Heine-Stieltjes correspondence and the polynomial approach to the standard pairing problem

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A new approach for solving the Bethe ansatz (Gaudin-Richardson) equations of the standard pairing problem is established based on the Heine-Stieltjes correspondence. For k pairs of valence nucleons on n different single-particle levels, it is found that solutions of the Bethe ansatz equations can be obtained from one $(k+1)\times(k+1)$ and one $(n-1)\times(k+1)$ matrices, which are associated with the extended Heine-Stieltjes and Van Vleck polynomials, respectively. Since the coefficients in these polynomials are free from divergence with variations in contrast to the original Bethe ansatz equations, the approach provides an efficient and systematic way to solve the problem, which by extension, can also be used to solve a large class of Gaudin-type quantum many-body problems, including an efficient angular momentum projection method for multiparticle systems.

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I. INTRODUCTION

It is well known that the pairing force, similar to that in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductors [1], is a key residual interaction of the nuclear shell model that is required to reproduce ground state and low-energy spectroscopic features of nuclei, in particular, binding energies, excitation spectra, low-lying collective states, odd-even staggering effects, single-particle occupancies, electromagnetic transition rates, transfer reaction amplitudes, level densities, moments of inertia, and so on [2–4]. Unlike electrons in solids, drawbacks of the application of BCS theory and its extensions to nuclei can be pronounced due to the fact that the number of valence nucleons under the influence of the pairing force is too few to be treated by such particle-number nonconservation (quasiparticle) approximations [5,6].

An exact solution of the standard pairing problem was first obtained by Richardson and is now referred to as the Richardson-Gaudin method [7,8]. Recently, extensions of the Richardson-Gaudin theory have also been made by using the Bethe ansatz methodology [9–13]. Its advantage lies in the fact that the huge matrix in the Fock subspace is reduced to a set of equations, called Bethe ansatz equations (BAEs), such that the number of these equations equals exactly the number of pairs of the valence particles involved. However, less attention has been paid to the Richardson solutions of the pairing problem in realistic applications mainly because the nonlinear BAEs involved are very difficult to be solved numerically, especially for large-size systems. While considerable efforts in designing algorithms to obtain solutions have revealed promising results [14–19], including an advantageous polynomial technique for a shell of two levels [12], an efficient procedure for solving the general problem obviously seems still unclear. Thus, a simple and clear approach to the problem is in demand.

In the present study, we suggest a new approach for solving the standard pairing problem. In particular, we derive polynomial solutions of the second-order Fuchsian equation. This, in turn, transforms the problem to one that involves the handling of only two matrix equations and hence, makes exact pairing solutions feasible even when more energy levels or heavy nuclei are considered.

II. POLYNOMIAL SOLUTIONS TO THE EXACT PAIRING PROBLEM

The Hamiltonian of the standard pairing model is given by

$$\hat{H} = \sum_j \epsilon_j \hat{n}_j - G \sum_{jj'} S_j^+ S_{j'}^-, \tag{1}$$

where the sums run over given j levels of total number n, $G > 0$ is the overall pairing strength, ϵ_j are nondegenerate single-particle energies, $\hat{n}_j = \sum_m a_{jm}^\dagger a_{jm}$ is the number operator for valence particles in the jth level, and $S_j^+ = \sum_{m>0}(-)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger$ ($S_j^- = (S_j^+)^\dagger$) are pair creation (annihilation) operators. The formalism is first presented for an even number of particles that are all paired (seniority-zero case), while the generalization to an additional odd unpaired particle is discussed in relation to the pairing eigenenergies.

According to the Richardson-Gaudin method, k-pair eigenstates of Eq. (1) can be written as

$$|k;x\rangle = S^+(x_1)S^+(x_2)\cdots S^+(x_k)|0\rangle, \tag{2}$$

where $|0\rangle$ is the pairing vacuum state satisfying $S_j^-|0\rangle = 0$ for all j, and $S^+(x_i) = \sum_j S_j^+/(x_i - 2\epsilon_j)$, in which x_i ($i = 1, 2, \ldots, k$) are spectral parameters to be determined. It can then be verified by using the corresponding eigenequation that Eq. (2) is the eigenstates of Eq. (1) only when the spectral parameters x_i satisfy the following set of BAEs:

$$1 - 2G\sum_j \frac{\rho_j}{x_i - 2\epsilon_j} - 2G \sum_{\substack{i'=1 \\ (\neq i)}}^{k} \frac{1}{x_i - x_{i'}} = 0, \tag{3}$$

where the first sum runs over all j levels and $\rho_j = -\Omega_j/2$ with $\Omega_j = j + 1/2$. For each $x^{(\xi)}$ solution, the corresponding eigenenergy is given by $E_k^{(\xi)} = \sum_{i=1}^k x_i^{(\xi)}$.

As shown by Heine and Stieltjes, there is a one-to-one correspondence between every set of the BAEs and a set of orthogonal polynomials, called the extended Heine-Stieltjes polynomials. Roots of these BAEs are zeros of the polynomials, which can be interpreted as stable equilibrium positions in the two-dimensional complex plane for a set

of free unit charges in an external electrostatic field [20]. This link between Richardson's BCS pairing model for nuclei and the corresponding electrostatic problem was established in Ref. [21] based on an earlier unpublished preprint of Gaudin, which was then made clearer in Ref. [22]. A much more general approach to the pairing model was shown in Refs. [23,24]. According to Heine-Stieltjes correspondence, for nonzero pairing strength G, the polynomials $y(x)$ with zeros corresponding to the solutions of Eq. (3) should satisfy the following second-order Fuchsian equation:

$$A(x)y''(x) + B(x)y'(x) - V(x)y(x) = 0. \tag{4}$$

Here, $A(x) = \prod_j (x - 2\epsilon_j)$ is a polynomial of degree n, the polynomial $B(x)$ is given as

$$B(x)/A(x) = \sum_j \frac{2\rho_j}{x - 2\epsilon_j} - \frac{1}{G}, \tag{5}$$

where the sum runs over all j levels and $V(x)$ are called Van Vleck polynomials [20] of degree $n-1$, which are determined according to Eq. (4). In the original electrostatic analog considered by Heine and Stieltjes [20], the parameters $\{\rho_j\}$ that specify fixed charges should all be positive for no external electrostatic field, $1/G \to 0$. Therefore, the polynomials $y(x)$ with negative $\{\rho_j\}$ and $1/G \neq 0$ are called extended Heine-Stieltjes polynomials. They approach the original Heine-Stieltjes polynomials with negative $\{\rho_j\}$ in the $G \to \infty$ limit.

In search for polynomial solutions of (4), we write

$$y(x) = \sum_{j=0}^{k} a_j x^j, \quad V(x) = \sum_{j=0}^{n-1} b_j x^j, \tag{6}$$

where $\{a_j\}$ and $\{b_j\}$ are the expansion coefficients to be determined. Substitution of Eq. (6) into Eq. (4) yields two matrix equations. Namely, the condition that the coefficients in front of x^i $(i = 0, \ldots, k)$ must be zero yields a $(k+1) \times (k+1)$ matrix $\mathbf{F}$ with $\mathbf{Fv} = b_0\mathbf{v}$, where the eigenvector $\mathbf{v}$ of $\mathbf{F}$ is simply given by the expansion coefficients $\mathbf{v} = \{a_0, \ldots, a_k\}$. In addition, the condition that the coefficients in front of x^i $(i = k+1, \ldots, n+k-1)$ must be zero yields another $(n-1) \times (k+1)$ upper-triangular matrix $\mathbf{P}$ with $\mathbf{Pv} = 0$, which provides a unique solution for b_i $(i = 1, \ldots, n-1)$ in terms of $\{a_j\}$. Entries of the two matrices are all linear in the coefficients $\{b_1, b_2, \ldots, b_{n-1}\}$. Matrices $\mathbf{F}$ and $\mathbf{P}$ can be easily constructed, for which a simple MATHEMATICA code is available [25].

A. Eigenvalues of the standard pairing model Hamiltonian: Pairing energies

Let the single-particle energies satisfy the interlacing condition $\epsilon_1 < \cdots < \epsilon_n$. Real parts of zeros of $y(x)$ satisfy the interlacing condition, $-\infty < \mathbf{Re}(x_1) < \mathbf{Re}(x_2) < \cdots < \mathbf{Re}(x_k) < +\infty$, where $\mathbf{Re}(x_i)$ lies in one of the $n+1$ intervals $(-\infty, \epsilon_1)$, (ϵ_1, ϵ_2), $\ldots$, $(\epsilon_{n-1}, \epsilon_n)$, and $(\epsilon_n, +\infty)$. It should be noted that many $\mathbf{Re}(x_i)$ of adjacent zeros may lie within the same interval. When $G \to \infty$, there will be only n intervals with $(-\infty, \epsilon_1)$ being removed.

The number of different such allowed configurations gives the possible solutions of $y(x)$ and the corresponding $V(x)$. The number of solutions of $y(x)$, excluding those with sum of zeros of $y(x)$ complex, should equal to the number of levels produced by the standard pairing model, which is given by

$$\eta(n,k) = \sum_{p_1=0}^{-2\rho_1} \cdots \sum_{p_n=0}^{-2\rho_n} \delta_{q,k}\,, \tag{7}$$

where $q = \sum_{i=1}^{n} p_i$. When $\rho_i = -1/2$ for any i, which corresponds to the case of the Nilsson mean-field plus pairing model, $\eta(n,k) = \binom{n}{k}$.

Furthermore, if we set $a_k = 1$ in $y(x)$, the coefficient a_{k-1} becomes equal to the negative sum of the $y(x)$ zeros. Hence, for each set of solutions, a_{k-1} yields the corresponding energy,

$$a_{k-1} = -\sum_{i=1}^{k} x_i = -E_k. \tag{8}$$

Therefore, the solution corresponding to the largest real a_{k-1} is that for the ground state of the system; the one corresponding to the next largest real a_{k-1} is that of the first excited state; and so on. In the standard pairing model, the solution with the same a_{k-1} is unique except complex conjugation and permutations within $\{x_i\}$. This is helpful for simplifying the calculation process, especially when only a few low-lying states are needed.

For odd-A systems, one of the particles in a system does not form a pair and blocks the level it occupies. When the j'-th level is blocked, its space dimensionality becomes $\rho_{j'} = -(\Omega_{j'} - 1)/2$. This together with $k = \frac{A-1}{2}$ for A total number of particles enter into Eq. (3), and hence, into Eq. (4). Pairing solutions, $x^{(\xi)}$, $\xi = 1, \ldots, \eta(n,k)$, are thus obtained for the $A-1$ system with the j'-th level blocked. The corresponding energy of the ground state is thereby given as $E_k^{(1)} = \sum_{i=1}^{k} x_i^{(1)} + \epsilon_{j'}$.

In contrast to the original BAEs of Eq. (3), the coefficients $\{a_j\}$ and $\{b_j\}$ in $\mathbf{F}$, $\mathbf{P}$, and $\mathbf{v}$ are free from divergence with variations, and, hence, one can use any standard recursive or iteration method to solve the problem with arbitrary initial values of these coefficients. Because solving the eigenequation $\mathbf{Fv} = b_0\mathbf{v}$, in which $\mathbf{F}$ is a $(k+1) \times (k+1)$ matrix, is the only CPU time-consuming operation involved, the CPU time needed in the process should always be reasonable for most realistic applications in nuclear physics. For example, on a 2×2.8 GHz CPU / 4 GB RAM desktop computer with Mac OS X, a single solution for $n = 10$ levels can be calculated by using MATHEMATICA v.8.0.4 in 0.32 seconds for $k = 5$, in 0.59 s for $k = 10$, and in 13.39 s for $k = 40$, which scales roughly as k^2. Similarly, a reasonable trend is observed with increasing number of levels, namely, for five pairs, the execution time is 1.24 s for $n = 15$ and 3.07 s for $n = 20$ levels.

B. Solutions and pairing energies for ^{110}Sn

To demonstrate the new approach, we consider a simple example of $k = 5$ pairs in the fifth harmonic oscillator (HO) shell, $1g_{7/2}$, $2d_{5/2}$, $2d_{3/2}$, $3s_{1/2}$, and $1h_{11/2}$, which is relevant,

TABLE I. First five sets of solutions to the extended Heine-Stieltjes polynomials Eq. (4) and the corresponding eigenenergies (in arbitrary units) of the standard pairing model Hamiltonian (1) in the case of $k = 5$ pairs in the fifth major shell with $n = 5$ single-particle levels, $1g_{7/2}$, $2d_{5/2}$, $2d_{3/2}$, $3s_{1/2}$, and $1h_{11/2}$. The single-particle energies used are $\epsilon_i = i$, and the overall pairing strength $G = 0.5$.

Zeros of the polynomials	$\sum_{i=1}^{5} x_i$
$x_1 = -1.4993$, $x_2 = -1.1412 - 2.1396\imath$,	-3.6158
$x_3 = -1.1412 + 2.1396\imath$, $x_4 = 0.0829 - 4.5018\imath$,	
$x_5 = 0.0829 + 4.5018\imath$	
$x_1 = -0.5078 - 1.0411\imath$, $x_2 = -0.5078 + 1.0411\imath$,	3.0299
$x_3 = 0.5469 - 3.3066\imath$, $x_4 = 0.5469 + 3.3066\imath$,	
$x_5 = 2.9517$	
$x_1 = -0.9234 - 1.0718\imath$, $x_2 = -0.9234 + 1.0718\imath$,	3.5444
$x_3 = 0.0573 - 3.3613\imath$, $x_4 = 0.0573 + 3.3613\imath$,	
$x_5 = 5.2767$	
$x_1 = -1.1244 - 1.0987\imath$, $x_2 = -1.1244 + 1.0987\imath$,	4.8379
$x_3 = -0.1739 - 3.4422\imath$, $x_4 = -0.1739 + 3.4422\imath$,	
$x_5 = 7.4346$	
$x_1 = -1.2032 - 1.1109\imath$, $x_2 = -1.2032 + 1.1109\imath$,	5.77020
$x_3 = -0.2619 - 3.4804\imath$, $x_4 = -0.2619 + 3.4804\imath$,	
$x_5 = 8.7004$	

for example, for applications to the tin isotopes, in this case ^{110}Sn, as well as to ^{154}Sm if the sixth shell is considered [18]. While these cases are difficult to be solved by employing directly the nonlinear BAEs (3), the $\eta(5, 5) = 71$ solutions $y(x)$ for ^{110}Sn are easily obtained in the present polynomial approach. We set single-particle energies to be of an equal spacing with $\epsilon_i = i$, and the overall pairing strength $G = 0.5$. First five sets of zeros of the corresponding polynomials $y(x)$ and a_{k-1} coefficients are listed in Table I.

We note that the use of $B(x)$ in Eq. (4) [and not $B(x)/A(x)$ of Eq. (5)] removes the singularities of the original BAEs (3) and, hence, results such as $x_i = 2\epsilon_j$, may appear among the solutions of Eq. (4). Additionally, in low-precision calculations we made for ^{110}Sn, we found some unphysical solutions with complex a_{k-1} that had to be discarded. This also resulted in a total number of solutions greater than $\eta(n, k)$ and therefore it is recommended that calculations explicitly include a check for singularities and make use of high precision. As very little is known about the polynomials with negative charges, further analytical studies are also essential.

C. Angular momentum projection method

As shown in our previous study [26], a new angular momentum projection method for multi-particle systems can be established based on the BAEs similar to Eq. (3). In fact, for n single-particle levels with angular momentum j_i $(i = 1, 2, \ldots, n)$, the multiparticle state with total angular momentum $J = \sum_i j_i - k$ can be written as

$$|\eta, J, M = J\rangle = J^-(x_1)J^-(x_2)\cdots J^-(x_k)|\text{h.w.}\rangle, \qquad (9)$$

where η is a quantum number needed to resolve the multi-occurrence of J, $|\text{h.w.}\rangle$ is the highest weight single-particle product state with $|j_1, m_1 = j_1, \cdots, j_n, m_n = j_n\rangle$, and

$$J^-(x) = \sum_{i=1}^{n} \frac{1}{x - 2\epsilon_i} J_i^-, \qquad (10)$$

where J_i^- is the angular momentum lowering operator acting only on the ith single-particle state $|j_i, m_i\rangle$, and ϵ_i $(i = 1, 2, \ldots, n)$ can be any set of unequal numbers [26]. The condition that $J^+|\eta, J, M = J\rangle = 0$, where J^+ is the total angular momentum raising operator, yields the same BAEs of Eq. (3) with the replacement of ρ_i by $-j_i$ in the $G \to \infty$ limit. Therefore, once the solutions of Eq. (3) in the $G \to \infty$ limit are obtained, the resultant sets of $\{x_i\}$ determine multiparticle states with a good angular momentum J. The number of these sets is exactly equal to the number of occurrence of J for the given system. In this case, the polynomials $y(x)$ that satisfy Eq. (4) become the original Heine-Stieltjes polynomials. This angular momentum projection is certainly much simpler than the projection operator technique [3] and that based on the permutation group method [27].

III. PAIRING GAPS FOR MEDIUM-MASS NUCLEI

The empirical like-particle pairing gap can be estimated by the third derivative of binding energies, BE, with respect to the number of valence like-particles [28], which for neutrons is

$$\Delta_{nn} \equiv \tfrac{1}{4}(BE(Z, N-2) - 3BE(Z, N-1) + 3BE(Z, N) - BE(Z, N+1)). \qquad (11)$$

This isolates the like-particle pairing interaction of the Nth and $(N-1)$st neutrons for an even-even $(Z, N-2)$-core and removes the average contribution of additional one- and two-body interactions (equivalently, the N and N^2 energy dependence).

To determine neutron pairing energies, the filter (11) is applied to the lowest pairing energies for Ca, Ni, and Sm isotopes, using two pairing approaches, namely, the polynomial approach presented here, referred as "HS pairing", and the BCS scheme ("BCS pairing") [1,5]. While other terms in the nuclear Hamiltonian may be important for reproducing binding energies, such as the average mean field (e.g., $-5.86N$ for Ca isotopes, $-8.57N$ for Ni isotopes, and $-5.80N$ for Sm isotopes) as well as the average two-body interaction $\sim N(N-1)/2$, these contributions are filtered out by Eq. (11) and hence, as expected, are irrelevant for pairing gap estimates.

To obtain BCS solutions, we solve the two BCS nonlinear equations [1,5] for a Lagrange multiplier λ and a "gap" parameter Δ, using the ("nonshifted") single-particle energies that enter in Eq. (1). The pairing energies are then calculated as, $E = 2\sum_j \epsilon_j v_j^2 \Omega_j - \frac{\Delta^2}{G} - G\sum_j \Omega_j v_j^4$ with probability amplitudes, $v_j^2 = \frac{1}{2}\{1 - (\epsilon_j - \lambda)[(\epsilon_j - \lambda)^2 + \Delta^2]^{-\frac{1}{2}}\}$. For odd-$A$ systems, the BCS equations are solved (and hence E obtained) for $k = \frac{A-1}{2}$ pairs and with $\Omega_{j'} - 1$ for the j' level blocked by the odd particle. The lowest energy of an odd-A system is approximated by the quasiparticle excitation, $E + [(\epsilon_{j'} - \lambda)^2 + \Delta^2]^{\frac{1}{2}}$.

For all applications, we employ the particle-hole formalism, which treats pairs of holes when more than half of the model

space is occupied. In this case, the single-particle energies used in Eq. (1) are replaced by $-\epsilon_j$.

To compare both pairing theories and how they agree with the experiment [29], we use the root-mean-square deviation measure, $\sigma = \sqrt{\sum_{\mu=1}^{\mathcal{N}} (\mathrm{th}_\mu - \mathrm{exp}_\mu)^2/\mathcal{N}}$, where th_μ are theoretical predictions, exp_μ are the corresponding experimental values, and $\mathcal{N}$ is the total number of nuclei considered.

A. Ca isotopes

Calculations for Ca isotopes are performed for a ^{40}Ca core and a model space of five j levels, $f_{5/2,7/2}$, $p_{1/2,3/2}$, $g_{9/2}$. The single-particle energies (s.p.e.) used in Eq. (1) are

$$\begin{aligned} \epsilon_{7/2} &= -2.50\ \mathrm{MeV}, \quad \epsilon_{3/2} = -0.56\ \mathrm{MeV}, \\ \epsilon_{5/2} &= 0.08\ \mathrm{MeV}, \quad \epsilon_{1/2} = 1.11\ \mathrm{MeV}, \\ \epsilon_{9/2} &= 1.95\ \mathrm{MeV}. \end{aligned} \tag{12}$$

These estimates are obtained from the ^{40}Ca and ^{41}Ca binding energies [29] and the ^{41}Ca energy spectrum [30]. The average single-particle energy ($\epsilon_{\mathrm{avg}} = -5.86$ MeV) is then subtracted to yield the set in Eq. (12). For the odd-A systems, the levels blocked by the odd particle are obtained from the experiment as follows: $f_{7/2}$ ($^{43-47}$Ca), $p_{3/2}$ ($^{49-53}$Ca), $f_{5/2}$ ($^{55-57}$Ca), and $p_{1/2}$ (^{59}Ca).

For pairing strength $G = 16/A$ MeV, pairing gaps (11) obtained using the present approach are found to reproduce the experimental data remarkably well (Fig. 1, left and Table II). This holds even for a small number of pairs and is a remarkable result given the simplicity of the Hamiltonian and the approximation for the single-particle energies, which are assumed not to vary with the particle number. The outcome also agrees with the BCS scheme, which, however, if kept simple and without invoking the number projection method yields wave functions that do not preserve the number of particles.

TABLE II. Root-mean-square deviations σ (in MeV) of the theoretical pairing gaps as compared to the experimental values [29] for the $^{42-49}$Ca, $^{58-77}$Ni, and $^{146-153}$Sm isotopes.

Isotopes	σ (MeV)	
	"HS pairing"	"BCS pairing"
$^{42-49}$Ca	0.155	0.210
$^{58-77}$Ni	0.079	0.137
$^{146-153}$Sm	0.364	

B. Ni isotopes

Calculations for Ni isotopes are performed for a ^{56}Ni core and a model space of four j-levels, $f_{5/2}$, $p_{1/2,3/2}$, $g_{9/2}$. The single-particle energies used in Eq. (1) are

$$\begin{aligned} \epsilon_{3/2} &= -1.68\ \mathrm{MeV}, \quad \epsilon_{5/2} = -0.91\ \mathrm{MeV}, \\ \epsilon_{1/2} &= -0.57\ \mathrm{MeV}, \quad \epsilon_{9/2} = 1.33\ \mathrm{MeV}. \end{aligned} \tag{13}$$

They are obtained from the ^{56}Ni and ^{57}Ni binding energies [29] and the ^{57}Ni energy spectrum [31]. The average single-particle energy ($\epsilon_{\mathrm{avg}} = -8.57$ MeV) is subtracted from the s.p.e.'s experimentally deduced to yield Eq. (13). For odd-A systems, the levels blocked by the odd particle are obtained from the experiment as following, $p_{3/2}$ ($^{57-61}$Ni), $p_{1/2}$ (^{63}Ni), $f_{5/2}$ (^{65}Ni), $p_{1/2}$ (^{67}Ni), and $g_{9/2}$ ($^{69-77}$Ni). As there are no experimental values for ^{75}Ni and ^{77}Ni, the lowest energy obtained by the theory is used, which predicts the level $g_{9/2}$ as the most probable to be occupied by the odd particle in both cases.

We find that a paring strength of $G = 23/A$ yields a close reproduction of experiment. Even in the case of complex systems like the isotopes of Ni, the pairing energies (11) calculated using the present approach closely follow the experimental trend (Fig. 1, right) and yield better results than the BCS scheme (Table II).

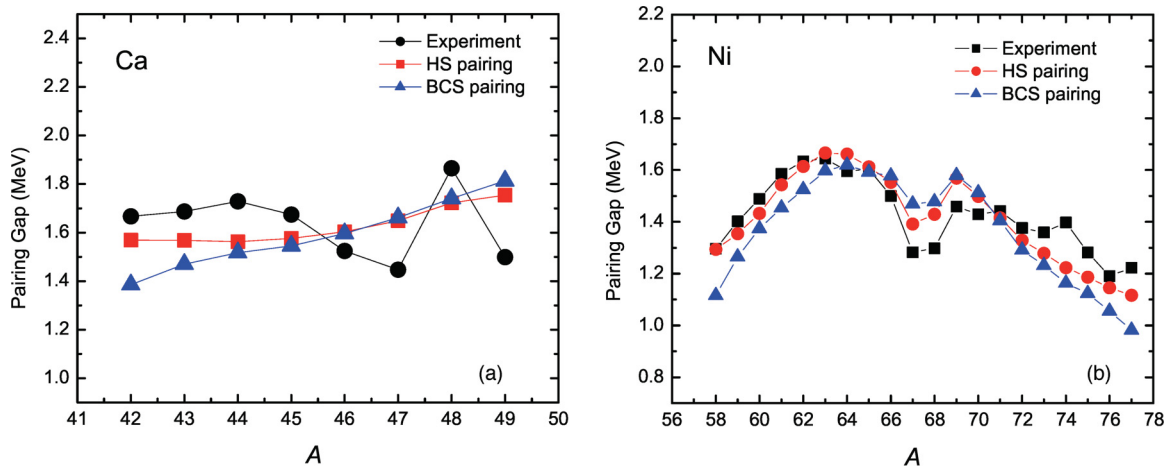

FIG. 1. (Color online) Pairing gaps in MeV as calculated in the present study ("HS pairing") and using the BCS approach ("BCS pairing"), and compared to experiment [29] for (a) Ca isotopes, ^{42}Ca to ^{49}Ca, using five j levels, $f_{5/2,7/2}$, $p_{1/2,3/2}$, $g_{9/2}$, and $G = 16/A$ MeV, and (b) Ni isotopes, ^{58}Ni to ^{77}Ni, using four j shells, $f_{5/2}$, $p_{1/2,3/2}$, $g_{9/2}$ and $G = 23/A$ MeV.

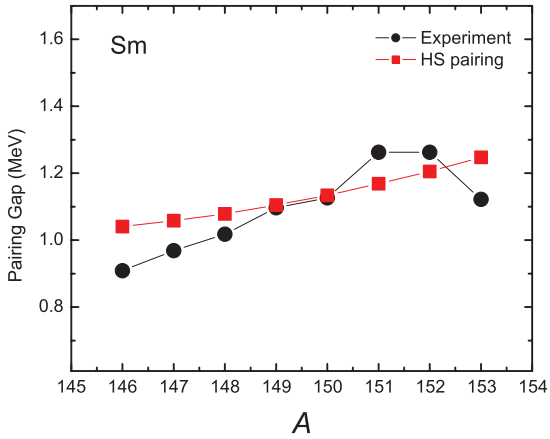

FIG. 2. (Color online) Pairing gaps in MeV as calculated in the present study ("HS pairing") and using the BCS approach ("BCS pairing"), and compared to experiment [29] for Sm isotopes, ^{146}Sm to ^{153}Sm, using six j levels in the $pfhi$ shell and $G = 24/A$ MeV (right).

C. Solutions for heavy nuclei: Sm isotopes

Larger valence spaces that are necessary for good descriptions of heavier systems are typically very difficult to be solved directly by employing the original BAE's. For isotopes of Sm, we show that the exact pairing solutions are made feasible using the present polynomial approach.

Calculations for Sm isotopes are performed for valence neutrons in the sixth HO shell with a model space of six j levels, $f_{5/2,7/2}$, $p_{1/2,3/2}$, $h_{9/2}$, $i_{13/2}$. The single-particle energies used for Eq. (1) are

$$\begin{aligned} \epsilon_{7/2} &= -1.056 \text{ MeV}, \quad \epsilon_{3/2} = -0.162 \text{ MeV}, \\ \epsilon_{13/2} &= 0.049 \text{ MeV}, \quad \epsilon_{9/2} = 0.368 \text{ MeV}, \\ \epsilon_{1/2} &= 0.552 \text{ MeV}, \quad \epsilon_{5/2} = 0.603 \text{ MeV}. \end{aligned} \tag{14}$$

These estimates are obtained from the ^{144}Sm and ^{145}Sm binding energies [29] and the ^{145}Sm energy spectrum [32]. The average single-particle energy ($\epsilon_{\text{avg}} = -5.80$ MeV) is subtracted. For the odd-A systems, experimental data indicates that the $f_{7/2}$ levels are most likely to be blocked by the odd particle for $^{145-149}$Sm. For $^{151-153}$Sm, the lowest theoretical energy corresponds to the odd particle occupying the $f_{7/2}$ level.

Calculations for a large number of pairs are possible when the polynomial approach is employed. We find that using $G = 24/A$ MeV yields a very close agreement of the theoretical paring gaps with the experimental counterparts (Fig. 2, right and Table II). This example shows that the polynomial approach can be straightforwardly applied for heavy nuclear systems and large model spaces, and hence, can provide exact pairing solutions in regions where the BAE's are impossible to handle.

In short, the comparison of the two pairing approaches reveals the superiority of the present "HS pairing" solutions as compared to the BCS ones, especially for a small number of particles. The polynomial approach makes exact pairing solutions feasible for many pairs in large model spaces.

IV. CONCLUSION

In summary, we have established a new approach for solving the standard pairing problem based on a robust mathematical foundation—the extended Heine-Stieltjes polynomials and the corresponding Van Vleck polynomials satisfying the polynomial solutions of the second-order Fuchsian equation. Thus, we reach the goal of the Richardson-Gaudin theory via the Heine-Stieltjes correspondence, which provides an exact solution to the pairing problem by solving only two matrix equations. This makes exact pairing solutions feasible even when more energy levels or heavy nuclei are considered. The approach can easily be extended and applied to solve a large class of Gaudin-type quantum many-body problems. A new efficient angular momentum projection method for multiparticle systems is thus proposed as a byproduct, of which the application to either boson or fermion systems will be studied elsewhere.

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