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Relating pseudospin and spin symmetries through chiral transformation with tensor interaction

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We address the behavior of the Dirac equation with scalar (S), vector (V), and tensor (U) interactions under the γ^5 discrete chiral transformation. By using this transformation, in a simple way, we can obtain solutions for the Dirac equation with spin ($\Delta = V - S = 0$) and pseudospin ($\Sigma = V + S = 0$) symmetries, which includes a tensor interaction. As an application, the Dirac equation with scalar, vector, and tensor Cornell radial potentials is considered, and the correct solution to this problem is obtained.

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Introduction. The pseudospin symmetry was introduced in nuclear physics many years ago [1,2] to account for the degeneracies of orbitals in single-particle spectra. It is known that the spin and pseudospin symmetries correspond to SU(2) symmetries of a Dirac Hamiltonian with vector (V) and scalar (S) potentials. Also, it is known that the spin symmetry occurs in the spectrum of a meson with one heavy quark [3] and antinucleon bound in a nucleus [4], and the pseudospin symmetry occurs in the spectrum of nuclei [5,6]. In Refs. [7–10], it was reported that pseudospin and spin symmetries were connected by charge conjugation. This was shown explicitly for harmonic oscillator potentials in (1 + 1) dimensions [9]. Also, in Ref. [9], a connection of pseudospin and chiral symmetries in (1 + 1) dimensions is shown. In recent years, some authors have extended the research field for pseudospin and spin symmetries by including a tensor interaction. The tensor interaction has been used in studies of nuclear properties with effective Lagrangians, which include relativistic mean-field theories [11], and in the relativistic Hartree approach model [12]. Those papers suggest that the tensor interaction could have a significant contribution to pseudospin splittings in nuclei. In Ref. [13], the authors show that the tensor interaction can strongly change the spin-orbit term. The connection of pseudospin and spin symmetries by charge conjugation, which includes a tensor interaction, has also been studied in Refs. [14,15]. However, a clear connection between pseudospin and spin symmetries, obtained by a discrete chiral transformation, which includes a tensor interaction, has not been established. Therefore, we believe that this connection deserves to be explored.

The main motivation of this Rapid Communication is inspired by the results obtained in Ref. [9]. As a natural extension, we address the behavior of the Dirac equation with scalar (S), vector (V), and tensor (U) interactions under the γ^5 discrete chiral transformation. By using this transformation, in a simple way, we can obtain solutions for the Dirac equation with spin ($\Delta = V - S = 0$) and pseudospin ($\Sigma = V + S = 0$) symmetries that include a tensor interaction. As an application, the Dirac equation with scalar, vector, and tensor Cornell radial potentials is considered. The radial equation for this problem is mapped into a Schrödinger-like equation embedded in a three-dimensional harmonic oscillator plus a Cornell potential. We use this opportunity to present the correct solution to this problem in a more transparent way.

Dirac equation with scalar, vector, and tensor interactions. The time-independent Dirac equation for a fermion with scalar (S), vector (V), and tensor (U) interactions is given by ($\hbar = c = 1$)

$$H\Psi = E\psi, \tag{1}$$

where

$$H = \vec{\alpha}\cdot\vec{p} + \beta[m + S(r)] + V(r) - i\beta\vec{\alpha}\cdot\hat{r}U(r). \tag{2}$$

By using the combinations $\Sigma = V + S$ and $\Delta = V - S$, we can rewrite the Hamiltonian (2) as

$$H = \vec{\alpha}\cdot\vec{p} + \beta m + \frac{I+\beta}{2}\Sigma + \frac{I-\beta}{2}\Delta - i\beta\vec{\alpha}\cdot\hat{r}U(r). \tag{3}$$

Chiral transformation. The chiral operator is the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and therefore, under the discrete chiral transformation, the spinor is transformed as $\Psi_\chi = \gamma^5\Psi$, and the transformed Hamiltonian $H_\chi = \gamma^5 H\gamma^5$ is

$$H_\chi = \vec{\alpha}\cdot\vec{p} - \beta[m + S(r)] + V(r) + i\beta\vec{\alpha}\cdot\hat{r}U(r). \tag{4}$$

We can see that the discrete chiral transformation changes the sign of the mass and of the scalar and tensor potentials because γ^5 commutes with $\vec{\alpha}$ and anticommutes with β. In term of the combinations Σ and Δ, this means that Σ turns into Δ and vice versa.

Equation of motion. Now, we follow the same procedure of Ref. [15] and use the projectors $P_\pm = (I \pm \beta)/2$. By applying $P_\pm$ to the left of the Dirac equation (1) and by defining $\Psi_\pm = P_\pm\Psi$, we obtain

$$\vec{\alpha}\cdot\vec{p}\Psi_\mp + \{V(r) \pm [m + S(r)]\}\Psi_\pm \mp i\vec{\alpha}\cdot\hat{r}U(r)\Psi_\mp = E\Psi_\pm, \tag{5}$$

or

$$[\vec{\alpha}\cdot\vec{p} - i\vec{\alpha}\cdot\hat{r}U(r)]\Psi_- = (E - m - \Sigma)\Psi_+, \tag{6}$$

$$[\vec{\alpha}\cdot\vec{p} + i\vec{\alpha}\cdot\hat{r}U(r)]\Psi_+ = (E + m - \Delta)\Psi_-. \tag{7}$$

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If S, V, and U are radial functions, the Dirac spinor is considered as

$$\Psi_{km}(\vec{r}) = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} \frac{if_k(r)}{r} Y_{km}(\hat{r}) \\ \frac{g_k(r)}{r} Y_{-km}(\hat{r}) \end{pmatrix}, \tag{8}$$

where f_k and g_k are the radial wave functions of the upper and lower components, respectively. Y_{km} are the so-called spinor spherical harmonics. Here, k is the quantum number of the total angular momentum j, and it is related to the orbital momentum l by $k = -(l+1) = -(j+1/2)$ for $j = l + 1/2$ and $k = l = +(j+1/2)$ for $j = l - 1/2$.

As shown in Ref. [14], by using the following property $\vec{\sigma}\cdot\hat{r}Y_{km} = -Y_{-km}$, Eqs. (6) and (7) can be reduced to two coupled first-order ordinary differential equations for the radial upper (f_k) and lower (g_k) components,

$$\left[\frac{d}{dr} + \frac{k}{r} - U(r)\right] f_k(r) = [E + m - \Delta] g_k(r), \tag{9}$$

$$\left[\frac{d}{dr} - \frac{k}{r} + U(r)\right] g_k(r) = -[E - m - \Sigma] f_k(r). \tag{10}$$

Under the discrete chiral transformations, the spinor (8) becomes

$$\Psi_\chi = \gamma^5 \Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}. \tag{11}$$

This last result means that γ^5 interchanges the upper and lower components, thus, f turns into $-ig$, g turns into if, and k turns into $-k$.

Spin and pseudospin symmetries. By using the expression for g_k obtained from Eq. (9) with $\Delta = 0$ and $E \neq -m$ and by inserting it in Eq. (10), we obtain

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + 2k\frac{U(r)}{r} - U'(r) - U^2(r)\right] f_k(r) = -(E - m - \Sigma)(E + m) f_k(r). \tag{12}$$

In a similar way, by using the expression for f_k obtained from Eq. (10) with $\Sigma = 0$ and $E \neq m$ and inserting it in Eq. (9), we obtain

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + 2k\frac{U(r)}{r} + U'(r) - U^2(r)\right] g_k(r) = -(E + m - \Delta)(E - m) g_k(r). \tag{13}$$

Therefore, either for $\Delta = 0$ with $E \neq -m$ or for $\Sigma = 0$ with $E \neq m$, the solutions for this problem can be found by solving a Schrödinger-like problem.

As discussed in the previous section, the discrete chiral transformation performs the changes $\Delta \to \Sigma$, $\Sigma \to \Delta$, $m \to -m$, $U \to -U$, $f \to -ig$, $g \to if$, and $k \to -k$. At this stage, note that, by applying these changes in Eq. (9) [or Eq. (12)], we obtain Eq. (10) [or Eq. (13)]. This means that we can take advantage of this kind of transformation and can obtain the solutions for $\Sigma = 0$ from the $\Delta = 0$ case. For instance, we can focus the discussion on this case $\Delta = 0$, $\Sigma = c_1 F_1(r)$, and $U = c_2 F_2(r)$, and the results for the case when $\Delta = c_1 F_1(r)$, $\Sigma = 0$, and $U = c_2 F_2(r)$ can be obtained easily by just changing the signs of m, c_2, and k in the relevant expressions.

Dirac equation with Cornell potentials. In the first instance, let us consider

$$\Delta = 0, \quad \Sigma = a_1 r + \frac{b_1}{r}, \quad U = a_2 r + \frac{b_2}{r}. \tag{14}$$

By substituting Eq. (14) into Eq. (12), we get

$$\frac{d^2 f_k(r)}{dr^2} + \left[\mathcal{E}^2 + \frac{a}{r} - br - cr^2 - \frac{\lambda(\lambda+1)}{r^2}\right] f_k(r) = 0, \tag{15}$$

where

$$\mathcal{E}^2 = E^2 - m^2 + 2a_2\left(k - b_2 - \tfrac{1}{2}\right), \tag{16}$$

$$a = -b_1(E + m), \tag{17}$$

$$b = a_1(E + m), \tag{18}$$

$$c = a_2^2, \tag{19}$$

$$\lambda = -\tfrac{1}{2} + \tfrac{1}{2}|2k + 1 - 2b_2|. \tag{20}$$

The solution for Eq. (15), with c necessarily real and positive, is the solution of the Schrödinger equation for the three-dimensional harmonic oscillator plus a Cornell potential. This novel potential was considered in Refs. [16,17], but the authors misunderstood the full meaning of the potential and made a few erroneous calculations. We use this opportunity to present the correct solution to this problem in a more transparent way.

The solution close to the origin, valid for all values of λ, can be written as being proportional to $r^{\lambda+1}$. By setting

$$f(r) = r^{\lambda+1}\exp\left(-\frac{\sqrt{c}}{2}r^2 - \frac{b}{2\sqrt{c}}r\right)\phi(r), \tag{21}$$

and by introducing the following new variable and parameters:

$$x = \sqrt[4]{c}\,r, \quad \omega = 2\lambda + 1, \quad \rho = \frac{b}{\sqrt[4]{c^3}}, \quad \tau = \frac{b^2 + 4c\varepsilon^2}{4\sqrt{c^3}}, \tag{22}$$

one finds that the solution for all r can be expressed as a solution of the biconfluent Heun differential equation [17],

$$x\frac{d^2\phi}{dx^2} + (\omega + 1 - \rho x - 2x^2)\frac{d\phi}{dx} + [(\tau - \omega - 2)x - \Theta]\phi = 0, \tag{23}$$

with

$$\Theta = \tfrac{1}{2}[\delta + \rho(\omega + 1)], \tag{24}$$

where $\delta = -\frac{2a}{\sqrt[4]{c}}$. The expressions for ω, τ, and δ are very different from that given in Refs. [16,17]. The reason for this disagreement is the mistakes in those references. The biconfluent Heun differential equation has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$. The solution, which is regular at the origin, is given by

$$N(\omega,\rho,\tau,\delta;x)=\sum_{j=0}^{\infty}\frac{\Gamma(\omega+1)}{\Gamma(\omega+1+j)}\frac{A_j}{j!}x^j, \tag{25}$$

where $\Gamma(z)$ is the γ function, $A_0=1$, $A_1=\Theta$ and the remaining coefficients of the series expansion for $\rho\neq 0$ satisfy the three-term recurrence relation,

$$A_{j+2}=[(j+1)\rho+\Theta]A_{j+1} - (j+1)(j+\omega+1)(\tau-\omega-2-2j)A_j. \tag{26}$$

The series is convergent for x in the range $[0,\infty)$ and tends to e^{x^2} as $x\to\infty$. In fact, ϕ presents polynomial solutions of degree n when $\tau=\omega+2+2n$ and $A_{n+1}=0$. Therefore, by using the condition $\tau=\omega+2+2n$, we obtain

$$\mathcal{E}^2=(2n+2\lambda+3)|a_2|-\frac{b^2}{4a_2^2}. \tag{27}$$

By substituting Eqs. (16) and (18) into Eq. (27), we obtain the spectrum for $\Delta=0$,

$$E=-\frac{m_E}{a_E}\pm\frac{m_E}{a_E}\sqrt{1-\frac{a_E}{m_E^2}\left[2a_2\left(k-b_2-\frac{1}{2}\right)-(2n+2\lambda+3)|a_2|+(a_E-2)m^2\right]}, \tag{28}$$

where $m_E=\frac{ma_1^2}{4a_2^2}$ and $a_E=1+\frac{a_1^2}{4a_2^2}$. Note that, at first view, Eq. (28) is independent of the value of b_1. Now, we focus attention on the condition $A_{n+1}=0$; this condition provides a constraint on the value of b_1. For instance, $n=0$ implies that $A_1=\Theta=0$. In this specific case, we obtain

$$b_1=-\frac{a_1}{2|a_2|}(1+|1-2b_2+2k|), \tag{29}$$

where the constraint (29) involves specific values of $a_1,\ a_2,\ b_2$, and the quantum number of the total angular momentum. At this stage, we can see a peculiar behavior of the parameter b_1. Initially, b_1 is arbitrary, but during the procedure to obtain the quantization condition, the value of b_1 is restricted by Eq. (29). This last result implies that the parameter b_1 in Eq. (14) should satisfy the constraint (29) to obtain the quantization condition. Therefore, we can conclude that the spectrum (28) depends implicitly on b_1 due to the fact that there is a link between the parameters $a_1,\ a_2,\ b_1,\ b_2$, and k.

Now, let us consider the case,

$$\Delta=a_1r+\frac{b_1}{r},\quad \Sigma=0,\quad U=a_2r+\frac{b_2}{r}. \tag{30}$$

As referred to before, we can take advantage of the discrete chiral transformation. We recall that this case can be obtained easily by just the changes $m\to -m,\ U(r)\to -U(r)$, and $k\to -k$ in the relevant expressions. We can see that the change in $U(r)$ implies that $a_2\to -a_2$ and $b_2\to -b_2$. Therefore, the spectrum for $\Sigma=0$ is given by

$$E=\frac{m_E}{a_E}\pm\frac{m_E}{a_E}\sqrt{1-\frac{a_E}{m_E^2}\left[2a_2\left(k-b_2+\frac{1}{2}\right)-(2n+2\lambda+3)|a_2|+(a_E-2)m^2\right]}, \tag{31}$$

where $\lambda=-\frac{1}{2}+\frac{1}{2}|1-2k+2b_2|$.

Conclusions. We have addressed the behavior of the Dirac equation with scalar (S), vector (V), and tensor (U) interactions under the γ^5 discrete chiral transformation. We showed that, in a simple way, it was possible to obtain solutions of the Dirac equation for $\Sigma=0$ (pseudospin symmetry) from the $\Delta=0$ (spin symmetry) case by using symmetry arguments. As an application, we have considered scalar, vector, and tensor Cornell radial potentials. For this case, the radial equation was mapped into a Scrödinger-like equation embedded in a three-dimensional harmonic oscillator plus a Cornell potential. We found the correct solution for this problem. Our results are definitely useful because they shed some light on this issue. Additionally, the correct solution for the Cornell potential may be useful due to wide applications in several physical problems.

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