_\rho] = Z_\rho^{-1}, \quad (8.25)$$

which leads to

$$Z_\rho = \left\{ 1 + s_{\pi\rho}^{(0)} \mathcal{P} \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{[m_\rho - W(q)]^2} \right\}^{-1}. \quad (8.26)$$

From here it appears that $Z_\rho < 1$, but because of the peculiar nature of the principal value integral [41] this is not necessarily so. Solving for the number one in (8.26) and putting the result in (8.24), we find

$$\begin{aligned} \text{Re } d[W(k+i\varepsilon)] &= Z_\rho^{-1} [W(k) - m_\rho] [1 + J(k)], \quad (8.27a) \\ J(k) &= [W(k) - m_\rho] s_{\pi\rho} \mathcal{P} \\ &\times \int_0^\infty dq \frac{q^4 \Lambda^2(q)}{[W(k) - W(q)][m_\rho - W(q)]^2}, \quad (8.27b) \end{aligned}$$

where

$$s_{\pi\rho} = Z s_{\pi\rho}^{(0)} = \frac{g_{\rho\pi\pi}^2}{3\pi^2 m_\rho^3}, \quad g_{\rho\pi\pi} = Z_\rho^{1/2} g_{\rho\pi\pi}^{(0)}. \quad (8.28)$$

Using these expressions in (8.20), we can write

$$X(k) = \frac{s_{\pi\rho} [k\Lambda(k)]^2}{[W(k) - m_\rho] [1 + J(k)] + i\Gamma(k)/2}, \quad (8.29a)$$

$$\Gamma(k) = (\pi/2) s_{\pi\rho} k^3 W(k) \Lambda^2(k). \quad (8.29b)$$

Evaluating (8.29b) at q_ρ , the momentum of each pion in the final state of the ρ decay, and using the fact that $\Lambda(q_\rho) = 1$, we find

$$\Gamma(q_\rho) = \frac{g_{\rho\pi\pi}^2 q_\rho^3}{6\pi m_\rho^2}, \quad q_\rho = [(m_\rho/2)^2 - m_\pi^2]^{1/2}. \quad (8.30)$$

In fitting our model to data we assume $m_u = m_d = 200.0$ MeV, and we take as our adjustable parameters the bare ρ mass $m_\rho^{(0)}$ and the bare coupling constant $g_{\rho\pi\pi}^{(0)}$. The interaction (8.4) depends on $g_{\rho\pi\pi}^{(0)}$ and on the parameters that appear in the cutoff function $\Lambda(q)$. The relativistic cutoff function is defined by (5.29b)–(5.31), and it is determined by the functions $F(\mathbf{q}, \mathbf{k})$ and $G(\mathbf{q}, \mathbf{k})$, which are given by (5.25) and (5.26). The meson wave functions, $\phi(k)$ and $\chi(k)$, which are given by (6.18), depend on the bare ρ mass $m_\rho^{(0)}$ through (6.20). The nonrelativistic cutoff function is given by (7.5b), (6.20), and (5.30). According to (5.32), (5.35), (7.4), and (7.5), the relativistic and nonrelativistic relations between the strength parameter γ that appears in the quark-pair creation operator (3.12) and the bare coupling constant $g_{\rho\pi\pi}^{(0)}$ are given by

$$\gamma = \frac{12\sqrt{6}\pi g_{\rho\pi\pi}^{(0)}}{E(q_\rho) m_q m_\rho^{3/2}}, \quad \gamma_{nr} = \frac{81\omega^{3/2} \exp(q_\rho^2/12\omega^2) g_{\rho\pi\pi}^{(0)}}{8\pi^{1/4} m_\rho^{3/2}}. \quad (8.31)$$

Our fit to the $\pi\pi$ p -wave phase shifts [42,43] is shown in Table I. The results for the parameters are given in Table II.

We see that there is a significant difference between the relativistic and nonrelativistic results for the bare ρ mass $m_\rho^{(0)}$. This suggests that relativistic effects can be important in determining the effect that strong coupling to decay channels has on the hadron spectrum. It is interesting to note that the

TABLE I. p -wave $\pi\pi$ phase shifts.

W (MeV)	$\delta_1^1(\circ)$ relativistic	$\delta_1^1(\circ)$ nonrelativistic	$\delta_1^1(\circ)$ experiment
689.89	34.6	34.3	34.8 ± 1.4
707.18	42.6	42.4	42.5 ± 1.4
731.38	56.9	57.0	55.7 ± 0.7
748.67	69.4	69.5	67.6 ± 0.7
810.90	113.	113.	$111. \pm 2.1$
831.65	123.	123.	$123. \pm 0.7$
848.94	130.	129.	$130. \pm 0.7$
869.68	136.	136.	$136. \pm 0.7$
890.43	141.	141.	$139. \pm 0.7$
911.17	144.	145.	$145. \pm 0.1$

differences between the relativistic and nonrelativistic values for the bare and renormalized coupling constants, $g_{\rho\pi\pi}^{(0)}$ and $g_{\rho\pi\pi}$, are quite small. The value of $g_{\rho\pi\pi}$ obtained here is rather close to the value $g_{\rho\pi\pi} = 6.199$ used in hadronic models of the pion-nucleon system [44]. The widths found for the ρ resonance are comparable to the value $\Gamma_\rho = 149.1 \pm 0.8$ given in the most recent review of particle physics [45]. The difference between the relativistic and nonrelativistic values of ω , which determines the strength of the quark-antiquark harmonic oscillator interaction, suggests that relativistic effects can be significant in determining the properties of the interactions between quarks and antiquarks. The fact that the relativistic and nonrelativistic values of the quark-pair creation operator strength parameter γ differ by over a factor of 2 also makes clear the importance of relativistic effects.

It should be noted when comparing values of γ with those obtained by Le Yaouanc *et al.* [6,7] that our quark-pair creation operator, given by (3.12), differs somewhat from theirs. They put a factor of 3 in their quark-pair creation operator to cancel the $1/3$ that arises from the color factor given by (5.2). Also in (3.12), where we have $m_q/\sqrt{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)}$, they have 1. Dividing our nonrelativistic result for γ in Table II by 3 gives $\gamma_{LeY} = 1.88$. It should be noted that γ_{nr} , given by (8.31), when divided by 3 agrees with the expression given by Le Yaouanc *et al.* [6] for the relation between γ and the $\rho\pi\pi$ coupling constant. Downum *et al.* [46] have obtained a value for γ . The relation between our γ and theirs is given by $\gamma = 3\sqrt{24\pi}\gamma_D$. By fitting a number of strong decays they decide on $\gamma = 10.4$, which differs from our nonrelativistic result by roughly a factor of 2. Clearly there is no reason to believe that γ is a universal constant.

TABLE II. Model parameters.

Parameter	Relativistic	Nonrelativistic
m_ρ (MeV)	775.5	775.5
$m_\rho^{(0)}$ (MeV)	855.7	776.3
$g_{\rho\pi\pi}^{(0)}$	5.704	5.595
$g_{\rho\pi\pi}$	6.033	6.065
Z_ρ	1.119	1.175
$\Gamma_\rho(q_\rho)$ (MeV)	152.7	154.4
ω (MeV)	157.4	135.5
γ	13.637	5.640

IX. DISCUSSION

There are a number of obvious extensions and applications of the present work that should be considered.

As mentioned in Sec. I, the results obtained here can also be applied to the flux-tube model for decays studied by Isgur *et al.* [9–12]. This will test relativistic effects when nonpoint quarks are considered, when it is no longer assumed that quark-pair creation occurs with equal probability everywhere in space, and when pair creation is suppressed at short distances.

Clearly our calculation of the ρ mass shift and decay width should be extended to more of the meson spectrum. Results for these shifts and width can be compared to those obtained by Törnqvist [13–16] using the UQM. It is interesting to note that our relation between the bare and physical ρ mass, given by (8.23), is similar in structure to the dispersion relations used in the UQM, i.e., (1.1) and (1.2).

The method developed here for meson decays can be extended to baryon decays. The essential ingredient in all of this is the transformation of hadron, quark-model, momentum-space wave functions from an arbitrary frame to a c.m. frame. The analysis of three-particle states presented in Ref. [34] makes it possible to obtain this transformation for three-quark wave functions, and hence for baryons. Relativistic results for baryon mass shifts and widths can be compared to the UQM results presented by Törnqvist and Zenczykowski [17–19] and Silvestre-Brac and Gignoux [20], as well as the extensive results obtained by Capstick and Roberts [22–24].

APPENDIX: WIGNER ROTATIONS

The Wigner rotations that are of interest to us are given by [3]

$$r_c[l_c(q), k] = l_c^{-1}(p)l_c(q)l_c(k), \quad p = l_c(q)k. \quad (A1)$$

In deriving formulas for this rotation it is convenient to work with the elements of SL(2, C), the covering group for the Lorentz transformations [3]. This group is the set of all 2×2 complex matrices with determinant +1. With this group four-vectors are represented by 2×2 Hermitian matrices,

$$X = x^\mu \sigma_\mu = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}, \quad (A2)$$

where the x^μ are the components of a four-vector x , σ_0 is the unit matrix, while σ_1, σ_2 , and σ_3 are the usual Pauli matrices. A Lorentz transformation is given by

$$X \rightarrow X' = \Lambda X \Lambda^\dagger, \quad \det(\Lambda) = +1. \quad (A3)$$

The four-momenta that appear in (A1) are given by

$$\begin{aligned} q &= (E(\mathbf{q}), \mathbf{q}), & E(\mathbf{q}) &= \sqrt{\mathbf{q}^2 + W^2}, & W &= \sqrt{q \cdot q}, \\ k &= (\varepsilon(\mathbf{k}), \mathbf{k}), & \varepsilon(\mathbf{k}) &= \sqrt{\mathbf{k}^2 + m^2}, & p &= (\varepsilon(\mathbf{p}), \mathbf{p}), \\ \varepsilon(\mathbf{p}) &= \sqrt{\mathbf{p}^2 + m^2}. \end{aligned} \quad (A4)$$

The rotationless boosts determined by these momenta are defined by

$$L_c(q) = \exp(\boldsymbol{\omega} \cdot \boldsymbol{\sigma}/2), \quad \boldsymbol{\omega} = \omega \hat{\mathbf{q}}, \quad \omega = \tanh^{-1}[|\mathbf{q}|/E(\mathbf{q})], \quad (\text{A5a})$$

$$L_c(k) = \exp(\boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2), \quad \boldsymbol{\eta} = \eta \hat{\mathbf{k}}, \quad \eta = \tanh^{-1}[|\mathbf{k}|/\varepsilon(\mathbf{k})], \quad (\text{A5b})$$

$$L_c(p) = \exp(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma}/2), \quad \boldsymbol{\lambda} = \lambda \hat{\mathbf{p}}, \quad \lambda = \tanh^{-1}[|\mathbf{p}|/\varepsilon(\mathbf{p})]. \quad (\text{A5c})$$

The $SL(2, \mathbb{C})$ representative of the Wigner rotation (A1) is of the form

$$D^{(1/2)}\{r_c[l_c(q), k]\} = \exp(i\boldsymbol{\zeta} \cdot \boldsymbol{\sigma}/2), \quad \boldsymbol{\zeta} = \zeta \mathbf{u}, \quad \mathbf{u} \cdot \mathbf{u} = 1, \quad (\text{A6})$$

where we note that $D^{(1/2)}$ is also an $SU(2)$ representative. In order to find $\boldsymbol{\zeta} = \zeta \mathbf{u}$ we solve the equation

$$L_c(p) \exp(i\boldsymbol{\zeta} \cdot \boldsymbol{\sigma}/2) = L_c(q)L_c(k). \quad (\text{A7})$$

We find

$$\cos(\zeta/2) = \cosh^{-1}(\lambda/2)[\cosh(\omega/2) \cosh(\eta/2) + (\hat{\mathbf{q}} \cdot \hat{\mathbf{k}}) \sinh(\omega/2) \sinh(\eta/2)], \quad (\text{A8a})$$

$$\sin(\zeta/2) = \cosh^{-1}(\lambda/2)[1 - (\hat{\mathbf{q}} \cdot \hat{\mathbf{k}})]^{1/2} \sinh(\omega/2) \sinh(\eta/2), \quad (\text{A8b})$$

$$\mathbf{u}(\mathbf{q}, \mathbf{k}) = \frac{\mathbf{q} \times \mathbf{k}}{|\mathbf{q} \times \mathbf{k}|} = \frac{\mathbf{q} \times \mathbf{k}}{[\mathbf{q}^2 \mathbf{k}^2 - (\mathbf{q} \cdot \mathbf{k})^2]^{1/2}}. \quad (\text{A8c})$$

Upon dividing (A8b) by (A8a) and using standard identities for hyperbolic functions we find that the angle for the Wigner rotation is given by

$$\zeta(\mathbf{q}, \mathbf{k}) = 2 \tan^{-1} \left\{ \frac{[\mathbf{q}^2 \mathbf{k}^2 - (\mathbf{q} \cdot \mathbf{k})^2]^{1/2}}{[E(\mathbf{q}) + W][\varepsilon(\mathbf{k}) + m] + \mathbf{q} \cdot \mathbf{k}} \right\}. \quad (\text{A9})$$

Clearly, ζ depends on $|\mathbf{q}|$, $|\mathbf{k}|$, and $\hat{\mathbf{q}} \cdot \hat{\mathbf{k}}$. Finally, we can write for our Wigner rotation

$$\begin{aligned} D^{(1/2)}\{r_c[l_c(q), k]\} &= \exp[i\boldsymbol{\zeta}(\mathbf{q} \cdot \mathbf{k}) \cdot \boldsymbol{\sigma}/2] \\ &= 1 \cos[\zeta(\mathbf{q} \cdot \mathbf{k})/2] \\ &\quad + i \hat{\boldsymbol{\zeta}}(\mathbf{q} \cdot \mathbf{k}) \cdot \boldsymbol{\sigma} \sin[\zeta(\mathbf{q} \cdot \mathbf{k})/2]. \end{aligned} \quad (\text{A10})$$

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