

Extension of the Siegert theorem for photon emission

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When the long-wavelength approximation is not performed, extensions of the Siegert theorem can take various forms. An alternative expression is derived for the electric multipole fields, which is inspired by earlier works on photonuclear reactions. It is applied to the Siegert transformation of electric transition operators for single photon emission. In the particular case of isospin forbidden $E1$ transitions, which can be useful for some important radiative capture reactions, an approximation of this electric transition operator is established and discussed.

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I. INTRODUCTION

The electric transition operators describing photon emission or absorption depend on a current operator that is not completely known. Corrections due to meson exchange currents are difficult to establish and must be consistent with the nucleon-nucleon force employed in the nuclear Hamiltonian. For these reasons, electric transition operators are better expressed by using the so-called Siegert theorem (i.e., using current conservation to replace the nuclear current as a function of the charge density of the nucleus). At the long-wavelength approximation, the Siegert hypothesis states that the main effects of meson exchange currents are then included in the operator [1]. Spontaneous photon emission usually occurs at energies where the long-wavelength approximation is valid.

Beyond the long-wavelength approximation, the Siegert transformation of the electric term can not be performed fully [2] and is not unique any more [3]. Even when the gauge is fixed, the electric multipoles of the electromagnetic field can be written in an infinite number of ways [3] as

$$A_{\mu}^{E\lambda}(\mathbf{r}) = \nabla\Phi_{\lambda\mu}(\mathbf{r}) + A_{\mu}^{\prime E\lambda}(\mathbf{r}). \quad (1)$$

Any modification of $\Phi_{\lambda\mu}$ can be compensated by a modification of $A_{\mu}^{\prime E\lambda}$. The compatibility with the long-wavelength approximation restricts $\Phi_{\lambda\mu}$ to forms possessing the limit, when the photon wave number k vanishes,

$$\Phi_{\lambda\mu}(\mathbf{r}) \underset{k \rightarrow 0}{\sim} r^{\lambda} Y_{\lambda\mu}(\Omega). \quad (2)$$

A part of the nuclear current is then eliminated from transition operators with the Siegert theorem,

$$\int \mathbf{J}_c(\mathbf{r}) \cdot \nabla\Phi_{\lambda\mu}(\mathbf{r}) d\mathbf{r} = \frac{i}{\hbar} \int \Phi_{\lambda\mu}(\mathbf{r}) [H, \rho(\mathbf{r})] d\mathbf{r}, \quad (3)$$

where $\mathbf{J}_c(\mathbf{r})$ is the convection current density, $\rho(\mathbf{r})$ is the charge density, and H is the Hamiltonian of the nuclear system. Physical results would be independent of the choice of $\Phi_{\lambda\mu}$ as long as the term involving $A_{\mu}^{\prime E\lambda}$ is taken into account, if exact energies and wave functions and currents consistent with the Hamiltonian are used [3].

For many applications involving spontaneous photon emission, the long-wavelength approximation is excellent and the residual term involving $A_{\mu}^{\prime E\lambda}$ can safely be neglected if condition (2) is satisfied. The situation is very different in photonuclear physics where higher photon energies are used [3]. In this case, different extensions of the Siegert theorem have been studied and compared. Friar and Fallieros have introduced a variant of the Siegert theorem where the non-Siegert part only depends on a magnetic density [4,5]. Some aspects of their approach were anticipated by Foldy [6]. These variants are said to differ by the gauge by some authors [3] although they are derived within the same Coulomb gauge. The various gauges referred to in Ref. [3] correspond to different choices of $\Phi_{\lambda\mu}$, rather than a change of gauge in the usual sense.

There is however a situation in single photon emission where the long-wavelength approximation may not be sufficient and the choice of $\Phi_{\lambda\mu}$ may play a role. This can occur when $E1$ transitions are forbidden by the isospin selection rule. Then the isoscalar part of the $E1$ operator vanishes at the long-wavelength approximation and higher-order terms may contribute. This occurs for $T_i = 0 \rightarrow T_f = 0$ transitions in $N = Z$ nuclei. The study of radiative capture reactions of $N = Z$ nuclei that occur in the context of nuclear astrophysics are important applications. Well known examples are the $d(d, \gamma)^4\text{He}$, $\alpha(d, \gamma)^6\text{Li}$, and $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$ reactions [7] where the forbidden $E1$ component is in competition with the allowed $E2$ component. A reliable evaluation of the nonresonant part of the $E1$ component is difficult to obtain in these cases.

The aim of the present paper is to establish a variant of the standard expression of electric multipoles of the electromagnetic field, inspired by the works of Foldy [6], and Friar and Fallieros [4], and to study its consequences for photon emission. This leads to alternative forms for the electric transition operators. Then the special case of isospin forbidden $E1$ transitions is analyzed with emphasis on the dominant isoscalar and isovector terms.

In Sec. II, some properties of the multipoles of the electromagnetic field are recalled and a variant of the electric multipoles is presented. Proofs are given in Appendix. In Sec. III, various expressions of the transition operators are presented, with or without Siegert transformation. The

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particular case of $E1$ transitions is discussed in Sec. IV. Concluding remarks are presented in Sec. V.

II. MULTIPOLES OF THE ELECTROMAGNETIC FIELD

The solutions of the Helmholtz equation for the electromagnetic field that are eigenfunctions of the orbital momentum operators L^2 and L_z are characterized by a multipolarity λ and a projection μ . The electric multipoles have parity $(-1)^{\lambda+1}$ and can be defined as [8]

$$\begin{aligned} A_{\lambda\mu}^E &= k^{-1}[\lambda(\lambda+1)]^{-1/2} \nabla \times \mathbf{L}\phi_{\lambda\mu} \\ &= \frac{i}{k[\lambda(\lambda+1)]^{1/2}} \left\{ \nabla \left[\frac{\partial}{\partial r} (r\phi_{\lambda\mu}) \right] + k^2 \mathbf{r}\phi_{\lambda\mu} \right\}, \end{aligned} \quad (4)$$

where $\mathbf{L} = -i\mathbf{r} \times \nabla$ is dimensionless and

$$\phi_{\lambda\mu}(k\mathbf{r}) = j_\lambda(kr)Y_{\lambda\mu}(\Omega). \quad (5)$$

Other definitions may differ by the normalization or by a phase factor [9,10]. The magnetic multipoles have parity $(-1)^\lambda$ and can be defined as [8]

$$A_{\lambda\mu}^M = [\lambda(\lambda+1)]^{-1/2} \mathbf{L}\phi_{\lambda\mu}. \quad (6)$$

The longitudinal multipoles have the same parity $(-1)^{\lambda+1}$ as the electric multipoles and read

$$A_{\lambda\mu}^L = k^{-1} \mathbf{p}\phi_{\lambda\mu}. \quad (7)$$

With the present normalizations and phases, the electric and magnetic multipoles are equal to the rotational of each other, divided by k . Their divergences vanish. This is not the case for the longitudinal multipoles, which do not play a role in the Coulomb gauge. In other gauges, the electric multipole potentials are linear combinations of $A_{\lambda\mu}^E$ and $A_{\lambda\mu}^L$ [11]. From now on, I only consider the Coulomb gauge.

A photon traveling in direction $\hat{\mathbf{k}} = (\theta_k, \varphi_k)$ is described in the Coulomb gauge by a polarized plane wave with circular polarizations ϵ_q orthogonal to \mathbf{k} ($q = \pm 1$). This wave can be expanded in partial waves as

$$\epsilon_q e^{i\mathbf{k}\cdot\mathbf{r}} = -\sqrt{2\pi} \sum_{\lambda\mu} i^\lambda \sqrt{2\lambda+1} (A_{\lambda\mu}^E + q A_{\lambda\mu}^M) D_{\mu q}^\lambda(\Omega_k), \quad (8)$$

where $D_{\mu q}^\lambda$ is a Wigner matrix element depending on

$$\Omega_k = (-\varphi_k, -\theta_k, 0). \quad (9)$$

A variant of expansion of the polarized plane wave has been analyzed by Foldy [6] in the context of photonuclear physics,

$$\epsilon_q e^{i\mathbf{k}\cdot\mathbf{r}} = \int_0^1 [\nabla(\epsilon_q \cdot \mathbf{r} e^{i\mathbf{s}\cdot\mathbf{r}}) - i\mathbf{s}\mathbf{r} \times (\mathbf{k} \times \epsilon_q) e^{i\mathbf{s}\cdot\mathbf{r}}] ds \quad (10)$$

(see Appendix for a brief proof). A partial wave expansion of the corresponding electric current has been performed by Friar and Fallieros [4]. These authors have used it to extend the Siegert theorem. The principle of their approach can be applied to the electromagnetic field itself to find a variant of $A_{\lambda\mu}^E$.

As proved in Appendix, the electric multipoles can also be written as

$$\begin{aligned} A_{\lambda\mu}^E &= \frac{i}{k} [\lambda(\lambda+1)]^{1/2} \nabla [Y_{\lambda\mu}(\Omega) G_\lambda(kr)] \\ &\quad - \frac{k}{[\lambda(\lambda+1)]^{1/2}} \mathbf{r} \times \mathbf{L} Y_{\lambda\mu}(\Omega) H_\lambda(kr), \end{aligned} \quad (11)$$

where

$$G_\lambda(z) = \int_0^1 s^{-1} j_\lambda(sz) ds \quad (12)$$

and

$$H_\lambda(z) = \int_0^1 s j_\lambda(sz) ds. \quad (13)$$

Expression (11) is at the root of the derivation in Ref. [4] but does not seem to be available in the literature. It is strictly equivalent to Eq. (4) as can be verified with Taylor series expansions. The notations G_λ and H_λ are inspired by, but different from, the functions g_λ and h_λ of Friar and Fallieros [4]. Function g_λ (respectively h_λ) is equal to G_λ (respectively H_λ) divided by the first term of its Taylor expansion [see Eqs. (30) and (31) below]. The first term of both Taylor expansions of g_λ and h_λ is thus unity.

III. TRANSITION MULTIPOLE OPERATORS

The transition multipole operators are defined [9,10] by

$$\mathcal{M}_\mu^{E\lambda} = \frac{1}{c} \sqrt{\frac{\lambda}{\lambda+1}} \frac{(2\lambda+1)!!}{k^\lambda} \int \mathbf{J} \cdot \mathbf{A}_{\lambda\mu}^E d\mathbf{r} \quad (14)$$

and

$$\mathcal{M}_\mu^{M\lambda} = -\frac{i}{c} \sqrt{\frac{\lambda}{\lambda+1}} \frac{(2\lambda+1)!!}{k^\lambda} \int \mathbf{J} \cdot \mathbf{A}_{\lambda\mu}^M d\mathbf{r}, \quad (15)$$

where \mathbf{J} is the current density. This current is the sum of a convection current \mathbf{J}_c depending on the motion of the protons and a magnetization current \mathbf{J}_m depending on the spins of all nucleons. It should also contain a component arising from meson exchange. However a correct and consistent treatment of that part is difficult, and depends on the choice of nuclear forces. This problem is partly solved with the Siegert hypothesis.

Assuming a system of A point nucleons with coordinates \mathbf{r}_j and momenta \mathbf{p}_j , the charge density is

$$\rho(\mathbf{r}) = e \sum_{j=1}^A g_{lj} \delta(\mathbf{r}_j - \mathbf{r}). \quad (16)$$

The factor $g_{lj} = \frac{1}{2} - t_{j3}$ depends on the third component t_{j3} of the isospin of nucleon j . For simplicity, the modifications related to Galilean invariance are postponed to Sec. IV. The convection current density reads

$$\mathbf{J}_c(\mathbf{r}) = \frac{e}{m_p} \sum_{j=1}^A g_{lj} \frac{1}{2} [\mathbf{p}_j \delta(\mathbf{r}_j - \mathbf{r}) + \delta(\mathbf{r}_j - \mathbf{r}) \mathbf{p}_j], \quad (17)$$

where m_p is the proton mass. The magnetization current density

$$\mathbf{J}_m(\mathbf{r}) = \nabla \times \boldsymbol{\mu}_m(\mathbf{r}) \quad (18)$$

depends on the density of intrinsic magnetic moment,

$$\boldsymbol{\mu}_m(\mathbf{r}) = \mu_N \sum_{j=1}^A g_{sj} \delta(\mathbf{r}_j - \mathbf{r}) \mathbf{S}_j, \quad (19)$$

where the spin operator \mathbf{S}_j of nucleon j is dimensionless, $\mu_N = e\hbar/2m_p$, and $g_{sj} = g_p(\frac{1}{2} - t_{j3}) + g_n(\frac{1}{2} + t_{j3})$ as a function of the proton (g_p) and neutron (g_n) gyromagnetic factors.

With Eqs. (4) and (14), one obtains the electric transition operators

$$\begin{aligned} \mathcal{M}_\mu^{E\lambda} = & \frac{e\hbar}{m_p c} \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda} \sum_{j=1}^A \left\{ \frac{g_{lj}}{k} \left[\left(k^2 \mathbf{r} + \nabla \frac{\partial}{\partial \mathbf{r}} \right) \phi_{\lambda\mu} \right]_j \cdot \nabla_j \right. \\ & \left. + \frac{1}{2} k g_{sj} [\mathbf{L} \phi_{\lambda\mu}]_j \cdot \mathbf{S}_j \right\}, \quad (20) \end{aligned}$$

where the subscript j means that the term is evaluated at \mathbf{r}_j . This expression is rather standard for photon emission [2,12] but is called the Partovi gauge [13] in studies of photonuclear reactions [3]. With Eqs. (6) and (15), the magnetic operators read

$$\begin{aligned} \mathcal{M}_\mu^{M\lambda} = & \frac{e\hbar}{m_p c} \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda} \sum_{j=1}^A \left\{ g_{lj} [\nabla \phi_{\lambda\mu}]_j \cdot \mathbf{L}_j \right. \\ & \left. + \frac{1}{2} g_{sj} \left[\left(k^2 \mathbf{r} + \nabla \frac{\partial}{\partial \mathbf{r}} \right) \phi_{\lambda\mu} \right]_j \cdot \mathbf{S}_j \right\}. \quad (21) \end{aligned}$$

The nuclear current \mathbf{J} or \mathbf{J}_c is related to the charge density ρ by the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = -\frac{i}{\hbar} [H, \rho(\mathbf{r})]. \quad (22)$$

According to the Siegert hypothesis, if the gradient of the current in the electric operators is transformed as a function of the charge density, one obtains an expression where the effects of exchange currents are partially included [4]. Using Eq. (22) in matrix elements $\langle f | \mathcal{M}_\mu^{E\lambda} | i \rangle$ and neglecting recoil effects leads to a Siegert form of the electric transition operators [2]

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E\lambda} = & \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda} \sum_{j=1}^A \left\{ e g_{lj} \left(\phi_{\lambda\mu} + r \frac{\partial \phi_{\lambda\mu}}{\partial r} \right)_j \right. \\ & + \frac{e\hbar k}{2m_p c} \left[g_{lj} \left(3\phi_{\lambda\mu} + r \frac{\partial \phi_{\lambda\mu}}{\partial r} + 2\phi_{\lambda\mu} r \frac{\partial}{\partial r} \right)_j \right. \\ & \left. \left. + g_{sj} (\mathbf{L} \phi_{\lambda\mu})_j \cdot \mathbf{S}_j \right] \right\}. \quad (23) \end{aligned}$$

A problem, however, is the occurrence of additional terms that make the calculation without long-wavelength approximation complicated. Results when the conservation equation (22) is used like in Eq. (23) or not used like in Eq. (20) are different for several reasons. First, the eigenenergies and eigenfunctions of the nuclear Hamiltonian are approximate. Second, the contribution of the meson exchange currents is missing. This

contribution should be consistent with realistic nuclear forces. According to the Siegert hypothesis, Eq. (23) is preferable because meson exchange effects are partially included.

An alternative expression for these effective operators based on Eq. (11) is given by

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E\lambda} = & \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda} \sum_{j=1}^A \left\{ e g_{lj} \lambda(\lambda+1) G_\lambda(kr_j) Y_{\lambda\mu}(\Omega_j) \right. \\ & + \frac{e\hbar k}{2m_p c} (g_{lj} H_\lambda(kr_j) [\lambda(\lambda+1) Y_{\lambda\mu}(\Omega_j) \\ & \left. + 2(\mathbf{L} Y_{\lambda\mu})_j \cdot \mathbf{L}_j] + g_{sj} (\mathbf{L} \phi_{\lambda\mu})_j \cdot \mathbf{S}_j) \right\} \quad (24) \end{aligned}$$

(see also Ref. [5]). The spin term is unchanged. But Eqs. (23) and (24) will not only give results different from the non-Siegert equations but also different from each other. Following Ref. [4], the non-Siegert corrections to the charge density term should be less important with Eq. (24). Calculations with this equation should be more realistic.

IV. ELECTRIC DIPOLE OPERATOR FOR ISOSPIN-FORBIDDEN TRANSITIONS

The $E1$ operator contains isoscalar and isovector components. At the long-wavelength approximation, it has a simple expression with the Siegert theorem. However, beyond the leading-order approximation, the expression of the charge density part depends on the chosen variant [3]. Although it is quite possible to use the exact expressions (23) and (24) in practical calculations, I now present truncated expressions that give an idea of how the various terms in the two variants differ. Since the energy of emitted photons is usually not large (≤ 5 MeV), these truncated expressions can often provide good approximations of the full ones.

From expression (23), one obtains the expansion truncated at the first order beyond the long-wavelength approximation

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E\lambda} \approx & \sum_{j=1}^A \left\{ e g_{lj} \left[r_j^\lambda - \frac{(\lambda+3)k^2 r_j^{\lambda+2}}{2(\lambda+1)(2\lambda+3)} \right] Y_{\lambda\mu}(\Omega_j) \right. \\ & + \frac{e\hbar k}{2m_p c} \frac{r_j^\lambda}{\lambda+1} \left[g_{lj} Y_{\lambda\mu}(\Omega_j) \left(\lambda+3+2r_j \frac{\partial}{\partial r_j} \right) \right. \\ & \left. \left. + g_{sj} (\mathbf{L} Y_{\lambda\mu})_j \cdot \mathbf{S}_j \right] \right\}. \quad (25) \end{aligned}$$

For $E1$, it becomes

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E1} \approx & \sum_{j=1}^A \left\{ e g_{lj} \left(r_j - \frac{k^2 r_j^3}{5} \right) Y_{1\mu}(\Omega_j) \right. \\ & + \frac{e\hbar k}{4m_p c} r_j \left[2g_{lj} Y_{1\mu}(\Omega_j) \left(2+r_j \frac{\partial}{\partial r_j} \right) \right. \\ & \left. \left. + g_{sj} (\mathbf{L} Y_{1\mu})_j \cdot \mathbf{S}_j \right] \right\}. \quad (26) \end{aligned}$$

In fact, the coordinates \mathbf{r}_j and momenta \mathbf{p}_j should be replaced by Galilean invariant expressions [6,14]

$$\mathbf{r}'_j = \mathbf{r}_j - \mathbf{R}, \quad (27)$$

$$\mathbf{p}'_j = \mathbf{p}_j - A^{-1}\mathbf{P}, \quad (28)$$

where \mathbf{R} and \mathbf{P} are the coordinate and momentum of the center of mass. Hence, the operator (26) is replaced by

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E1} \approx & -e \sum_{j=1}^A t_{j3} r'_j Y_{1\mu}(\Omega'_j) - e \frac{k^2}{10} \sum_{j=1}^A r_j'^3 Y_{1\mu}(\Omega'_j) \\ & + \frac{e\hbar k}{8m_p c} \sum_{j=1}^A r'_j \left[\frac{2i}{\hbar} Y_{1\mu}(\Omega'_j) \mathbf{r}'_j \cdot \mathbf{p}'_j \right. \\ & \left. + (g_p + g_n)(LY_{1\mu})'_j \cdot \mathbf{S}_j \right], \quad (29) \end{aligned}$$

where all higher-order isovector corrections are neglected and $\mathbf{L}'_j = \mathbf{r}'_j \times \mathbf{p}'_j$. The first term in the square bracket of Eq. (26) vanishes for the same reason that there is no isoscalar term at the long-wavelength approximation. The first two terms of Eq. (29) have been used to evaluate the role of $E1$ transitions in the $^{16}\text{O}(\alpha, \gamma)^{20}\text{Ne}$ reaction [15]. It is not clear whether the remaining terms give a large contribution or not.

The advantage of expression (24) is that the term outside the Siegert part should be less important and could be neglected [3]. With

$$G_\lambda(z) = \frac{z^\lambda}{\lambda(2\lambda+1)!!} \left[1 - \frac{\lambda}{2(\lambda+2)(2\lambda+3)} z^2 + \dots \right] \quad (30)$$

and

$$H_\lambda(z) = \frac{z^\lambda}{(\lambda+2)(2\lambda+1)!!} \left[1 - \frac{\lambda+2}{2(\lambda+4)(2\lambda+3)} z^2 + \dots \right], \quad (31)$$

the operator (24) can be approximated as

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E\lambda} \approx & \sum_{j=1}^A \left\{ e g_{lj} \left[r_j^\lambda - \frac{\lambda k^2 r_j^{\lambda+2}}{2(\lambda+2)(2\lambda+3)} \right] Y_{\lambda\mu}(\Omega_j) \right. \\ & + \frac{e\hbar k}{2m_p c} \frac{r_j^\lambda}{\lambda+1} \left[g_{lj} \frac{\lambda(\lambda+1)}{\lambda+2} Y_{\lambda\mu}(\Omega_j) \right. \\ & \left. \left. + (LY_{\lambda\mu})_j \cdot \left(\frac{2g_{lj}}{\lambda+2} \mathbf{L}_j + g_{sj} \mathbf{S}_j \right) \right] \right\}. \quad (32) \end{aligned}$$

For $E1$, neglecting all higher-order isovector corrections leads to the Galilean invariant expression

$$\begin{aligned} \widetilde{\mathcal{M}}_\mu^{E1} \approx & -e \sum_{j=1}^A t_{j3} r'_j Y_{1\mu}(\Omega'_j) - e \frac{k^2}{60} \sum_{j=1}^A r_j'^3 Y_{1\mu}(\Omega'_j) \\ & + \frac{e\hbar k}{8m_p c} \sum_{j=1}^A r'_j (LY_{1\mu})'_j \cdot \left[\frac{2}{3} \mathbf{L}'_j + (g_p + g_n) \mathbf{S}_j \right]. \quad (33) \end{aligned}$$

One observes a large difference between the coefficients of the first isoscalar term in Eqs. (29) and (33) (i.e., $1/10$ vs $1/60$). This indicates that neglecting the remaining spin-independent isoscalar terms in Eq. (29) is not a good approximation. An

advantage of expression (33) is that the matrix elements of the correction terms should be small if they act on a wave function with a largely dominant component with zero total orbital momentum and intrinsic spin. In this case, the first or first two terms of Eq. (33) should give a reasonable approximation.

V. CONCLUDING REMARKS

In this note, a variant of the electric multipole fields [Eq. (11)] is established starting from an expansion used by Foldy [6] and employing techniques derived by Friar and Fallieros [4] for photonuclear reactions. A variant of the Siegert theorem is then discussed in the case of photon emission leading to an expression for the electric transition operators [Eq. (24)], without making use of the long-wavelength approximation. Going beyond the long-wavelength approximation is usually not crucial for spontaneous nuclear emissions of photons. However it should be useful in microscopic studies of the isospin forbidden $E1$ component of some low-energy radiative capture reactions.

Equation (33) shows that several types of terms may contribute to isospin forbidden $E1$ transitions. Contributions of the well known isovector term arise from the existence of small $T = 1$ components in the wave functions of the final fused nucleus and/or the initial colliding nuclei. The contribution from the dominant isoscalar term (at order k^2) comes from the larger $T = 0$ components. A comparison of Eqs. (29) and (33) shows that the first isoscalar term is quite different. Equation (33) should provide a better approximation. Corrections coming from the neutron-proton mass difference are also possible. However, the np mass difference only contributes to the isovector term and can thus most probably be neglected.

The $d(d, \gamma)^4\text{He}$ capture reaction offers a nice opportunity to make a detailed study of isospin forbidden $E1$ transitions. It has been studied in an *ab initio* model with realistic forces [16]. The cross sections are however limited to the contribution of $E2$ transitions. Scattering wave functions are available for the $J^\pi = 1^-$ scattering wave [17]. In the $d + d$ case, the final ^4He state on which the transition operator acts has mostly zero total orbital momentum and intrinsic spin. The first two terms of expansion (33) should provide a good approximation. An exact calculation with all terms of Eq. (33) is possible.

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APPENDIX: PROOFS OF EQS. (10) AND (11)

For information, I give a proof of Eq. (10) [6]. It is based on an integration by parts,

$$\begin{aligned} \epsilon_q e^{i\mathbf{k}\cdot\mathbf{r}} &= \int_0^1 \epsilon_q e^{i\mathbf{s}\mathbf{k}\cdot\mathbf{r}} ds + \int_0^1 s \frac{d}{ds} (\epsilon_q e^{i\mathbf{s}\mathbf{k}\cdot\mathbf{r}}) ds \\ &= \int_0^1 e^{i\mathbf{s}\mathbf{k}\cdot\mathbf{r}} \nabla (\epsilon_q \cdot \mathbf{r}) ds + i(\mathbf{k} \cdot \mathbf{r}) \int_0^1 s \epsilon_q e^{i\mathbf{s}\mathbf{k}\cdot\mathbf{r}} ds \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 [\nabla(\boldsymbol{\epsilon}_q \cdot \mathbf{r} e^{i\mathbf{s}\cdot\mathbf{k}\cdot\mathbf{r}}) - i\mathbf{s}(\boldsymbol{\epsilon}_q \cdot \mathbf{r})\mathbf{k}e^{i\mathbf{s}\cdot\mathbf{k}\cdot\mathbf{r}} \\
&\quad + i\mathbf{s}(\mathbf{k} \cdot \mathbf{r})\boldsymbol{\epsilon}_q e^{i\mathbf{s}\cdot\mathbf{k}\cdot\mathbf{r}}] d\mathbf{s} \\
&= \nabla[(\boldsymbol{\epsilon}_q \cdot \mathbf{r})\alpha(\mathbf{k} \cdot \mathbf{r})] - \frac{1}{2}i\mathbf{r} \times (\mathbf{k} \times \boldsymbol{\epsilon}_q)\beta(\mathbf{k} \cdot \mathbf{r})
\end{aligned} \tag{A1}$$

with, in the notation of Ref. [4],

$$\alpha(z) = \int_0^1 e^{isz} ds = \frac{e^{iz} - 1}{iz} \tag{A2}$$

and

$$\beta(z) = 2 \int_0^1 s e^{isz} ds = -2i\alpha'(z). \tag{A3}$$

The proof of Eq. (11) closely follows the treatment in Ref. [4]. We assume that wave vector \mathbf{k} is in the z direction. The expansion of the first term of Eq. (A1) reads

$$\begin{aligned}
(\boldsymbol{\epsilon}_q \cdot \mathbf{r})\alpha(\mathbf{k} \cdot \mathbf{r}) &= -i\boldsymbol{\epsilon}_q \cdot \nabla_{\mathbf{k}} \int_0^1 s^{-1} e^{i\mathbf{s}\cdot\mathbf{k}\cdot\mathbf{r}} ds \\
&= -\sqrt{2\pi}k^{-1} \sum_{\lambda} i^{\lambda+1} \sqrt{\lambda(\lambda+1)(2\lambda+1)} \\
&\quad \times Y_{\lambda q} G_{\lambda}(kr),
\end{aligned} \tag{A4}$$

where Eq. (12) and

$$\boldsymbol{\epsilon}_q \cdot (\nabla_{\mathbf{k}} Y_{\lambda\mu}^*)(0, 0) = (k\sqrt{8\pi})^{-1} \sqrt{\lambda(\lambda+1)(2\lambda+1)} \delta_{\mu q} \tag{A5}$$

have been used [4].

As noted by Foldy [6] and explicitly treated in Ref. [4], the second term of Eq. (A1) is a mixing of electric and magnetic contributions. Function $\frac{1}{2}\boldsymbol{\epsilon}_q\beta$ can be separated according to

the $(-1)^\lambda$ or $(-1)^{\lambda+1}$ parity into an electric part

$$\frac{1}{2}\boldsymbol{\epsilon}_q\beta^E = -q\sqrt{2\pi} \sum_{\lambda=1}^{\infty} i^\lambda \sqrt{2\lambda+1} Y_q^{\lambda\lambda} H_\lambda(kr) \tag{A6}$$

and a magnetic part

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\epsilon}_q\beta^M &= \sqrt{2\pi} \sum_{\lambda=0}^{\infty} i^\lambda (\sqrt{\lambda+2} Y_q^{\lambda+1\lambda} + \sqrt{\lambda-1} Y_q^{\lambda-1\lambda}) H_\lambda(kr) \\
&= \sqrt{2\pi} \sum_{\lambda=1}^{\infty} i^{\lambda+1} [\sqrt{\lambda} Y_q^{\lambda\lambda+1} H_{\lambda+1}(kr) \\
&\quad - \sqrt{\lambda+1} Y_q^{\lambda\lambda-1} H_{\lambda-1}(kr)],
\end{aligned} \tag{A7}$$

where $Y_q^{j\lambda}$ is a vector spherical harmonics with $j \geq |q| = 1$, and $\beta = \beta^E + \beta^M$. With

$$\mathbf{r} \times (\mathbf{k} \times \boldsymbol{\epsilon}_q) = -iqk\mathbf{r} \times \boldsymbol{\epsilon}_q, \tag{A8}$$

one obtains for \mathbf{k} along the z axis

$$\frac{1}{2}\mathbf{r} \times (\mathbf{k} \times \boldsymbol{\epsilon}_q)\beta^E = \sqrt{2\pi}k \sum_{\lambda} i^{\lambda+1} \sqrt{2\lambda+1} \mathbf{r} \times Y_q^{\lambda\lambda} H_\lambda(kr) \tag{A9}$$

and

$$\begin{aligned}
\frac{1}{2}\mathbf{r} \times (\mathbf{k} \times \boldsymbol{\epsilon}_q)\beta^M &= q\sqrt{2\pi}kr \sum_{\lambda} \frac{i^{\lambda+1}}{\sqrt{2\lambda+1}} [\lambda H_{\lambda+1} - (\lambda+1)H_{\lambda-1}] Y_q^{\lambda\lambda} \\
&= -q\sqrt{2\pi} \sum_{\lambda} i^{\lambda+1} \sqrt{2\lambda+1} Y_q^{\lambda\lambda} j_\lambda(kr).
\end{aligned} \tag{A10}$$

Equation (11) follows from Eqs. (8), (A1), (A4), (A9), and

$$Y_m^{\lambda\lambda} = [\lambda(\lambda+1)]^{-1/2} L Y_{\lambda m}. \tag{A11}$$

With Eqs. (8), (A1), (A10), and (A11), one recovers Eq. (6).

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