Universal structure of the three-body system

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A nonlocal energy-dependent two-body quasipotential (E2Q) is defined within the framework of the three-body Faddeev formalism. The Fourier transform of this E2Q generates an energy-dependent Yukawa-type local potential. After an appropriate average of the potential with respect to the energy, a variety of local potentials with different ranges are obtained. These include a Yukawa potential, a Van der Waals potential, and a $1/r^2$ potential. An interesting potential appears in the πNN system, which gives rise to the one-pion-exchange NN interaction. It is also found that the Yukawa potential is automatically accompanied by an additional longer-range interaction. This potential could give rise to an infinite number of bound states near the threshold above the deuteron bound state with a more interesting physics than the Efimov effect.

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I. INTRODUCTION

After the three-body Faddeev equations [1] were proposed, it was pointed out that the exact theory of the scattering of a particle from a two-component composite target reduces to the multichannel Lippmann-Schwinger (LS)-type equation [2–5], also known as the Faddeev-Lovelace equations. However, such a LS-type equation consists of a nonlocal energy dependent two-body quasipotential (E2Q) and a two-body propagator. The E2Q or the nonlocal energy-dependent three-body potential is characterized by a particle transfer between a state $\{(ab)c\}$ with a compound quasiparticle (ab) and the elementary particle (c), and the other state $\{(bc)a\}$ or $\{(ca)b\}$.

The zero-energy limit of the three-body Faddeev equation was extensively investigated in combination with the Efimov effect, which arises with certain two-body interactions [6], especially when the two-body potential has a large scattering length. Such a two-body interaction could lead to a plethora of three-particle bound states emerging one after the other. Efimov physics has been widely studied theoretically (see, for example, Refs. [8–13]), while attempts to observe such a phenomenon in nature have not been fully successful. There are some reports for experimental evidence for the existence of Efimov states [7,14–16], and recently, the strength of the potential was varied artificially, and Kraemer *et al.* claim evidence for Efimov quantum states in an ultracold gas of cesium atoms [7]. However, one may be curious about what kind of *reality* exists in this phenomenon? Or, to put it in another way, what kind of *physics* could one extract?

In general, there are two types of mechanisms that generate a three-body binding. One is that the three particles combine simultaneously with $E \to -E_B$; the other is that two particles make a pair first [with the binding energy $\epsilon_B(>0)$], and the pair absorbs the other particle next, which means $E \equiv E_{\rm cm} - \epsilon_B \to (-\eta_B - \epsilon_B) \equiv -E_B$. Here, $E_{\rm cm}$, E, E_B, and η_B are the three-body center of mass energy, the three-body free energy, the three-body binding energy, and the separation energy, respectively.

Therefore, in the former type, if we take $E \to 0_-$ (i.e., E reaches 0 from the negative side of it) with $\epsilon_B = 0$, then $E_{\rm cm} \to 0_-$ where the Efimov levels exist. In the latter case, $E_{\rm cm} \to 0_-$, also occurs when $E \to -(\epsilon_B)_-$. One then wonders if the latter case creates Efimov-like states as well, although $\epsilon_B \neq 0$ or the scattering length is not infinity. Both cases can be investigated using a specific energy-dependent ($\mathcal{E} \equiv E_{\rm cm}$) E2Q [17].

In Sec. II, we present the E2Q and its Fourier transform. Since the E2Q as a function of the coordinate r is energy-dependent, we average it with respect to the energy by using a proper weight function, the so-called Laplace transform. This is described in Sec. III. As a result, the energy-independent E2Q becomes a $1/r^2$-type function for $r \to \infty$, which is described by a power series in a mass ratio. In Sec. IV, the lowest order is investigated by solving the Schrödinger equation to obtain a sequence of binding energies. A root mean square (rms) sequence is also found. Our theory is applied to the πNN three-body system in Sec. V, where we predict that the one-pion-exchange Yukawa potential is automatically accompanied by an additional longer range interaction. Such a potential generates a level condensation near zero energy in which the rms radius becomes very large. Finally, our discussions are presented in Sec. VI.

II. ENERGY-DEPENDENT TWO-BODY QUASIPOTENTIAL AND ITS FOURIER TRANSFORM

A. Definition of E2Q

In this paper, we assume, for convenience, that two of the particles are identical (particles $\alpha \equiv \beta$), but the theory could be easily extended to the three identical particle case.

To start with, we recall the Born term of the AGS equation [3,4] for three particles a, b, and c ($a \equiv b$).

$$\begin{aligned} Z_{\alpha n,\beta m}(\mathbf{q}_\alpha, \mathbf{q}_\beta; E) &= g_{\alpha n}(\mathbf{p}_\alpha) G_0(\mathbf{q}_\alpha, \mathbf{q}_\beta;\, E) g_{\beta m}(\mathbf{p}_\beta)\bar{\delta}_{\alpha\beta} \\ &= \frac{g_{\alpha n}(\mathbf{p}_\alpha) g_{\beta m}(\mathbf{p}_\beta)\bar{\delta}_{\alpha\beta}}{E - q_\alpha^2/2m_\alpha - q_\beta^2/2m_\beta - (\mathbf{q}_\alpha + \mathbf{q}_\beta)^2/2m_\gamma}, \end{aligned} \quad (1)$$

where α, β, and γ are the three-body channels—channel 1, channel 2, and channel 3—denoted by a_1-(b_2c_3), b_2-(c_3a_1), and c_3-(a_1b_2), respectively. Equation (1) results from the use of separable two-body potentials with form factors $g_1(\mathbf{p}_1)$, $g_2(\mathbf{p}_2)$, and $g_3(\mathbf{p}_3)$, where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the two-body relative momenta; $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are the three-body relative momenta while the subscripts n and m correspond to physical states for the corresponding channel and m_1, m_2, and m_3 are the particle masses. The $Z_{\alpha n,\beta m}(\mathbf{q}_\alpha, \mathbf{q}_\beta; E)$ is a function in the energy-momentum $(E; \mathbf{q}_\alpha, \mathbf{q}_\beta, \mathbf{q}_\gamma)$ plane [in short, $(E; q_\alpha)$ plane], while $\bar{\delta}_{\alpha\beta} \equiv 1 - \delta_{\alpha\beta}$. It should be noted that Eq. (1) is valid for the positive three-body free energy, $E > 0$, the integral variable range being $0 \leqslant q''_\alpha < \infty$, in the $(E; q''_\alpha)$ plane and corresponds to the two-body subenergy region of $-\infty < z''_\alpha \leqslant E$ for the on-shell condition of $E = q''^2_\alpha/2\mu_\alpha + z''_\alpha$ with the reduced mass: $\mu_\alpha = m_\alpha(m_\beta + m_\gamma)/(m_\alpha + m_\beta + m_\gamma)$. One could say that the variation of q''_α covers the intermediate energy spectrum in the region of the two-body subsystem $-\infty < z''_\alpha \leqslant E$. However, for the negative energy $E \leqslant 0$, the variation $0 \leqslant q''_\alpha < \infty$ covers the region $-\infty < z''_\alpha \leqslant -|E|$ only while the two-body bound states in $-|E| < z''_\alpha \leqslant 0$ are not covered in Eq. (1). On the other hand, in the energy-momentum plane between a particle and a pair $(E_{\rm cm}, q'')$ with $E_{\rm cm} = E + \epsilon_B$, the two-body subsystem could gain more energy $-|E| < z''_\alpha \leqslant -|E| + \epsilon_B$ than $-\infty < z''_\alpha \leqslant -|E|$ for the variation $0 \leqslant q'' < \infty$ in the intermediate region where the energy breaks the pair virtually and ignites a particle transfer in the E2Q model. This fact reminds us that the energy region $-|E| < z''_\alpha \leqslant -|E| + \epsilon_B$ should be compensated in Eq. (1) to satisfy the *completeness* of the intermediate state.

In order to satisfy the above requirement, we introduce the E2Q by using a new energy-momentum plane $(E_{\rm cm}; q) \equiv (E_{\rm cm}; \mathbf{q}, \mathbf{q}')$. The new plane is defined analogously to $(E; q_\alpha)$ plane, i.e.,

$$\begin{aligned} E &\to \mathcal{E} \equiv E_{\rm cm} = E + \epsilon_B; \\ \mathbf{q}_\alpha &\to -\mathbf{q}, \mathbf{q}_\beta \to \mathbf{q}', \\ \mathbf{q}_\gamma &= -\mathbf{q}_\alpha - \mathbf{q}_\beta \to \mathbf{q} - \mathbf{q}' \\ \mathbf{p}_\alpha &= \mathbf{q}_\beta + \frac{m_\beta}{m_\beta + m_\gamma}\mathbf{q}_\alpha \to \mathbf{q}' - \mathbf{q}/\Lambda_\beta = \mathbf{p} \\ \mathbf{p}_\beta &= -\mathbf{q}_\alpha - \frac{m_\alpha}{m_\gamma + m_\alpha}\mathbf{q}_\beta \to \mathbf{q} - \mathbf{q}'/\Lambda_\alpha = \mathbf{p}', \end{aligned} \tag{2}$$

with $\Lambda_\alpha = (1 + m_\gamma/m_\alpha) = 1 + \Delta_\alpha$, and $\Lambda_\beta = (1 + m_\gamma/m_\beta) = 1 + \Delta_\beta$. Here, $\mathbf{q} - \mathbf{q}' = \mathbf{q}_\gamma$ is the momentum transfer of the virtual particle with mass m_γ in E2Q, where the three-body center-of-mass system is taken as $\mathbf{q}_\alpha + \mathbf{q}_\beta + \mathbf{q}_\gamma = -\mathbf{q} + \mathbf{q}' + (\mathbf{q} - \mathbf{q}') = 0$.

Therefore, by using the new energy-momentum variables of Eq. (2), the E2Q could be formulated for the Feynman diagram (Fig. 1), which denotes the time-delayed virtual three particles lines. For the E2Q $Z_{\alpha n,\beta m}$, we now have

$$\begin{aligned} &Z_{\alpha n,\beta m}(-\mathbf{q}, \mathbf{q}'; E) \\ &= \frac{g_{\alpha n}(\mathbf{p}) g_{\beta m}(\mathbf{p}')\bar{\delta}_{\alpha\beta}}{E + \epsilon_B - q^2/2m_\alpha - q'^2/2m_\beta - (\mathbf{q} - \mathbf{q}')^2/2m_\gamma} \\ &= \frac{-2g_{\alpha n}(\mathbf{p}) m_\gamma g_{\beta m}(\mathbf{p}')\bar{\delta}_{\alpha\beta}}{-2m_\gamma(E + \epsilon_B) + \Lambda_\alpha q^2 + \Lambda_\beta q'^2 - 2\mathbf{q}\mathbf{q}'}, \end{aligned} \tag{3}$$

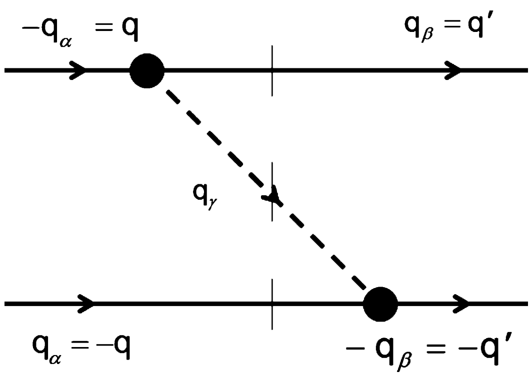

FIG. 1. The E2Q of one particle transfer Feynman diagram for an example of nonrelativistic πNN system. Solid lines denote nucleons, and the dashed line is the pion.

with $\mathbf{q}\mathbf{q}' = qq'x$. We may write Eq. (3) in a compact form as

$$Z_{\alpha n,\beta m}(-\mathbf{q}, \mathbf{q}'; E) = \frac{C^\gamma_{\alpha n,\beta m}(\mathbf{q}, \mathbf{q}')}{2qq'(\chi' - x)}, \tag{4}$$

where the form factor function $C^c_{a,b}$ is given by

$$C^\gamma_{\alpha n,\beta m}(\mathbf{q}, \mathbf{q}') \equiv -2g_{\alpha n}(\mathbf{p}) m_\gamma g_{\beta m}(\mathbf{p}')\bar{\delta}_{\alpha\beta}, \tag{5}$$

and the energy-momentum term χ' by

$$\chi' = \frac{-2m_\gamma(E + \epsilon_B) + \Lambda_\alpha q^2 + \Lambda_\beta q'^2}{2qq'}. \tag{6}$$

Although, the E2Q is defined for $E < 0$, it coincides with the AGS Born for $\epsilon_B = 0$. Therefore, the singularity at $E \to 0_-$ with $\epsilon_B = 0$, could cause the Efimov effect. It is also noted that the E2Q has a singularity at $\mathcal{E} = (E + \epsilon_B) \to 0_-$, or $E \to -(\epsilon_B)_- \neq 0$, which corresponds to the case of finite scattering length "a" in the Efimov's $(1/a - E)$ diagram [6].

B. Fourier Transform

Let us assume, without loss of generality, that $m_\gamma < m_\alpha = m_\beta$. Then, we have

$$\begin{aligned} \Delta_\alpha &= \Delta_\beta \equiv \Delta < 1 \\ \Lambda_\alpha &= \Lambda_\beta \equiv \Lambda, \end{aligned} \tag{7}$$

and $\Lambda_\alpha = \Lambda_\beta = \Lambda \equiv 1 + \Delta$. Therefore, Eq. (6) becomes

$$\chi' = \frac{-2m_\gamma(E + \epsilon_B)/\Lambda + q^2 + q'^2}{2qq'}\Lambda$$

$$= \frac{\sigma^2 + q^2 + q'^2}{2qq'}\Lambda \equiv \chi\Lambda \tag{8}$$

$$= \chi + \Delta\chi, \tag{9}$$

with

$$\chi \equiv \frac{\sigma^2 + q^2 + q'^2}{2qq'}. \tag{10}$$

Following the idea mentioned in the last paragraph of the previous subsection, we would like to investigate the case $\mathcal{E} = E + \epsilon_B \to 0_-$ [i.e., $\mathcal{E}$ reaches 0 from the negative side, or $E \to -(\epsilon_B)_-]$, then we have for $E \leqslant -\epsilon_B$,

$$\sigma^2 \equiv -\frac{2m_\gamma(E+\epsilon_B)}{\Lambda} \geqslant 0. \tag{11}$$

The assumption $\epsilon_B = 0$ leads (see our discussion bellow) to the Efimov effect. In contrast, the case $\epsilon_B \neq 0$ leads, as we shall see, to other interesting physical properties for the system.

In Eq. (4), the term $(\chi' - x)^{-1}$ is given by

$$\frac{1}{\chi'-x} = \frac{1}{(\chi'-\chi)+(\chi-x)} = \frac{2qq'}{\Delta[\sigma^2+q^2+q'^2]+[\sigma^2+(\mathbf{q}-\mathbf{q}')^2]}. \tag{12}$$

Here, by using well-known momentum transfer $\mathbf{q}'-\mathbf{q} = 2\mathbf{K}$, and the momentum sum $\mathbf{q}'+\mathbf{q} = \mathbf{Q}$, Eq. (12) is rewritten as,

$$\frac{1}{\chi'-x} = \frac{2qq'}{\Delta(\sigma^2+2K^2+Q^2/2)+(\sigma^2+4K^2)} = \frac{2qq'}{(\sigma^2+4K^2)(1+\Delta H)} = \frac{1}{(\chi-x)(1+\Delta H)}, \tag{13}$$

with

$$H \equiv \frac{\sigma^2+2K^2+Q^2/2}{\sigma^2+4K^2} = \frac{1}{2}\left[1+\frac{\sigma^2+Q^2}{\sigma^2+4K^2}\right]. \tag{14}$$

If we assume $\mathbf{Q} = \mathbf{0}$, then H becomes,

$$H = \frac{1}{2}\left[1+\frac{\sigma^2}{\sigma^2+4K^2}\right]. \tag{15}$$

Therefore, $1/(1+\Delta H)$ can be expanded with respect to ΔH, and then we obtain $Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E)$ is given by

$$Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E) = \frac{C^\gamma_{\alpha n,\beta m}(\mathbf{q},\mathbf{q}')}{\sigma^2+4K^2}\sum_{j=0}^{\infty}\left(\frac{-\Delta}{2}\right)^j\left[1+\frac{\sigma^2}{\sigma^2+4K^2}\right]^j. \tag{16}$$

Since the form factor function $C^\gamma_{\alpha n,\beta m}(\mathbf{q},\mathbf{q}')$ is a monotonic function with respect to the variable K in small σ value, then we can take $C^\gamma_{\alpha n,\beta m}(\mathbf{q},\mathbf{q}')$ as a constant $C_{\alpha n,\beta m}$. Therefore, by using Eq. (A3) in the Appendix, we obtain for Eq. (16)

$$Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E) = \frac{C_{\alpha n,\beta m}}{\sigma^2+4K^2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}{}_{j+k}C_k\left(\frac{-\Delta}{2}\right)^{j+k}\right]\left[\frac{\sigma^2}{\sigma^2+4K^2}\right]^k = \frac{C_{\alpha n,\beta m}}{\sigma^2+4K^2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}{}_{j+k}C_k z^{j+k}\right]y^k, \tag{17}$$

with

$$z = -\Delta/2 \tag{18}$$

and

$$y = \frac{\sigma^2}{\sigma^2+4K^2}. \tag{19}$$

By using the "binomial formula" Eq. (A5) and Eq. (18) we obtain

$$\sum_{j=0}^{\infty}{}_{j+k}C_k z^{j+k} = \frac{z^k}{(1-z)^{k+1}} = \frac{2(-\Delta)^k}{(2+\Delta)^{k+1}}. \tag{20}$$

Therefore, Eq. (17) becomes

$$Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E) = \frac{2C_{\alpha n,\beta m}}{(2+\Delta)(\sigma^2+4K^2)}\sum_{k=0}^{\infty}\left(\frac{-\Delta}{2+\Delta}\right)^k\left(\frac{\sigma^2}{\sigma^2+4K^2}\right)^k. \tag{21}$$

The Fourier transform of the above result in Eq. (21) is given by

$$\begin{aligned}\mathcal{F}\{Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E)\} &= \langle \mathbf{x}_\alpha | Z_{\alpha n,\beta m}(\mathbf{q}_\alpha,\mathbf{q}_\beta;E)|\mathbf{x}_\beta\rangle \\ &= \frac{2C_{\alpha n,\beta m}}{2+\Delta}\sum_{k=0}^{\infty}\left(\frac{-\Delta}{2+\Delta}\right)^k\iint\frac{d\mathbf{q}_\alpha}{(2\pi)^3}\frac{d\mathbf{q}_\beta}{(2\pi)^3} \\ &\quad\times\frac{\sigma^{2k}}{(\sigma^2+4K^2)^{k+1}}e^{i\mathbf{q}_\alpha\mathbf{x}_\alpha}e^{-i\mathbf{q}_\beta\mathbf{x}_\beta} \\ &= \frac{2C_{\alpha n,\beta m}}{2+\Delta}\sum_{k=0}^{\infty}\left(\frac{-\Delta}{2+\Delta}\right)^k\iint\frac{d\mathbf{K}}{(2\pi)^3}\frac{d\mathbf{Q}}{(2\pi)^3} \\ &\quad\times\frac{\sigma^{2k}}{(\sigma^2+4K^2)^{k+1}}e^{-i\mathbf{QR}}e^{-i\mathbf{Kr}},\end{aligned} \tag{22}$$

where we adopted some well-known coordinate relations and the set Eqs. (2),

$$2\mathbf{K} \equiv \mathbf{q}_\alpha+\mathbf{q}_\beta = \mathbf{q}'-\mathbf{q}, \tag{23}$$

$$\mathbf{Q} = \mathbf{q}'+\mathbf{q}, \tag{24}$$

$$\mathbf{q}' = \frac{\mathbf{Q}}{2}+\mathbf{K} = \mathbf{q}_\beta, \tag{25}$$

$$\mathbf{q} = \frac{\mathbf{Q}}{2}-\mathbf{K} = -\mathbf{q}_\alpha \tag{26}$$

$$q^2+q'^2 = \left(\frac{\mathbf{Q}}{2}-\mathbf{K}\right)^2+\left(\frac{\mathbf{Q}}{2}+\mathbf{K}\right)^2 = 2\left(K^2+\frac{Q^2}{4}\right) \tag{27}$$

$$\begin{aligned}\mathbf{q}_\alpha\mathbf{x}_\alpha-\mathbf{q}_\beta\mathbf{x}_\beta &= -\mathbf{q}\mathbf{x}_\alpha-\mathbf{q}'\mathbf{x}_\beta \\ &= -\left(\frac{\mathbf{Q}}{2}-\mathbf{K}\right)\mathbf{x}_\alpha-\left(\frac{\mathbf{Q}}{2}+\mathbf{K}\right)\mathbf{x}_\beta \\ &= -\mathbf{Q}\left(\frac{\mathbf{x}_\alpha+\mathbf{x}_\beta}{2}\right)-\mathbf{K}(\mathbf{x}_\beta-\mathbf{x}_\alpha) \\ &= -\mathbf{QR}-\mathbf{Kr},\end{aligned} \tag{28}$$

with

$$\mathbf{R} \equiv \frac{\mathbf{x}_\beta + \mathbf{x}_\alpha}{2}, \tag{29}$$

$$\mathbf{r} \equiv \mathbf{x}_\beta - \mathbf{x}_\alpha, \tag{30}$$

$\mathbf{R}$ being the center-of-mass coordinate with the corresponding momentum $\mathbf{Q}$, and $\mathbf{r}$ is the relative coordinate with the corresponding momentum $\mathbf{K}$. In terms of the new coordinates, the Jacobian is unity, i.e.,

$$d\mathbf{q}_\alpha d\mathbf{q}_\beta = \frac{\partial(\mathbf{q}_\alpha, \mathbf{q}_\beta)}{\partial(\mathbf{K}, \mathbf{Q})} d\mathbf{K} d\mathbf{Q} = d\mathbf{K} d\mathbf{Q}. \tag{31}$$

Therefore, the Fourier transform with respect to $\mathbf{Q}$ of Eq. (22) is as follows,

$$\mathcal{F}\{Z_{\alpha n,\beta m}(-\mathbf{q}, \mathbf{q}'; E)\} = \frac{2C_{\alpha n,\beta m}}{2+\Delta} \sum_{k=0}^{\infty} \left(\frac{-\Delta}{2+\Delta}\right)^k \delta(\mathbf{R}) \times \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{\sigma^{2k}}{(\sigma^2 + 4K^2)^{k+1}} e^{-i\mathbf{K}\mathbf{r}}, \tag{32}$$

with the center of mass (CM) δ-function factor,

$$\delta(\mathbf{R}) = \int e^{-i\mathbf{Q}\mathbf{R}} \frac{d\mathbf{Q}}{(2\pi)^3}. \tag{33}$$

Therefore, the integral part of the Eq. (32) is

$$J(\sigma; k) = \int \frac{e^{-i\mathbf{K}\mathbf{r}}}{(\sigma^2 + 4K^2)^{k+1}} \frac{d\mathbf{K}}{(2\pi)^3}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty \frac{\sin Kr}{(\sigma^2 + 4K^2)^{k+1}} K dK \tag{34}$$

$$= \frac{1}{2\pi^2 r} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial(\sigma^2)^k} \left(\frac{\pi}{8} e^{-\sigma r/2}\right). \tag{35}$$

Therefore, the Fourier transform of Eq. (3) is given [omitting the CM δ-function factor $\delta(\mathbf{R})$ in Eq. (32)] by

$$\begin{aligned}
&\mathcal{F}\{Z_{\alpha n,\beta m}(-\mathbf{q}, \mathbf{q}'; E)\} \\
&= \frac{2C_{\alpha n,\beta m}}{2+\Delta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\Delta}{2+\Delta}\right)^k \sigma^{2k} \\
&\quad \times \frac{1}{2\pi^2 r} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial(\sigma^2)^k} \left(\frac{\pi}{8} e^{-r\sigma/2}\right) \\
&= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)} \sum_{k=0}^{\infty} \left(\frac{\Delta}{2+\Delta}\right)^k \frac{\sigma^{2k}}{k!} \frac{\partial^k}{\partial(\sigma^2)^k} \left(\frac{e^{-r\sigma/2}}{r}\right) \\
&= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)} U(\Delta, \sigma; r),
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
U(\Delta, \sigma; r) &= \sum_{k=0}^{\infty} \left(\frac{\Delta}{2+\Delta}\right)^k \frac{\sigma^{2k}}{k!} \frac{\partial^k}{\partial(\sigma^2)^k} \left(\frac{e^{-r\sigma/2}}{r}\right) \equiv \sum_{k=0}^{\infty} U^{(k)}(\Delta, \sigma; r) = \left\{\frac{1}{r} e^{-r\sigma/2} + \left(\frac{\Delta}{2+\Delta}\right) \frac{\sigma^2}{1!} \left(\frac{-1}{4\sigma}\right) e^{-r\sigma/2}\right. \\
&\quad \left. + \left(\frac{\Delta}{2+\Delta}\right)^2 \frac{\sigma^4}{2!} \left(\frac{(-1)^2(r\sigma/2+1)}{(2\sigma)^3}\right) e^{-r\sigma/2} + \left(\frac{\Delta}{2+\Delta}\right)^3 \frac{\sigma^6}{3!} \left(\frac{(-1)^3(\sigma^2 r^2/2 + 3r\sigma + 6)}{(2\sigma)^5}\right) e^{-r\sigma/2} + \cdots\right\} \\
&= \left\{\frac{1}{r} e^{-r\sigma/2} + \left(\frac{\Delta}{2+\Delta}\right) \frac{\sigma}{1!} \frac{(-1)}{4} e^{-r\sigma/2} + \left(\frac{\Delta}{2+\Delta}\right)^2 \frac{\sigma}{2!} \left(\frac{(-1)^2(r\sigma/2+1)}{2^3}\right) e^{-r\sigma/2}\right. \\
&\quad \left. + \left(\frac{\Delta}{2+\Delta}\right)^3 \frac{\sigma}{3!} \left(\frac{(-1)^3(\sigma^2 r^2/2 + 3r\sigma + 6)}{2^5}\right) e^{-r\sigma/2} + \cdots\right\}.
\end{aligned} \tag{37}$$

This means that the first term is an energy-dependent Yukawa-type potential and the higher terms are proportional to an exponential function with respect to r and σ (with $\sigma^2 = -2m_\gamma(E + \epsilon_B)/\Lambda$). It should be mentioned that in the zero-energy limit, with which we are concerned, it becomes a Coulomb like potential.

III. AN ENERGY AVERAGE BY A LAPLACE TRANSFORM OF THE E2Q

It is perhaps difficult to understand that a Coulomb-type potential can be generated in the three-hadron system, because historically the three-hadron Faddeev calculations didn't reveale such a feature. The origin of this curious phenomenon is caused by the energy dependence of the particle transfer interaction in the E2Q for $E_{\rm cm} \to 0$, or the AGS's Born term and the kernel in the Faddeev formalism for $E \to 0$ with $\epsilon_B \to 0$.

An energy-independent potential is difficult to obtain from a three-body formalism. We propose here to take the "statistical average" by using the *probability density function* with respect to the possible energy range, which also represents effects of the structure or the form factors of the composite particles,

$$P_\sigma = \sigma^{2\gamma+1} e^{-a\sigma} \Big/ \int_0^\infty \sigma^{2\gamma+1} e^{-a\sigma} d\sigma \equiv \frac{\sigma^{2\gamma+1} e^{-a\sigma}}{\rho}, \tag{38}$$

with

$$\rho = \int_0^\infty \sigma^{2\gamma+1} e^{-a\sigma} d\sigma = \frac{\Gamma(2\gamma+2)}{a^{2\gamma+2}}, \tag{39}$$

where the weight function (or the probability density function) $\sigma^{2\gamma+1}$ is adopted, and $e^{-a\sigma}$ is the damping factor when $\sigma = \sqrt{-2m_\gamma(E + \epsilon_B)/\Lambda}$ or the three-body energy $|E|$ increases. The weight function stems from the dispersion theory; however, its description is beyond the scope of this paper and details concerning this function will be presented elsewhere.

Using the probability density function, the expectation value of the energy-dependent potential becomes energy independent. This is the Laplace transform or the Euler integral of the second kind in the first term of Eq. (37). Therefore, by using Eqs. (38) and (39), the first order potential with $k = 0$ in (37) is

$$\mathcal{L}\{U^{(0)}(\Delta,\sigma;r)\} = \frac{1}{\rho}\int_0^\infty \sigma^{2\gamma+1}e^{-a\sigma}\frac{e^{-\sigma r/2}}{r}d\sigma = \frac{a^{2\gamma+2}}{r(r/2+a)^{2\gamma+2}}, \tag{40}$$

where, if one takes $\gamma = 3/2$ in Eq. (40), we obtain a Van der Waals-type potential, which for $r \gg a$ is of the London-type similar to the potential between two atoms, and of the Yukawa-type for $a \gg r$,

$$\mathcal{L}\{U^{(0)}(\Delta,\sigma;r)\} = \frac{a^5}{r(r/2+a)^5} = \frac{a_0^5}{r(r+a_0)^5} \approx \begin{cases} \dfrac{a_0^5}{r^6} & \text{for} \quad r \gg a, \\ \dfrac{e^{-5r/a_0}}{r} & \text{for} \quad a \gg r, \end{cases} \tag{41}$$

with $2a = a_0$, and also the Casimir-type for $\gamma = 2$, with $r \gg a$, and the Yukawa-type for $a \gg r$,

$$\mathcal{L}\{U^{(0)}(\Delta,\sigma;r)\} = \frac{a^6}{r(r/2+a)^6} = \frac{a_0^6}{r(r+a_0)^6} \approx \begin{cases} \dfrac{a_0^6}{r^7} & \text{for} \quad r \gg a, \\ \dfrac{e^{-6r/a_0}}{r} & \text{for} \quad a \gg r. \end{cases} \tag{42}$$

Such a short-range Yukawa-type potential comes from a particle transfer in the three particle system, although the residual terms of Eq. (37) contribute to the shorter range.

If and only if the weight function is replaced by

$$\sigma^{2\gamma+1}e^{-a\sigma} \to \delta(\sigma - 2\mu_0) \tag{43}$$

with the meson range $1/\mu_0$, then the potential is a Yukawa potential [18,19],

$$\mathcal{L}\{U^{(0)}(\Delta,\sigma;r)\} = \frac{1}{\rho}\int_0^\infty \delta(\sigma-2\mu_0)\frac{e^{-\sigma r/2}}{r}d\sigma = \frac{e^{-\mu_0 r}}{r}, \tag{44}$$

since in this case

$$\rho = \int_0^\infty \delta(\sigma - 2\mu_0)d\sigma = 1. \tag{45}$$

However, the condition Eq. (43) seems to be *arbitrary* in this case.

In Eq. (16), if the form factor $C^c_{a,b}$, which is defined in Eq. (5) is energy independent and monotonic for very small momentum transfer $K \approx 0$, the weight function $\sigma^{2\gamma+1}$ could be a constant, which means $\gamma = -1/2$ in Eq. (40).

In such a case, the energy- or σ-independent E2Q of Eq. (36) is given, using the coupling constant $C_{\alpha n,\beta m}/[8\pi(2+\Delta)]$ for the three-body center of mass system by

$$\begin{aligned} V_{\alpha n,\beta m}(\Delta;r) &\equiv \mathcal{LF}\{Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E)\} \\ &= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)}\mathcal{L}\{U(\Delta,\sigma;r)\} \qquad (46) \\ &= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)}\frac{1}{\rho}\int_0^\infty U(\Delta,\sigma;r)e^{-a\sigma}d\sigma \qquad (47) \\ &= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)}\left\{\frac{a_0}{r(r+a_0)} + \frac{(-1)}{1!2^2}\left(\frac{\Delta}{2+\Delta}\right)\frac{2a_0}{(r+a_0)^2}\right. \\ &\quad + \frac{(-1)^2}{2!2^3}\left(\frac{\Delta}{2+\Delta}\right)^2\left[\frac{2a_0}{(r+a_0)^2} + \frac{2^2ra_0}{(r+a_0)^3}\right] \\ &\quad + \frac{(-1)^3}{3!2^5}\left(\frac{\Delta}{2+\Delta}\right)^3\left[\frac{6\cdot 2a_0}{(r+a_0)^2} + \frac{6\cdot 2^2ra_0}{(r+a_0)^3}\right. \\ &\quad \left.\left. + \frac{3\cdot 2^3r^2a_0}{(r+a_0)^4}\right] + \cdots\right\}. \qquad (48) \end{aligned}$$

This potential goes to $1/r^2$ limit for $r \to \infty$, while for $r < a_0$ it reaches the Yukawa-type potential plus an exponential one. The latter case is not quantitatively correct for the very low energy case; however, it is correct for the long range case, i.e.,

$$\begin{aligned} \lim_{r\to\infty} V_{\alpha n,\beta m}(\Delta;r) &= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)}\left\{1 - \frac{1}{2}\left(\frac{\Delta}{2+\Delta}\right) + \frac{3}{8}\left(\frac{\Delta}{2+\Delta}\right)^2 - \frac{5}{16}\left(\frac{\Delta}{2+\Delta}\right)^3 + \cdots\right\}\frac{a_0}{r^2} \\ &= \frac{C_{\alpha n,\beta m}}{8\pi(2+\Delta)}\frac{Sa_0}{r^2} \equiv \frac{V_0 a_0}{r^2}, \qquad (49) \end{aligned}$$

where the constant S depends on the masses of the three particles. For the three identical particles case with $\Delta = m_\gamma/m_\alpha = 1$ then $S = 0.8634$. For the $NN\pi$-system where $\Delta = m_\gamma/m_\alpha = M_\pi/M_N = 0.14703$, we obtain $S = 0.96741$. This means that the first-order potential approximation in Eq. (49) is 86.34% correct for the three equal mass case, while for the $NN\pi$ system in one pion transfer is 96.74% correct.

IV. PROPERTIES OF THE $1/R^2$ POTENTIAL

Adopting the elastic channel in the corresponding state, we can reduce Eq. (48) with a simple coupling constant $V_0 \equiv C_{\alpha n,\beta m}/[8\pi(2+\Delta)]$ and by omitting for simplicity channel notations,

$$\begin{aligned} V(\Delta;r) &\equiv \mathcal{LF}\{Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E)\} \\ &\approx \mathcal{LF}\{Z(-\mathbf{q},\mathbf{q}';E)\} \\ &= V_0\mathcal{L}\{U(\Delta,\sigma;r)\} \equiv \sum_{k=0}^\infty V^{(k)}(\Delta;r). \qquad (50) \end{aligned}$$

In order to make our discussion clearer, let us consider as an example the nonrelativistic πNN-system at very low energy. In such a case the heavy particle transfer channels can be omitted because the kinematic coupling coefficients become smaller than the elastic one. Then, we obtain the first-term potential of Eq. (37) or the Yukawa-type of Eqs. (40), (48), and (50) by using $2a = a_0$:

$$V^{(0)}(\Delta; r) = \frac{V_0 a}{r(r/2 + a)} = \frac{V_0 a_0}{r(r + a_0)}. \tag{51}$$

Therefore, the behavior of this potential is given for small r, or for $r \ll a_0$, as

$$V^{(0)}(\Delta; r) = \frac{V_0 a_0}{r a_0(1 + r/a_0)} \approx V_0 \frac{e^{-r/a_0}}{r}. \tag{52}$$

This potential suggests an important meson exchange potential in which case the range is a_0 or the meson mass $m = \mu_0 \approx 1/a_0$ for the Yukawa potential.

For large r, or $r \gg a_0$, one has

$$V^{(0)}(\Delta; r) = \frac{V_0 a_0}{r^2(1 + a_0/r)} \approx \frac{V_0 a_0}{r^2}. \tag{53}$$

The latter potential with $a_0 \ll r$ shows a long-range form with r^{-2} in this simple approximation. It should be noted that the reliability of the first-order approximation could be estimated in terms of the $\Delta/(2 + \Delta)$ value in Eq. (49).

The Schrödinger equation for the potential of Eq. (51) is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\psi_l(r) + \left[\frac{V_0 a_0}{r(r + a_0)} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] \times \psi_l(r) = \mathcal{E}\psi_l(r), \tag{54}$$

where m is the reduced mass between a pair and the rest particle, or $m = M_N/2$ in the $(\pi N)N$-system. $\mathcal{E} \equiv E + \epsilon_B$ is the three-body energy, which is composed of the three-body free energy and a pair-binding energy or resonant energy. If a resonant pair is given by a pair interaction with a very large scattering length, then the system satisfies the Efimov condition. However, Eq. (54) is also correct for a bound pair with the binding energy $\epsilon_B (\geqslant 0)$ plus a particle such as N-(πN) and n-d systems, etc., by $\mathcal{E} = E + \epsilon_B$. Therefore, being free from the restriction on two-body interactions for appearance of the Efimov states, our approach spreads over a more wide field than the one of Efimov and generates three-body bound states $\mathcal{E} = -E_n$ one after the other.

In order to look at the long-range behavior, we can reduce the equation to

$$\left[\frac{d^2}{dr^2} - \kappa^2 - \frac{\lambda}{r^2}\right]\chi_l(r) = 0 \tag{55}$$

or to the Bessel's function differential equation

$$\left[\frac{d^2}{dr^2} + \left(\beta^2 - \frac{\nu^2 - 1/4}{r^2}\right)\right]\chi_l(r) = 0, \tag{56}$$

where $\kappa^2 = -\beta^2 = -2m\mathcal{E}/\hbar^2 > 0$ for negative energies, which are verified by the effective potential factor: $\lambda = \nu^2 - 1/4 = l(l+1) - 2m|V_0|a_0/\hbar^2 < 0$ with $V_0 < 0$ and $a_0 > 0$. Let us define ν^2, which must be negative to have the binding energies, by

$$\begin{aligned}\nu^2 &= \tfrac{1}{4} + \lambda = \tfrac{1}{4} + l(l+1) - 2m|V_0|a_0/\hbar^2 \\ &= \left(l + \tfrac{1}{2}\right)^2 - 2m|V_0|a_0/\hbar^2.\end{aligned} \tag{57}$$

For the case $\lambda < -1/4$, or $(l + \frac{1}{2})^2 < 2m|V_0|a_0/\hbar^2$,

$$\begin{aligned}\nu &= i\sqrt{|\lambda + 1/4|} \\ &= i\sqrt{2m|V_0|a_0/\hbar^2 - \left(l + \frac{1}{2}\right)^2} \equiv i\mu,\end{aligned} \tag{58}$$

where

$$\mu^2 = \frac{2m}{\hbar^2}|V_0|a_0 - \left(l + \frac{1}{2}\right)^2. \tag{59}$$

In this case we have the solution for the modified Bessel function [20],

$$\chi(r) = \sqrt{\kappa r}Z_\nu(i\kappa r) \equiv \sqrt{\kappa r}K_{i\mu}(\kappa r). \tag{60}$$

Because of the pure imaginary index $\nu = i\mu$, the modified Bessel function is

$$K_{i\mu}(\kappa r) = \frac{\pi}{2\sin i\mu\pi}\{I_{-i\mu}(\kappa r) - I_{i\mu}(\kappa r)\}. \tag{61}$$

For small κr, it becomes

$$\begin{aligned}I_{i\mu}(\kappa r) &= \frac{(\kappa r/2)^{i\mu}}{\Gamma(1 + i\mu)} + \dots \\ &= \frac{(\kappa r/2)^{i\mu}}{|\Gamma(1 + i\mu)|e^{i\arg\Gamma(1+i\mu)}} + \dots \\ &= \frac{e^{i\{\mu\log(\kappa r/2) - \arg\Gamma(1+i\mu)\}}}{|\Gamma(1 + i\mu)|}\{1 + o(\kappa r)\}.\end{aligned} \tag{62}$$

Therefore, we have

$$\begin{aligned}K_{i\mu}(\kappa r) &= \frac{\pi}{2\sin i\mu\pi}\frac{1}{|\Gamma(1 + i\mu)|} \\ &\quad \times [e^{-i\{\mu\log(\kappa r/2) - \arg\Gamma(1+i\mu)\}} \\ &\quad - e^{i\{\mu\log(\kappa r/2) - \arg\Gamma(1+i\mu)\}}]\{1 + o(\kappa r)\} \\ &= \frac{\pi}{\sinh\pi\mu}\frac{1}{|\Gamma(1 + i\mu)|} \\ &\quad \times \sin\left(\mu\log\frac{\kappa r}{2} - \arg\Gamma(1 + i\mu)\right)\{1 + o(\kappa r)\}.\end{aligned} \tag{63}$$

In order to smoothly connect $K_{i\mu}(\kappa r)$ with the full-wave function $\psi_l(\kappa r)$ and the potential $V(r) = V_0 a_0/[r(r + a_0)]$, the logarithmic derivative should be employed at sufficiently large $r = a$. For the modified Bessel function, it reduces for small κr to

$$\begin{aligned}&\kappa r\frac{d\{\sqrt{\kappa r}K_{i\mu}(\kappa r)\}/d(\kappa r)}{\sqrt{\kappa r}K_{i\mu}(\kappa r)} \\ &\quad = \frac{1}{2} + \mu\cot\left[\mu\log\frac{\kappa r}{2} - \arg\Gamma(1 + i\mu)\right].\end{aligned} \tag{64}$$

The periodicity of "cot" leads to the relationship

$$\mu\log\frac{\kappa a}{2} = C - n\pi \qquad \{n = 1, 2, \dots\}. \tag{65}$$

Therefore,

$$\kappa = \frac{2}{a}e^{(C - n\pi)/\mu} \equiv \kappa_n. \tag{66}$$

Therefore, the binding energy $-E_n$ (for $n = 1, 2, \ldots$) is

$$-E_n = \frac{\kappa_n^2}{2m} = \left(\frac{2}{ma^2}e^{2C/\mu}\right)e^{-2\pi n/\mu}. \tag{67}$$

This means that one needs n different boundary conditions to fit the full-wave function smoothly, corresponding to the different energies E_n.

From this relationship the energy ratio has a typical structure for E_n and Eq. (56),

$$\frac{E_n}{E_{n+1}} = e^{2\pi/\mu}, \tag{68}$$

or

$$E_{n+1} = E_n e^{-2\pi/\mu}. \tag{69}$$

In order to fix the parameter μ, for example, we will consider the $NN\pi$ three-body system, where the NN bound state is given by the π exchange potential. Let us take $E_0 = -2.2246$ MeV for the deuteron binding energy. It should be stressed here that the deuteron bound state and other two-body properties are generated by the short-range potential. Therefore, the additional long-range potential $V_0a_0/r(r+a_0) - V_0e^{-r/a_0}/r \approx V_0a_0/r^2$ should not change these characteristics. One can smoothly connect the additional potential to the traditional core potential at a sufficiently long range.

For clarity, we adopt the parameters V_0 and a_0 to fit the deuteron binding energy, although one knows that the $^3S_1$-$^3D_1$ coupling, with the potential elements $V_{ll'} = V_{00}$, V_{02}, V_{20}, V_{22}, is important to reproduce the deuteron binding energy (these issues, however, belong to the traditional short-range potentials). The additional potential mainly contributes to the diagonal part, but the repulsive centrifugal potential in the element V_{22} will be weakened by our additional attractive V_2a_2/r^2 potential (with D-wave parameters V_2, a_2). As a result, the leading term of the long-range potential should be V_{00}. In this context, we can choose the central potential: $V_0 = -50.577$ MeV·fm, and $a_0 = 1.4295$ fm, then we obtain $V_0a_0 = -72.3$ MeV·fm^2$= -0.36641$ fm, and $2m = M_N = 938.90$ MeV $= 4.7583$ fm^{-1} for $\hbar = c = 1$. Therefore, Eq. (59) becomes

$$\begin{aligned}\mu^2 &= 4.7583 \times 0.36641 - \left(l + \tfrac{1}{2}\right)^2\\ &= 1.7435 - \left(l + \tfrac{1}{2}\right)^2 \geqslant 0.\end{aligned} \tag{70}$$

This suggests that only $l = 0$ supports a bound state, and we find $\mu = 1.2221 > 0$. However, it is obvious that the bound state is not an Efimov state because the present example does not require an infinite value of the scattering length. From the value $\mu = 1.2221$, the factors $e^{2\pi/\mu}$ and $e^{\pi/\mu}$ are obtained:

$$e^{2\pi/\mu} = 170.98 \tag{71}$$

$$e^{\pi/\mu} = 13.076. \tag{72}$$

The rms radius is calculated by using the wave function of Eq. (60). Let us define

$$\begin{aligned}\xi[c] &\equiv \int_0^\infty dx x^c K_a(x)^2\\ &= \frac{\sqrt{\pi}\Gamma([1+c]/2)\Gamma([1-2a+c]/2)\Gamma([1+2a+c]/2)}{4\Gamma(1+c/2)}.\end{aligned} \tag{73}$$

In order to obtain the rms radius, we adopt

$$\xi[3] = \frac{1}{\kappa^3}\int_0^\infty d(\kappa r)(\kappa r)^3[K_{i\mu}(\kappa r)]^2, \tag{74}$$

$$\xi[1] = \frac{1}{\kappa}\int_0^\infty d(\kappa r)(\kappa r)[K_{i\mu}(\kappa r)]^2. \tag{75}$$

Therefore, the rms radius is given for the modified Bessel wave function:

$$\begin{aligned}\langle r^2\rangle_n &= \frac{\int_0^\infty dr\chi_l^*(r)r^2\chi_l(r)}{\int_0^\infty dr\chi_l^*(r)\chi_l(r)} = \frac{\xi[3]}{\xi[1]}\\ &= \frac{2}{3\kappa^2}(1+\mu^2) = \frac{(1+\mu^2)}{-3mE_n},\end{aligned} \tag{76}$$

$$\langle r\rangle_n \equiv \sqrt{\langle r^2\rangle_n} = \sqrt{\frac{(1+\mu^2)}{3m|E_n|}} = \sqrt{\frac{2(1+\mu^2)}{3M_N|E_n|}} \equiv r_n. \tag{77}$$

Therefore, we obtain the ratio relation between the rms radius-ratio and the energy-ratio by using Eq. (76),

$$\frac{E_n}{E_{n+1}} = \left(\frac{r_{n+1}}{r_n}\right)^2 = e^{2\pi/\mu}. \tag{78}$$

Thus,

$$r_{n+1} = r_n e^{\pi/\mu}. \tag{79}$$

From this formula, we find that the rms radius becomes larger, corresponding to the shallower binding energies in the case $e^{\pi/\mu} = 13.076 \geqslant 1$.

These formulas for the binding energy and rms radius were discussed by Sawada for the nucleon-monopole system with the $1/r^2$ potential in 1993 [21]. If the condition $e^{2\pi/\mu} \geqslant 1$ is satisfied, then the series of binding energies for the bound states will be smaller and smaller, while the rms radius becomes larger and larger. However, our original potential Eq. (51) doesn't satisfy Eq. (76) for small quantum numbers n, because the modified Bessel function leads to $r_0 = 5.56619$ fm. This value is much larger than $r_0 = 2.5155$ fm, which is obtained using our potential with Eq. (51).

Since the above discussions are general, one may apply the formalism to various systems, such as the 3α cluster model for the ^{12}C nucleus, the $3N$ systems for ^{3}H and ^{3}He nuclei, etc. [22,23].

V. AN APPLICATION FOR THE πNN THREE-BODY SYSTEM

The Faddeev-Lovelace equation has the form of the many-channel two-particle Lippmann-Schwinger equation, if one identifies the Born term $Z_{\alpha n,\beta m}$ with the potential and $\tau_{\gamma s}$

with the propagator [3]. The equation was applied to the $3N$ and $N\pi\pi$ systems. Later on, it was generalized for the relativistic case [24,25]. Thomas calculated π-d scattering using the Faddeev-Lovelace equation for the three-body πNN system and obtained a good fit with the experimental data [26]. Although the major objection to this work was that it did not provide a covariant theory, Thomas claimed the issue should not be very important below 100 MeV for the πNN system.

In this section, we would like to evaluate the binding energy of the πNN three-body system, especially the shallow binding energy where the root mean square radius is rather large. This means that the higher momentum components are small, so that relativistic effects could be smaller than in the triton case.

Assuming that the nucleon is a composite particle (πN), then the two-nucleon bound state could be investigated by the three-body rearrangement scattering amplitude, $N_1 + (N_2\pi) \to N_2 + (\pi N_1)$. This process could construct "the meson exchange potential," where the meson transfer could be written by the momentum transfer between two nucleons under the three-body threshold.

Following the Thomas formalism, the Born term is given, using the pion energy ω_π and mass m_π, and for $0 < E$, by

$$\begin{aligned} Z_{N\pi,N\pi}(\mathbf{q}_\alpha,\mathbf{q}_\beta;E) &= g_{N\pi}(\mathbf{p}_\alpha)G_0(\mathbf{q}_\alpha,\mathbf{q}_\beta;\,E)g_{N\pi}(\mathbf{p}_\beta)\overline{\delta}_{\alpha\beta} \\ &= \frac{g_{N\pi}(\mathbf{p}_\alpha)g_{N\pi}(\mathbf{p}_\beta)\overline{\delta}_{\alpha\beta}}{E - q_\alpha^2/2m_\alpha - q_\beta^2/2m_\beta - (\omega_\pi - m_\pi)}, \end{aligned} \tag{80}$$

with

$$\omega_\pi = \sqrt{q_\gamma^2 + m_\pi^2} = \sqrt{(\mathbf{q}_\alpha + \mathbf{q}_\beta)^2 + m_\pi^2} \tag{81}$$

$$\approx m_\pi + \frac{(\mathbf{q}_\alpha + \mathbf{q}_\beta)^2}{2m_\pi} \quad \text{for} \quad q_\gamma \ll m_\pi. \tag{82}$$

Therefore, at the very low energy limit, Eq. (80) is essentially equivalent with Eq. (1), and, moreover, it gives Eq. (3) for the ($N\pi$)-bound N-($N\pi$)-interaction or nucleon-nucleon interaction for the case $E \leqslant -\epsilon_{N\pi}$ or $\mathcal{E} \leqslant 0$, i.e.,

$$Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E) = \frac{C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}')}{2qq'(\chi' - x)}, \tag{83}$$

with

$$C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}') \equiv -2g_{N\pi}(\mathbf{p})m_\pi g_{N\pi}(\mathbf{p}'), \tag{84}$$

and

$$\chi' = \frac{-2m_\pi(E+\epsilon_{N\pi})/\Lambda + q^2 + q'^2}{2qq'}\Lambda \tag{85}$$

$$= \frac{\sigma^2 + q^2 + q'^2}{2qq'}\Lambda \equiv \chi\Lambda = \chi + \Delta\chi \tag{86}$$

$$\sigma^2 \equiv -\frac{2m_\pi(E+\epsilon_{N\pi})}{\Lambda} = -\frac{2m_\pi\mathcal{E}}{\Lambda} \geqslant 0. \tag{87}$$

$$\Lambda = (1 + m_\pi/M_N) = 1 + \Delta = 1 + 0.147, \tag{88}$$

where $-\epsilon_{N\pi} = -m_\pi$ is the πN binding energy.

In the framework of the above discussion, the πNN three-body system reveals a characteristic long range property of the nucleon-nucleon (or *three-body NN*) interaction, which should be discriminated from the two-body NN subsystem (not treated in this paper). In addition, this $1/r^2$ potential could have an infinite number of energy levels at $\mathcal{E} \leqslant 0$, which belongs to the S-wave *three-body NN* bound states. These bound states are *not* equivalent to the Efimov states because any two-body subsystems in πNN system do not have infinite scattering length.

It is known that the most important state in the NN system is the triplet state and that the deuteron bound state mainly depends on the D state. However, from the discussion under Eq. (69), if and only if we fit the S wave parameters V_0 and a_0 to the deuteron binding energy, the first excited state ($n = 1$) of the deuteron could be

$$E_1 \approx E_0/e^{2\pi/\mu} = -2.2246/170.98 \approx -13\ \text{keV}, \tag{89}$$

$$r_1 \approx r_0 \times e^{\pi/\mu} = 1.97 \times 13.076 \approx 26\ \text{fm}. \tag{90}$$

Here, for the lower quantum number states (or $n = 1, 2, 3$) we may have slight inaccuracies, because one needs the full wave function to fit the deuteron binding energy. Nevertheless, it should be pointed out that such an energy region ($|E| \leqslant 50$ keV) for the bound states and the phase shift below the energy 100 keV have not been measured yet [27].

VI. SUMMARY AND DISCUSSION

We introduced the E2Q in the negative three-body free energy, where the virtual particle transfer between pairs exists. After the Fourier transform of this E2Q, an energy average with respect to the probability density function P_σ leads to a $1/r^2$-type potential for $a_0 \ll r$ and $\gamma = -1/2$, where the form factor is monotonic for our reference energy-momentum region, while for $\gamma = 3/2$ we obtain the Van der Waals-type potential. For this reason, one could say that the distribution function contains the characteristics of the two-body form factors. There is a possibility that the Yukawa potential is accompanied by long-range potentials, which are of the $1/r^2$ type or of the Van der Waals type, depending on the energy-dependent particle transfer interaction.

We applied our prediction to the three-body $NN\pi$ system, where $N_1 + (N_2\pi) \to N_2 + (\pi N_1)$ channel generates the $1/r^2$ potential at the longer range, which is more important than the heavy particle transfer, $N_1 + (N_2\pi) \to \pi + (N_1N_2)$, because the latter channel rather contributes to the shorter range potential, and therefore is neglected in this paper.

The deuteron binding energy is used for the energy scaling and estimating the series of the higher states, $E_{n+1} = e^{-2\pi/\mu}E_n$, and also the rms radius, $r_{n+1} = e^{\pi/\mu}r_n$. If these shallow binding energies and large rms radii exist, the reaction energy could be within the keV region or the chemical reaction energies, and the size of the compound nucleus could be enlarged from the picometer to the atomic radius.

One may feel uncomfortable with such a classical meson theoretical treatment that ignores the quark degrees of freedom in the modern QCD theory. However, the effects of the constituent quarks could be realized mainly in the repulsive hard or soft core of the very short-range NN interaction.

Besides, our discussion in this paper concerns the longer range region where one (or more) pion exchange occurs.

Even if the $1/r^2$ potential does not support bound states when the condition $\mu^2 \geqslant 0$ in Eq. (59) is not fulfilled, this potential could interfere with the centrifugal and the Coulomb potentials.

The three-body force at very low energy could originate from this long-range potential, because the explanation in terms of the Δ-isobar-origin of the Fujita-Miyazawa three-body force is hard to be justified in the very low energy region. In addition, the $1/r^2$ potential could affect not only the final-state interaction in the break-up experiments with respect to nn, pp, and np, but also a very wide field of other nuclei, such as the neutron- or proton-rich nuclei [28], unknown *nuclear materials* (not nuclear matter) in the universe, and, moreover, the atomic- and molecular physics.

Finally, by means of the particle transfer type potentials, the $1/r^2$-type potential structure could universally occur, not only in three-body systems but also in many-body systems [29–31]. $1/r^2$ is a long-range potential next to the Coulomb; however, it belongs to the strong interaction, and thus it could pave the way to a *pico-size science* together with the *deep* (short-range) Coulomb potential.

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APPENDIX: BINOMIAL SERIES

In order to prove Eq. (20), let us consider some relations for the case $|x| < 1$:

$$\begin{aligned}\sum_{k=0}^{\infty} x^k(1+y)^k &= 1 + x(1+y) + x^2(1+y)^2 \\ &\quad + x^3(1+y)^3 + \cdots + x^n(1+y)^n + \cdots \\ &= 1 + x + xy + x^2[{}_2C_0y^0 + {}_2C_1y^1 + {}_2C_2y^2] \\ &\quad + x^3[{}_3C_0y^0 + {}_3C_1y^1 + {}_3C_2y^2 + {}_3C_3y^3] \\ &\quad + \cdots + x^n\sum_{l=0}^{n} {}_nC_l y^l + \cdots \end{aligned} \tag{A1}$$

$$\begin{aligned} &= y^0\{1 + x + x^2 + x^3 + \cdots\} \\ &\quad + y\{{}_1C_1x + {}_2C_1x^2 + {}_3C_1x^3 + \cdots\} \\ &\quad + y^2\{{}_2C_2x^2 + {}_3C_2x^3 + {}_4C_2x^4 \cdots\} \\ &\quad + y^3\{{}_3C_3x^3 + {}_4C_3x^4 + {}_5C_3x^5 \cdots\} \\ &\quad + y^4\{{}_4C_4x^4 + {}_5C_4x^5 + {}_6C_4x^6 + \cdots\} \\ &\quad + \cdots + y^l\sum_{k=0}^{\infty} {}_{k+l}C_l x^{k+l} + \cdots . \end{aligned} \tag{A2}$$

Therefore, we have

$$\sum_{k=0}^{\infty} x^k(1+y)^k = \sum_{l=0}^{\infty}\left[\sum_{k=0}^{\infty} {}_{k+l}C_l x^{k+l}\right] y^l \tag{A3}$$

$$= \sum_{l=0}^{\infty} \frac{x^l}{(1-x)^{l+1}} y^l, \tag{A4}$$

where we used the formula

$$\sum_{k=0}^{\infty} {}_{k+l}C_l x^{k+l} = \frac{x^l}{(1-x)^{l+1}}, \tag{A5}$$

which is obtained by the formula of the "negative binomial series" with $s - 1 = n$,

$$\begin{aligned}\frac{1}{(1-x)^s} &= \sum_{k=0}^{\infty} {}_{k+s-1}C_k x^k \equiv \sum_{k=0}^{\infty} {}_{k+s-1}C_{s-1} x^k \\ &= \frac{1}{x^n}\sum_{k=0}^{\infty} {}_{k+n}C_n x^{k+n} \equiv \frac{1}{(1-x)^{n+1}}. \end{aligned} \tag{A6}$$

Using the above relations, we obtain Eq. (A5). Letting $x = -\Delta/2$ in Eq. (A5), we obtain for Eqs. (17) and (20),

$$\begin{aligned}\sum_{k=0}^{\infty} {}_{k+n}C_n x^{k+n} &= \frac{x^n}{(1-x)^{n+1}} \\ &= \left(\frac{2}{2+\Delta}\right)\left(\frac{-\Delta}{2+\Delta}\right)^n. \end{aligned} \tag{A7}$$

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