

**Shear viscosities from the Chapman-Enskog and the relaxation time approaches**

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The interpretation of the measured elliptic and higher order collective flows in heavy-ion collisions in terms of viscous hydrodynamics depends sensitively on the ratio of shear viscosity to entropy density. Here we perform a quantitative comparison between the results of shear viscosities from the Chapman-Enskog and relaxation time methods for selected test cases with specified elastic differential cross sections: (i) the nonrelativistic, relativistic and ultrarelativistic hard sphere gas with angle and energy independent differential cross section, (ii) the Maxwell gas, (iii) chiral pions, and (iv) massive pions for which the differential elastic cross section is taken from experiments. Our quantitative results (i) reveal that the extent of agreement (or disagreement) depends sensitively on the energy dependence of the differential cross sections employed, and (ii) stress the need to perform quantum molecular dynamical (URQMD) simulations that employ Green-Kubo techniques with similar cross sections to validate the codes employed and to test the accuracy of other methods.

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**I. INTRODUCTION**

The study of relativistic heavy-ion collisions up to 200 GeV per particle center-of-mass energy at the Brookhaven National Laboratory (BNL), and up to 7 TeV per particle at the Large Hadron Collider (LHC) at CERN, has required the development of special theoretical tools to unravel the complex space-time evolution of the matter created in these collisions. In view of the large multiplicities of hadrons (predominantly pions, kaons, etc.) observed in these collisions [1], there is much interest in the description of these collisions from the initial stages in which quark and gluon degrees of freedom are liberated to the final stages in which hadrons materialize [2]. In a hydrodynamical description of the system's evolution, local thermal equilibrium is presumed to prevail in the quark-gluon phase, the mixed phase, and the pure hadronic phase. Thereafter, hadrons cease to interact (i.e., freeze-out) and reach the detectors. Electromagnetic probes, such as photons and dileptons, produced in matter are expected to reveal the properties of the dense medium in which they are produced and from which they escape without any interactions [3]. Highly energetic probes such as jets shed light on the energy loss of quarks in an interacting dense medium [4,5]. In addition, spectral properties (i.e., longitudinal and transverse momentum distributions) of the produced hadrons have revealed interesting collective effects in their flow patterns [3].

A theoretical understanding of the variety of phenomena observed and expected in these very high energy collisions is clearly a daunting task. As a first-pass attempt, however, relativistic ideal hydrodynamics has been fruitfully employed in the description of the basic facts [6–9]. Detailed compar-

isons of the predictions of ideal hydrodynamics with data have been made, and the merits and demerits of the theoretical description identified [10–13]. As a result, much attention has recently been focused on improved developments of viscous relativistic hydrodynamics. In addition to the specification of initial conditions and the knowledge of the equation of state that are the central inputs to ideal hydrodynamics, the knowledge of transport properties such as shear and bulk viscosities, diffusion coefficients, etc. is crucial to viscous hydrodynamics [14–16].

Our objective in this paper is to quantify the extent to which results from different approximation schemes for shear viscosities agree (or disagree) by choosing some classic examples in which the elastic scattering cross sections are specified. The two different approximation schemes chosen for this study are the Chapman-Enskog and the relaxation time methods. These test studies are performed for the following cases:

- (1) a hard sphere gas (nonrelativistic, relativistic, and ultrarelativistic) with angle and energy independent differential cross section  $\sigma = a^2/4$ , where  $a$  is the hard sphere radius,
- (2) the Maxwell gas [ $\sigma(g, \theta) = m\Gamma(\theta)/2g$  with  $m$  being the mass of the heat bath particles,  $\Gamma(\theta)$  is an arbitrary function of  $\theta$ , and  $g$  is the relative velocity],
- (3) chiral pions [for which the  $t$ -averaged cross section  $\sigma = s/(64\pi^2 f_\pi^4)(3 + \cos^2 \theta)$ , where  $s$  and  $t$  are the usual Mandelstam variables and  $f_\pi$  is the pion-decay constant], and
- (4) massive pions (for which the differential elastic cross section is taken from experiments). Where possible, analytical results are obtained in either the nonrelativistic or extremely relativistic cases.

The organization of this paper is as follows. In Sec. II, the formalism and working formulas in the Chapman-Enskog and relaxation time methods are summarized. Applications to

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the above mentioned test cases are considered in Sec. III. A comparison of results from the two methods is performed on Sec. IV. Our results are summarized in Sec. V, which also contains our conclusions. The Appendix contains some details regarding the collision frequency in the nonrelativistic limit.

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## II. FORMALISM

In this section, formalisms used to calculate shear viscosity using elastic cross sections are described. In the nonrelativistic regime (such as encountered in atomic and molecular systems), classic works can be found in Refs. [17,18]. Elementary discussions can be found in Refs. [19,20]. In the relativistic regime (as found in cosmology, many astrophysical settings, and relativistic heavy-ion collisions), the book on relativistic kinematics by de Groot [21] serves as a good reference. For performing quantitative calculations, the original articles referred to in this book are more useful. The relevant articles will be referred to as and when necessary.

In heavy-ion physics, particles of varying masses are produced, the predominant ones being pions (of mass  $\sim 140$  MeV), kaons (of mass  $\sim 500$  MeV), etc., the probabilities decreasing with increasing mass due to energetic considerations. Heavier mass mesons (and baryons and antibaryons with masses in excess of the nucleon  $\sim 940$  MeV) up to 5 GeV are also produced, albeit in relatively smaller abundances than pions and kaons. The system is thus a mixture of varying masses evolving in time from a high temperature (say in the range 200–500 MeV) at formation to 100–150 MeV at freeze-out. Thus varying degrees of relativity (gauged in terms of the individual relativity parameters  $z_i = m_i/T$ ) are encountered in the mixture. This situation, of varying relativity, is special to heavy-ion physics. Thus, a general formalism capable of handling a mixture with varying relativity parameters in time as the system expands is necessary. In this section, formalisms that address a one-component system in which particles undergo elastic processes only will be summarized.

It must be stressed that the formalisms used in this work are not new, but the application of these formalisms to test cases is new to the extent that a detailed comparison between two commonly used methods is provided. For the sake of clarity and completeness, the formalisms used in this work are summarized below along with working formulas. This section thus sets the stage for the ensuing sections in which applications relevant for heavy-ion physics will be considered.

### A. The Chapman-Enskog approximation

In this section, the formalism as developed in Ref. [22] is followed and described to reveal the essentials. We begin with the relativistic transport equation appropriate for a nondegenerate system,

$$p_\alpha \partial^\alpha f = \int (f' f'_1 - f f_1) \sigma F d\Omega' d\omega_1, \quad (1)$$

using the following notation:  $x_\alpha$  and  $p_\alpha$  are the space-time and energy-momentum four-vectors [metric:  $g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ ]. The abbreviations  $f \equiv f(x, p)$ ,  $f' \equiv f'(x, p')$ ,  $f_1 \equiv f(x, p_1)$ , and  $f'_1 \equiv f'(x, p'_1)$  denote Lorentz invariant distribution functions. The differential cross section  $\sigma \equiv \sigma(P, \Theta)$  is defined in the center-of-mass (c.m.) frame with  $P = [-(p^\alpha + p_1^\alpha)(p_\alpha + p_{1\alpha})]^{1/2}$  as the magnitude of the total four-momentum. The invariant flux is denoted by  $F = [(p_\alpha p_1^\alpha)^2 - (mc)^4]^{1/2}$ ,  $d\Omega'$  refers to the angles of  $\vec{p}'$  in the c.m. frame, and  $d\omega_1 = d^3 p_1/p_1^0$ .

For a situation not too far from equilibrium, one may write

$$f = f^0(1 + \phi), \quad (2)$$

where the deviation function  $|\phi| \ll 1$  and  $f^0$  is the Boltzmann distribution function for local equilibrium.<sup>1</sup>

$$f^0 = \rho z \exp(U_\alpha p^\alpha / kT) / [4\pi (mc)^3 K_2(z)], \quad (3)$$

where  $\rho \equiv \rho(x)$  and  $T \equiv T(x)$  are the particle-number density and temperature in a proper coordinate system,  $U \equiv U(x)$  is the four-velocity of the hydrodynamic particle flux ( $U_\alpha U^\alpha = -c^2$ ), and  $K_2(z)$  is the modified Bessel function with  $z = mc^2/kT$ . In the first Chapman-Enskog approximation, the function  $\phi(x, p)$  satisfies the equation

$$p_\alpha \partial^\alpha f^0 = -f^0 \mathcal{L}[\phi], \quad (4)$$

where  $\mathcal{L}[\phi]$  is the linearized collision integral and is given by

$$\mathcal{L}[\phi] = \int f_1^0 (\phi + \phi_1 - \phi' - \phi'_1) \sigma F d\Omega' d\omega_1. \quad (5)$$

The solution to Eq. (1) has the general structure

$$\begin{aligned} \phi = & A \partial_\alpha U^\alpha - B \Delta_{\alpha\beta} p^\beta \Delta^{\alpha\beta} (T^{-1} \partial_\beta T + c^{-2} D U_\beta) \\ & + C \langle p_\alpha p_\beta \rangle \langle \partial^\alpha U^\beta \rangle, \end{aligned} \quad (6)$$

where the notations  $D \equiv U^\alpha \partial_\alpha$ ,  $\Delta^{\alpha\beta} = g^{\alpha\beta} + c^{-2} U^\alpha U^\beta$ ,  $\langle t_{\alpha\beta} \rangle = \Delta_{\alpha\beta\gamma\delta} t^{\gamma\delta}$ , and  $\Delta_{\alpha\beta\gamma\delta} = (\Delta_{\alpha\gamma} \Delta_{\beta\delta} + \Delta_{\alpha\delta} \Delta_{\beta\gamma})/2 - \Delta_{\alpha\beta} \Delta_{\gamma\delta}/3$  have been used. The scalar functions  $A$ ,  $B$ , and  $C$ , which depend on  $p_\alpha U^\alpha(x)$ ,  $\rho(x)$ ,  $U^\alpha(x)$ , and  $T(x)$ , obey the integral equations

$$\mathcal{L}[A] = (-1/kT)Q, \quad (7)$$

$$\mathcal{L}[B \Delta_{\alpha\beta} p^\beta] = (1/kT)(p_\gamma U^\gamma + mh) \Delta_{\alpha\beta} p^\beta, \quad (8)$$

$$\mathcal{L}[C \langle p_\alpha p_\beta \rangle] = (-1/kT) \langle p_\alpha p_\beta \rangle, \quad (9)$$

where

$$\begin{aligned} Q = & -(mc)^2/3 + c^{-2} p_\alpha U^\alpha [(1 - \gamma)mh + \gamma kT] \\ & + c^{-2} [(4/3) - \gamma] (p_\alpha U^\alpha)^2. \end{aligned} \quad (10)$$

Above,  $\gamma = c_p/c_v$  is the ratio of specific heats, and  $h = c^2 K_3(z)/K_2(z)$  is the enthalpy at equilibrium. The energy-momentum tensor

$$T^{\alpha\beta} \equiv c \int p^\alpha p^\beta f \frac{d^3 p}{p^0} \quad (11)$$

<sup>1</sup>The generalization to Bose-Einstein and Fermi-Dirac statistics can be found in Ref. [23].

can now be calculated with  $f = f^o(1 + \phi)$ . In addition to the equilibrium energy-momentum tensor, the result features terms involving energy flow and the viscous pressure tensor, which are defined as

$$I_q^\alpha \equiv -U^\beta T_{\beta\gamma} \Delta^{\gamma\alpha}, \quad \Pi^{\alpha\beta} \equiv P^{\alpha\beta} - p \Delta^{\alpha\beta}, \quad (12)$$

where  $P^{\alpha\beta}$  is the pressure tensor defined as  $P^{\alpha\beta} \equiv \Delta^{\alpha\gamma} T_{\gamma\epsilon} \Delta^{\epsilon\beta}$ . By employing Eq. (2) in Eqs. (11) and (12), one can get

$$I_q^\alpha = -\lambda \Delta^{\alpha\beta} (\partial_\beta T + c^2 T D U_\beta), \quad (13)$$

$$\Pi^{\alpha\beta} = -2\eta \langle \partial^\alpha U^\beta \rangle - \eta_v \Delta^{\alpha\beta} \partial_\gamma U^\gamma. \quad (14)$$

The shear viscosity ( $\eta_s$ ) is given by

$$\eta_s = -\frac{1}{10} c \int C \langle p_\alpha p_\beta \rangle \langle p^\alpha p^\beta \rangle f^0 d\omega. \quad (15)$$

The above inhomogeneous integral equations for the transport coefficients can be reduced to sets of algebraic equations by expanding the unknown scalar function  $C(\tau)$ , where  $\tau = -(p_\alpha U^\alpha + mc^2)/kT$ , in terms of orthogonal polynomials, e.g., the Laguerre functions  $L_n^\alpha(\tau)$  with appropriate values of  $\alpha$  [half integers (0) for massive (massless) particles].

### 1. Shear viscosity of a one-component gas

Beginning with Eq. (9),

$$\mathcal{L}[C \langle p_\alpha p_\beta \rangle] = (-1/kT) \langle p_\alpha p_\beta \rangle, \quad (16)$$

the first approximation to the shear viscosity can be obtained explicitly by (i) multiplying both sides of the above equation with

$$(p^{(0)} f^{(0)} L_n^{5/2}(\tau))^{-1} \langle p^\alpha p^\beta \rangle$$

and integrating over momentum, (ii) introducing the quantity  $\gamma_n$  defined by

$$\gamma_n \equiv \frac{c}{\rho k^2 T^2} \int f^{(0)} L_n^{5/2}(\tau) \langle p_\alpha p_\beta \rangle \langle p^\alpha p^\beta \rangle \frac{d^3 p}{p^0} \quad (17)$$

and applying  $\gamma_n$  to Eq. (16), and writing the results in terms of the bracket expression as

$$[C \langle p_\alpha p_\beta \rangle, L_n^{5/2}(\tau) \langle p^\alpha p^\beta \rangle] = \frac{m^2 k T}{\rho} \gamma_n \quad (n = 0, 1, \dots). \quad (18)$$

We now write  $C$  as an expansion involving the generalized Laguerre polynomial as

$$C(\tau) = \sum_{m=0}^{\infty} c_m L_m^{5/2}(\tau) \quad (19)$$

so that Eq. (18) can be written as

$$\sum_{m=0}^{\infty} c_m c_{mn} = \frac{1}{\rho k T} \gamma_n \quad (n = 0, 1, \dots), \quad (20)$$

where

$$c_{mn} = \frac{1}{(mkT)^2} [L_m^{5/2}(\tau) \langle p_\alpha p_\beta \rangle, L_n^{5/2}(\tau) \langle p^\alpha p^\beta \rangle] \quad (m, n = 0, 1, \dots). \quad (21)$$

Note that  $c_{mn} = c_{nm}$ . The  $r$ th approximation to the coefficient  $c_m^{(r)}$  is obtained by truncating the sum in Eq. (20) to  $r$  terms; that is,

$$\sum_{m=0}^{r-1} c_m^{(r)} c_{mn} = \frac{1}{\rho k T} \gamma_n \quad (n = 0, 1, \dots, r-1). \quad (22)$$

Finally, the shear viscosity can be written as

$$\eta = \frac{1}{10} (kT)^2 \sum_{m=0}^{\infty} c_m \gamma_m. \quad (23)$$

The first, second, and third approximations to shear viscosity are

$$[\eta_s]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}}, \quad (24)$$

$$[\eta_s]_2 = \frac{1}{10} kT \frac{\gamma_0^2 c_{00} - 2 \gamma_0 \gamma_1 c_{01} + \gamma_1^2 c_{00}}{c_{00} c_{11} - c_{01}^2}, \quad (25)$$

$$[\eta_s]_3 = \frac{\rho (kT)^2}{10} (c_0 \gamma_0 + c_1 \gamma_1 + c_2 \gamma_2), \quad (26)$$

where

$$\gamma_0 = -10\hat{h}, \quad (27)$$

$$\gamma_1 = -[\hat{h}(10z - 25) - 10z], \quad (28)$$

$$c_{00} = 16 \left( w_2^{(2)} - \frac{1}{z} w_1^{(2)} + \frac{1}{3z^2} w_0^{(2)} \right), \quad (29)$$

$$c_{01} = 8 \left[ 2z(w_2^{(2)} - w_3^{(2)}) + (-2w_1^{(2)} + 3w_2^{(2)}) + z^{-1} \left( \frac{2}{3} w_0^{(2)} - 9w_1^{(2)} \right) - \frac{11}{3z^2} w_0^{(2)} \right], \quad (30)$$

$$c_{11} = 4 \left[ 4z^2(w_2^{(2)} - 2w_3^{(2)} + w_4^{(2)}) + 2z(-2w_1^{(2)} + 6w_2^{(2)} - 9w_3^{(2)}) + \left( \frac{4}{3} w_0^{(2)} - 36w_1^{(2)} + 41w_2^{(2)} \right) + z^{-1} \left( -\frac{44}{3} w_0^{(2)} - 35w_1^{(2)} \right) + \frac{175}{3z^2} w_0^{(2)} \right], \quad (31)$$

with

$$z = \frac{mc^2}{kT} \quad \text{and} \quad \hat{h} = \frac{K_3(z)}{K_2(z)}, \quad (32)$$

and the quantity  $w_i^{(s)}$  is so-called the relativistic omega integral which is defined as

$$w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)^2} \int_0^\infty d\psi \sinh^7 \psi \cosh^i \psi K_j(2z \cosh \psi) \times \int_0^\pi d\Theta \sin \Theta \sigma(\psi, \Theta) (1 - \cos^s \Theta). \quad (33)$$

In the third-order calculation, one more equation is needed to get the relation between the coefficients  $c_n$  and the coefficients  $c_{mn}$  which are shown in Eq. (22). The quantity  $\sigma(\psi, \Theta)$  is the transport cross section and  $j = \frac{5}{3} + \frac{1}{2}(-1)^i$ ; the others symbols are

$$g = \frac{1}{2}(p_1 - p_2) \quad \text{and} \quad P = (-p_\alpha p^\alpha)^{1/2}, \quad (34)$$

$$\sinh \psi = \frac{g}{mc} \quad \text{and} \quad \cosh \psi = \frac{P}{2mc}. \quad (35)$$

## 2. Massless particles

For nearly massless particles such as neutrinos and light quarks for which  $m/T \rightarrow 0$ , the formalism described earlier can be simplified as discussed in Ref. [24] and is summarized below. The reason for addressing the ultrarelativistic case is twofold: (1) For temperatures such that  $z_i = m_i/T \rightarrow 0$ , as is the case for light quarks in the context of heavy-ion collisions, it serves as a first orientation toward the magnitudes of viscosities, and (2) test cases for validating Green-Kubo calculations can be set up in this limit.

We start again with the relativistic transport equation for a one-component system of nondegenerate particles:

$$p^\alpha \partial_\alpha f(x, P) = \int (f f_1 - f' f'_1) \sigma F d\Omega' dw_1, \quad (36)$$

where  $f = f(x, p)$ ,  $\sigma = \sigma(\Theta, P)$  is the scattering cross section for  $p + p_1 \rightarrow p' + p'_1$  in the center-of-momentum frame. Other symbols are

$$F \equiv [(p^\alpha p_{1\alpha})^2 - p^\alpha p_\alpha p^{1\beta} p_{1\beta}]^{1/2} = -p^\alpha p_{1\alpha} = \frac{1}{2}P, \quad (37)$$

$$d\Omega' \equiv \sin \theta' d\theta' d\phi', \quad (38)$$

$$dw_1 \equiv \frac{d^3 p_1}{p_1^0}, \quad (39)$$

where  $\theta'$  and  $\phi'$  are the polar angles of the three-momentum  $\vec{p}'$  in the center-of-mass frame.

For massless particles, the equilibrium distribution function can be written as

$$f_{\text{eq}} = \frac{nc^3}{8\pi (k_B T)^3} \exp[p^\alpha U_\alpha / (k_B T)], \quad (40)$$

where  $n$  is the number density of particles,  $c$  is the speed of light,  $k$  is the Boltzman constant,  $T$  is the temperature, and  $U$  is the flow velocity. In the first Enskog approximation, the perturbed distribution function of the system can be written as

$$f(x, p) = f^{(0)}(x, p)[1 + \phi(x, p)], \quad (41)$$

where  $f^{(0)}(x, p)$  is the local equilibrium distribution function and  $\phi(x, p)$  is the deviation function. Using Eq. (41), one can linearize Eq. (36) to get

$$(p^\alpha U_\alpha + 4k_B T) p^\beta \Delta_{\beta\gamma} (T^{-1} \partial^\gamma T + c^{-2} D U^\gamma) + \langle p^\alpha p^\beta \rangle + \langle \partial_\alpha U^\beta \rangle = -k_B T \mathcal{L}[\phi], \quad (42)$$

where  $\mathcal{L}$  is the linearized operator defined by

$$\mathcal{L} \equiv \frac{1}{2} \int f_1^{(0)} \delta(F) \sigma P^2 d\Omega' dw_1 \quad (43)$$

with

$$\delta(F) \equiv F(p) + F(p_1) - F(p') - F(p'_1). \quad (44)$$

In Eq. (42), the angular bracket  $\langle \dots \rangle$  is for the operation

$$\langle A_{\alpha\beta} \rangle \equiv \frac{1}{2} \Delta_\alpha^\beta (A_{\gamma\delta} + A_{\delta\gamma}) \Delta_\beta^\delta - \frac{1}{3} \Delta_{\alpha\beta} \Delta_{\gamma\delta} A^{\gamma\delta}. \quad (45)$$

The general form of the deviation function (for elastic collisions) is

$$\phi(x, p) = -B_\alpha \Delta^{\alpha\beta} (T^{-1} \partial_\beta T + c^{-2} D U_\beta) + C_{\alpha\beta} \langle \partial^\alpha U^\beta \rangle. \quad (46)$$

In the case of shear viscosity, one needs to solve for the coefficients  $C_{\alpha\beta}$  which satisfy

$$\mathcal{L}[C_{\alpha\beta}] = -(kT)^{-1} \langle p_\alpha p_\beta \rangle. \quad (47)$$

In order to get an expression for the shear viscosity, one can use the distribution function in Eq. (41) in the viscous pressure tensor which is defined as

$$\pi^{\alpha\beta} \equiv P^{\alpha\beta} - p \Delta^{\alpha\beta}, \quad \text{where} \quad P^{\alpha\beta} \equiv \Delta^{\alpha\beta} T_{\gamma\delta} \Delta^{\delta\beta}. \quad (48)$$

As a result,

$$\pi^{\alpha\beta} = -2\eta_s \langle \partial^\alpha U^\beta \rangle, \quad (49)$$

where

$$\eta_s \equiv \frac{1}{10} c \int C_{\alpha\beta} \langle p_\alpha p_\beta \rangle f^{(0)} dw, \quad (50)$$

with  $C_{\alpha\beta} = C \langle p^\alpha p^\beta \rangle$ . The coefficient  $C$  can be written in terms of associated Laguerre polynomials:

$$C(\tau) = \sum_{n=0}^{\infty} c_n L_n^5(\tau), \quad (51)$$

where

$$L_n^\alpha(\tau) \equiv \sum_{i=1}^n \binom{n+\alpha}{i+\alpha} \frac{(-\tau)^i}{i!}, \quad \text{where} \quad \tau \equiv -p^\alpha U_\alpha / kT. \quad (52)$$

These functions satisfy the relations

$$\int_0^\infty L_m^\alpha(\tau) L_n^\alpha(\tau) \tau^\alpha \exp(-\tau) d\tau = \left[ \frac{\Gamma(n+\alpha+1)}{n!} \right] \delta_{0m}. \quad (53)$$

Inserting Eq. (51) in Eq. (47), one gets

$$\sum_0^\infty c_n \mathcal{L}[L_n^5(\tau) \langle p_\alpha p_\beta \rangle] = -(kT)^{-1} L_n^5(\tau) \langle p_\alpha p_\beta \rangle. \quad (54)$$

We introduce the notation

$$c_{mn} \equiv [L_m^5(\tau) \langle p^\alpha p^\beta \rangle, L_n^5(\tau) \langle p_\alpha p_\beta \rangle], \quad (55)$$

where the bracket operation means

$$[F, G] \equiv \frac{1}{8} c n^{-2} \int \delta(F) \delta(G) f^{(0)} f_1^{(0)} \sigma P^2 dw dw_1 d\Omega'. \quad (56)$$

Equation (54) can be written as

$$\sum_{n=0}^{r-1} c_{mn} c_n = -40 n^{-1} c^{-2} (kT)^2 \delta_{0m}. \quad (57)$$

Hence, one can write the shear viscosity for massless particles as

$$\eta_s = -40 n c^{-2} (kT)^3 c_0. \quad (58)$$

In Eq. (57), the coefficients  $c_{mn}$  are calculated from

$$c_{mn} = \sum_{i=1}^{19} c_{mn,i} \quad (59)$$

with

$$\begin{aligned} c_{mn,i} = & \frac{\gamma}{(2\beta c)^2} \sum_{r=0}^m \sum_{s=0}^n \binom{m+5}{r+5} \binom{n+5}{s+5} (-2)^{-r-s} \\ & \times \sum_{p=0}^r \sum_{q=0}^s (-1)^{p+q} \frac{t!}{p!(r-p)!} \frac{u!}{q!(s-q)!} \frac{[(t+u)/2]!}{(t+u+1)!} \\ & \times \sum_{k=0}^{[M/2]} \frac{2^{M-2k}}{k!(k+|t-u|/2)!(M-2k)!} \\ & \times \sum_{k=0}^{[w/2]} \binom{[w/2]}{0} (2v+2l-1)!! \tilde{w}_{r+s-l-v+9, l+v+\delta}^{M-2k+\epsilon}, \end{aligned} \quad (60)$$

where  $\tilde{w}_{ij}^k$  is the omega integral for massless particles :

$$\begin{aligned} \tilde{w}_{ij}^k \equiv & \frac{\pi}{2^4 \beta} \int_{-1}^1 d \cos \Theta \int_0^\infty dP \sigma(\Theta, P) (1 - \cos^k \Theta) \\ & \times (\beta c P)^i K_j(\beta c P). \end{aligned} \quad (61)$$

The quantity  $\delta$  is given by  $\delta \equiv (r+s) \bmod(2)$  and the quantity  $M$  is given by  $M = \min(t, u)$ . The rest of the variables needed are listed in Table I, which is reproduced from Ref. [24] in which details of the derivation that leads to the form shown in Eq. (60) for  $c_{mn,i}$  are given.

In the first-order approximation, the required coefficients are

$$c_{00} = \left( \frac{1}{3} \tilde{w}_{63}^2 + \frac{1}{2} \tilde{w}_{72}^2 + \frac{1}{4} \tilde{w}_{81}^2 \right) / (\beta c)^2, \quad (62)$$

$$c_{01} = \left( 2\tilde{w}_{63}^2 + 3\tilde{w}_{72}^2 - \frac{1}{2}\tilde{w}_{74}^2 + \frac{3}{2}\tilde{w}_{81}^2 - \frac{1}{2}\tilde{w}_{83}^2 - \frac{1}{8}\tilde{w}_{92}^2 \right) / (\beta c)^2, \quad (63)$$

$$\begin{aligned} c_{11} = & \left( 12\tilde{w}_{63}^2 + 18\tilde{w}_{72}^2 - \frac{3}{4}\tilde{w}_{74}^2 + 9\tilde{w}_{81}^2 - 3\tilde{w}_{83}^2 \right. \\ & \left. - \frac{15}{16}\tilde{w}_{92}^2 + \frac{1}{16}\tilde{w}_{10,1}^2 \right) / (\beta c)^2. \end{aligned} \quad (64)$$

The scheme outlined above has been utilized to calculate the shear viscosity of neutrinos in Ref. [24] and of chiral pions

by Prakash *et al.* [25]. In the next section, an application of this scheme to calculate  $\eta_s$  with a constant cross section will be presented. This application will be utilized to validate the Green-Kubo calculations.

### 3. Deviations from the ultrarelativistic limit

The ultrarelativistic limit corresponds to the relativity parameter  $z = mc^2/kT \rightarrow 0$ , in situations when either the mass tends to vanish or when the temperature is very large compared to the mass. In the context of relativistic heavy-ion collisions, low-mass quarks, such as the  $u$  and  $d$  quarks with current quark masses  $\leq 10$  MeV in conditions of temperatures above the phase transition temperature of  $kT \sim 200$  MeV, fall into the category of  $z \ll 1$ . In the hadronic phase, pions of masses  $\sim 140$  MeV in the temperature range of 100–200 MeV, however, fall in the borderline regime of the intermediate relativistic regime. It is therefore of some interest to gauge how deviations from the ultrarelativistic regime affect the transport coefficients. In this section, we summarize the work of Ref. [26] in which effects of slight deviations from the ultrarelativistic case were established in the case of hard spheres. Thereafter, the formalism for arbitrary interactions is developed. The case of the hard spheres will serve as a testbed for calculations of transport coefficients from the Green-Kubo formulas in which the mass of the particle is set to a small value for computational ease. Our development for arbitrary interactions will further aid the validation of such calculations in the relativistic regime.

### 4. The hard sphere gas

Here the calculation of Ref. [26] for the hard sphere gas with a constant differential cross section  $\sigma_0 = a^2/4$ , where  $a$  is the radius of the particle, is summarized.

The first step is to rewrite the relativistic omega integral in Eq. (33), with  $x = \cosh \psi$ , as

$$w_i^{(s)} = \frac{\pi z^3 c f^{(s)} a^2}{2K_2^2(z)} \int_1^\infty dx (x^2 - 1)^3 x^i K_j(2zx), \quad (65)$$

where

$$j = 5/2 + 1/2(-1)^i \quad \text{and} \quad f^{(s)} \equiv \frac{[2s+1+(-1)^{s+1}]}{(s+1)}. \quad (66)$$

By changing the integration variable and by employing binomial coefficients to express the third power, one can rewrite the above equation as

$$\begin{aligned} w_i^{(s)} = & \left[ \pi z^{-i-4} c f^{(s)} a^2 / 2K_2^2(z) \right] \sum_{k=0}^3 \binom{3}{k} (-1)^k z^{2k} \\ & \times \int_z^\infty dx x^{i-2k+6} K_j(2x). \end{aligned} \quad (67)$$

In the limit of  $z \ll 1$ , the two modified Bessel functions have the behaviors

TABLE I. The values of  $\gamma$ ,  $\epsilon$ ,  $t$ ,  $u$ ,  $v$ , and  $w$  as a function of  $i$ ,  $p$ ,  $q$ ,  $r$ , and  $s$  for the case of shear viscosity.

$i$	$\gamma$	$(-1)^p$	$(-1)^q$	$\epsilon$	$t - p$	$u - q$	$2v - p - q - 2$	$w + p + q - r - s$
1	1	+	+	2	0	0	0	0
2	2	-	-	1	0	0	2	0
3	2	+	+	1	1	1	2	0
4	-2	+	-	1	1	0	1	1
5	-2	-	+	1	0	1	1	1
6	2/3	+	+	0	0	0	0	4
7	-4/3	-	+	0	1	0	1	3
8	-4/3	+	-	0	0	1	1	3
9	2/3	+	+	0	2	0	2	2
10	8/3	-	-	0	1	1	2	2
11	2/3	+	+	0	0	2	2	2
12	-2	+	+	0	0	0	0	2
13	2	-	+	0	1	0	1	1
14	2	+	-	0	0	1	1	1
15	-4/3	+	-	0	2	1	3	1
16	-4/3	-	+	0	1	2	3	1
17	2/3	+	+	0	2	2	4	0
18	-2	-	-	0	1	2	2	0
19	1	+	+	0	0	0	0	0

$$K_2(z) = \frac{2}{z^{-2}} \left[ 1 - \frac{1}{4}z^2 - \frac{1}{16}z^4 \ln z + \frac{1.731863}{32}z^4 + \dots \right], \quad (68)$$

$$K_3(z) = \frac{8}{z^{-3}} \left[ 1 - \frac{1}{8}z^2 + \frac{1}{64}z^4 + \dots \right]. \quad (69)$$

Also, in the limit of  $z \rightarrow 0$ , the integral involving the modified Bessel function can be written as

$$\int_0^\infty dx x^\mu K_\nu(2x) = \frac{1}{4} \Gamma[(\mu + \nu + 1)/2] \Gamma[(\mu - \nu + 1)/2]. \quad (70)$$

The above relation is true for  $\mu \pm \nu + 1 > 0$ . Using these relations, the omega integral in Eq. (67) reads as

$$w_i^{(s)} = \frac{1}{32} \pi z^{-1} c f^{(s)} a^2 \Gamma[(i + j + 7)/2] \Gamma[(i - j + 7)/2] \times \left( 1 + \frac{(i^2 - j^2 + 10i + 1)}{(i^2 - j^2 + 10i + 25)} \frac{z^2}{2} + \dots \right). \quad (71)$$

Thermodynamic quantities such as the enthalpy,  $h$ , and the ratio of specific heats,  $\gamma$ , can be evaluated in the  $z \rightarrow 0$  limit by applying the properties of modified Bessel function in Eqs. (68) and (69) to read as

$$h = 4c^2 z^{-1} \left[ 1 + \frac{1}{8}z^2 + \frac{1}{16}z^4 \ln z + \frac{1}{32} \left( \frac{3}{2} - 1.731863 \right) z^4 + \dots \right], \quad (72)$$

$$\gamma = \frac{4}{3} \left[ 1 + \frac{1}{24}z^2 + \frac{1}{64}z^4 \ln z + \frac{1}{32} \left( \frac{43}{18} - 1.731863 \right) z^4 + \dots \right]. \quad (73)$$

Then the shear viscosities in this regime read as

$$[\eta_s]_p = \frac{mc}{\pi a^2} z^{-1} F_p (1 + F'_p z^2 + \dots), \quad (74)$$

where the subscript  $p$  refers to the  $p$ th approximation and the values of the various coefficients in the above equations are listed in Table II for  $p = 1, 2$ , and 3.

Note that, for massless particles,

$$[\eta_s]_p = \frac{k_B T}{\pi a^2} \frac{1}{c} F_p. \quad (75)$$

In the next section, calculations that attest to the rapid convergence of the coefficient  $F_p$  are carried to much higher order in  $p$ . These results will be of much utility in validating ultrarelativistic molecular dynamical simulations of shear viscosity.

## B. Reduction to the nonrelativistic case

In the nonrelativistic limit, i.e.,  $z = m/k_B T \gg 1$ , the results above can be further simplified. As for  $z \gg 1$ ,

$$K_n(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + \frac{(4n^2 - 1)}{2!} \frac{1}{z} + \dots \right), \quad (76)$$

the reduced enthalpy

$$\hat{h} \rightarrow 1 \quad \text{and} \quad \gamma_0 \rightarrow -10. \quad (77)$$

TABLE II. Values of the coefficients appearing in Eq. (74) for the hard sphere gas.

$p$	1	2	3
$F_p$	1.2	1.2588	1.2642
$F'_p$	0.05	0.0424	0.0403

In addition, the relativistic omega integral  $w_i^{(s)}$  in Eq. (33) can be transformed into its nonrelativistic counterpart  $\Omega_i^{(s)}$  as follows. Introducing the dimensionless quantity

$$\phi = \frac{g}{\sqrt{mc^2 k_B T}} = \frac{mc \sinh \psi}{\sqrt{mc^2 k_B T}}, \quad (78)$$

whereby

$$\cosh \psi = \sqrt{1 + z^{-1} \phi^2}, \quad (79)$$

the integral over  $\psi$  in Eq. (33) can be written as

$$I_\psi = \frac{2\pi k_B T}{m K_2^2(z)} \int_0^\infty d\phi \phi^7 (1 + z^{-1} \phi^2)^{i-1} \times K_j(2z\sqrt{1 + z^{-1} \phi^2}). \quad (80)$$

The use of a binomial expansion and the expansion of  $K_n(z)$  for  $z \gg 1$ ,

$$K_n(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + \frac{(4n^2 - 1)}{2!} \frac{1}{z} + \dots \right), \quad (81)$$

reduces the above integral to

$$I_\psi = 2 \sqrt{\frac{\pi k_B T}{m}} \int_0^\infty d\phi \phi^7 e^{-\phi^2}. \quad (82)$$

Inserting this result in Eq. (33), the omega integral in the nonrelativistic limit is

$$\Omega_2^{(s)} = 2 \sqrt{\frac{\pi k_B T}{m}} \int d\phi \phi^7 e^{-\phi^2} \times \int_0^\pi d\theta \sin \theta \sigma(\phi, \theta) (1 - \cos^s \theta). \quad (83)$$

Note that the magnitudes of the omega integrals in Eqs. (33) and (83) are determined by a combination of different physical factors: the thermal weight, collisions with large relative momenta, and the relative momentum dependence of the transport cross section. These omega integrals also feature in the calculation of the shear viscosity in higher order formulations; expressions for viscosity in higher order approximations may be found in Ref. [22].

A further simplification occurs in the nonrelativistic limit with

$$c_{00} \approx 16 w_2^{(2)} \rightarrow 16 \Omega_2^{(2)}, \quad (84)$$

as the second and third terms in Eq. (29) are suppressed by  $z$  being large. Thus, the shear viscosity takes the simple form

$$\eta_s = \frac{5 k_B T}{8 \Omega_2^{(2)}}. \quad (85)$$

### C. The relaxation time approximation

In the relaxation time approximation, the main assumption is that the effect of collisions is always to bring the perturbed distribution function close to the equilibrium distribution function, that is,  $f(\mathbf{x}, \mathbf{p}) \rightarrow f^{(0)}(\mathbf{x}, \mathbf{p})$ . In other words, the effect of collisions is to restore the local equilibrium distribution

function exponentially with a relaxation time  $\tau_0$  which is of order the time required between particle collisions [19]:

$$D_c f(\mathbf{x}, \mathbf{p}) = - \frac{f(\mathbf{x}, \mathbf{p}) - f^{(0)}(\mathbf{x}, \mathbf{p})}{\tau_0}. \quad (86)$$

In the relativistic case, we follow closely the formalism described in the review article by Kapusta [27] and develop working formulas for the calculation of shear viscosity using Maxwell-Boltzmann statistics [27–29]. (Bose-Einstein and Fermi-Dirac cases will be considered later.) We restrict our attention to the case involving two-body elastic reactions  $a + b \rightarrow c + d$  in a heat bath containing a single species of particles. In what follows, we use the notation employed in Ref. [29]. Differences from earlier notation in this section are few, and should not cause any confusion.

In the relaxation time approximation, the shear viscosity is given by [29]

$$\eta_s = \frac{1}{15T} \int_0^\infty \frac{d^3 p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f_a^{eq}, \quad (87)$$

where  $w_a(E_a)$  is the collision frequency and  $f_a^{eq}$  is the equilibrium distribution function of particles  $a$  with momenta  $p_a$  and energy  $E_a$ :

$$f_a(\mathbf{x}, \mathbf{p}_a, t) = \frac{1}{e^{(E_a - \mu_a)/T} - (-1)^{2s_a}}, \quad (88)$$

where  $\mu_a$  is the chemical potential of the particle, and  $f_a$  is normalized such that integration over momenta yields the density  $n(\mathbf{x}, t)$ .

The collision frequency is given by

$$w_a(E_a) = \sum_{bcd} \frac{1}{2} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a, b|c, d) f_b^{eq}, \quad (89)$$

where the quantity  $W(a, b|c, d)$  is defined as

$$W(a, b|c, d) = \frac{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}{2E_a 2E_b 2E_c 2E_d} |\mathcal{M}|^2. \quad (90)$$

Above,  $|\mathcal{M}|^2$  is the squared transition amplitude for the two-body reaction  $a + b \rightarrow c + d$  and  $f_b^{eq}$  is the distribution function of particles  $b$ . Utilizing the above expression, one can write

$$w_a(E_a) = \sum_{bcd} \frac{1}{2} \int \frac{d^3 p_b}{(2\pi)^3} f_b^{eq} I_E, \quad (91)$$

where

$$I_E \equiv \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \frac{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}{2E_c 2E_d} |\mathcal{M}|^2. \quad (92)$$

The exit channel integrals in the above equation can be manipulated in the center of mass (c.m.) frame to feature the differential cross section. In the c.m. frame,  $\sqrt{s} = E_{cm} = E_c^{cm} + E_d^{cm} = 2E_c^{cm} = 2E_d^{cm} = 2(m^2 + q_{cm}^2)^{1/2}$ . Performing the integration over  $\vec{p}_d$  in the c.m. frame,

$$I_E = \int \frac{d^3 p_c}{(2\pi)^3 2E_c^{cm}} \frac{(2\pi) \delta(\sqrt{s} - E_{cm})}{2E_c^{cm}} |\mathcal{M}|^2. \quad (93)$$

The integration over  $p_c$  can be effected through its connection to  $q_{cm}$  and  $E_{cm}$ ,

$$q_{cm}^2 = \frac{E_{cm}^2}{4} - m^2, \quad (94)$$

so that  $I_E$  can be rewritten as

$$I_E = 2\sqrt{s(s-4m^2)} \int d\Omega \sigma(\Omega), \quad (95)$$

where

$$\sigma(\Omega) = \frac{1}{64\pi^2 s} |\mathcal{M}|^2 \quad (96)$$

is the differential cross section in the c.m. frame. The collision frequency in Eq. (89) thus takes the form

$$w_a(E_a) = \int \frac{d^3 p_b}{(2\pi)^3} \frac{\sqrt{s(s-4m^2)}}{2E_a 2E_b} f_b^{eq} \sigma_T, \quad (97)$$

where  $\sigma_T$  is the total cross section.

As we can see from the above equation, interactions appear in the collision frequency through the total cross section. Here we see the difference with the Chapman-Enskog approximation, which features a transport cross section that favors right-angled collisions in the c.m. frame. We shall see that this difference is at the root of differences in results between the two approaches in the following sections.

#### D. Reduction to the nonrelativistic case

We turn now to reduce Eq. (97) for nonrelativistic particles. Recalling that  $s = 4(m^2 + q_{cm}^2)$ ,

$$\sqrt{s(s-4m^2)} = \sqrt{4(m^2 + q_{cm}^2)4q_{cm}^2} \approx 4m |\vec{q}_{cm}| \quad (98)$$

in the nonrelativistic limit. Also,  $E_a \approx m$  and  $E_b \approx m$  for equal mass particles. Thus,

$$w_a(E_a) = \int \frac{d^3 p_b}{(2\pi)^3} \sigma_T f_b^{eq} |\vec{v}_a - \vec{v}_b|, \quad (99)$$

which can be written in the form found in textbooks (see, e.g., Ref. [20]),

$$w_a(v_a) = \int_0^\infty d^3 v_b \sigma_T f_b^{eq} |\vec{v}_a - \vec{v}_b|, \quad (100)$$

after a suitable change of variables and normalization of  $f_b^{eq}$ .

Finally, employing the nonrelativistic expressions  $p_a = m v_a$  and  $E_a = m + p_a^2/(2m)$ , the shear viscosity in Eq. (87) takes the form

$$\eta_s = \frac{1}{30\pi^2} \frac{m^5}{T} \int d v_a v_a^6 \frac{f_a}{w_a(v_a)}. \quad (101)$$

### III. APPLICATIONS

#### A. The hard sphere gas (nonrelativistic)

This system is characterized by a constant differential cross section,  $\sigma_0 = a^2/4$ , where  $a$  is the hard sphere radius.

#### 1. The Chapman-Enskog approximation

Utilizing the hard spheres cross section, the nonrelativistic omega integral in Eq. (83) becomes

$$\Omega_2^{(2)} = 8\sigma_0 \sqrt{\frac{\pi k_B T}{m}}. \quad (102)$$

Use of this result in Eq. (85) yields the shear viscosity

$$\eta_s = \frac{5}{64} \sqrt{\frac{m k_B T}{\pi}} \frac{1}{a^2}. \quad (103)$$

#### 2. The relaxation time approximation

Here, the shear viscosity is calculated by combining the results in Eqs. (100) and (101). For a constant differential cross section, the collision frequency can be expressed as [20] (see Appendix for details of the derivation)

$$w_a(v_a) = n \sigma_T \sqrt{\frac{2k_B T}{\pi m}} \left[ e^{-\zeta^2} + (2\zeta + \zeta^{-1}) \int_0^\zeta dt e^{-t^2} \right], \quad (104)$$

where the dimensionless variable

$$\zeta = \sqrt{\frac{m}{2k_B T}} v. \quad (105)$$

The shear viscosity, from Eq. (101), is

$$\eta_s = \frac{1}{30\pi^2} \frac{m^5}{T} \int d v_a \frac{v_a^6}{n \sigma_T} \sqrt{\frac{\pi m}{2k_B T}} \times \frac{f_a^{eq}}{\left[ e^{-\zeta^2} + (2\zeta + \zeta^{-1}) \int_0^\zeta dt e^{-t^2} \right]}. \quad (106)$$

In the non-relativistic limit

$$f_a^{eq} = \exp(\mu/k_B T) e^{-\zeta^2}, \quad (107)$$

where  $\mu$  is the chemical potential. Hence

$$\eta_s = \frac{1}{30\pi^2} \frac{m^5}{T} \frac{z}{n \sigma_T} \sqrt{\frac{\pi m}{2k_B T}} \times \int d v_a v_a^6 \frac{e^{-\zeta^2}}{\left[ e^{-\zeta^2} + (2\zeta + \zeta^{-1}) \int_0^\zeta dt e^{-t^2} \right]}. \quad (108)$$

A numerical quadrature of the above integral yields the result  $0.463282(\frac{2k_B T}{m})^{7/2}$ . The shear viscosity is then given by

$$\eta_s = \frac{3.706256 \sqrt{\pi}}{30\pi^2} \frac{\exp(\mu/k_B T)}{n \sigma_T} \frac{m^2 c^4 k_B^2 T^2}{\hbar^3 c^3}, \quad (109)$$

where the factor of  $(\hbar c)^3$  has been inserted to get the correct unit of viscosity. For Boltzmann statistics

$$n = \exp(\mu/k_B T) \left( \frac{m c^2 k_B T}{2\pi \hbar^2 c^2} \right)^{3/2}, \quad (110)$$

so that

$$\eta_s = \frac{0.34942}{4\sqrt{\pi}} \sqrt{\frac{m k_B T}{\pi}} \frac{1}{a^2}. \quad (111)$$



### B. The hard sphere gas (ultrarelativistic)

In this section, we calculate the shear viscosity for massless particles using a constant differential cross section  $\sigma_0 = a^2/4$  for which the total cross section  $\sigma_T = \pi a^2$ , where  $a$  is the radius of the sphere. For point particles such as quarks, the quantity  $a$  can be regarded as an effective length scale that serves to define a scattering cross section.

### C. The Chapman-Enskog approximation

We start by simplifying the omega integral in Eq. (61):

$$w_{ij}^k \equiv \frac{\pi}{2^4 \beta} \int_{-1}^1 d \cos \Theta \int_0^\infty dP \sigma(\Theta, P) (1 - \cos^k \Theta) \times (\beta c P)^i K_i(\beta c P).$$

For  $s = 2$  and a constant cross section  $\sigma(\Theta, P) = \sigma_0$ , we can rewrite the above equation as

$$w_{\mu\nu}^2 = \frac{\pi \sigma_0}{12 \beta} \int_0^\infty dp (\beta c p)^\mu K_\nu(\beta c p). \quad (112)$$

The integration can be performed through a change of variable:

$$\beta c p = 2x \quad \Rightarrow \quad dp = \frac{2}{\beta c} dx. \quad (113)$$

Then the omega integral can be written as

$$w_{\mu\nu}^2 = \frac{\pi \sigma_0}{\beta^2 c} \frac{2^{\mu-1}}{3} \int_0^\infty dx x^\mu K_\nu(2x). \quad (114)$$

We can now use the identity

$$\int_0^\infty dx x^\mu K_\nu(2x) = \frac{1}{4} \Gamma[(\mu + \nu + 1)/2] \Gamma[(\mu - \nu + 1)/2] \quad (115)$$

to write the omega integral for massless particles with a constant cross section as

$$w_{\mu\nu}^2 = \frac{\pi \sigma_0}{\beta^2 c} \frac{2^{\mu-1}}{12} \Gamma[(\mu + \nu + 1)/2] \Gamma[(\mu - \nu + 1)/2]. \quad (116)$$

The shear viscosity is

$$\eta_s = -4nc^{-2} (k_B T)^3 c_0, \quad (117)$$

where  $c_0$  can be calculated from Eq. (54). For the first-order calculation, the coefficient  $c_0$  is

$$[c_0]_1 = -\frac{40}{cn^2} (k_B T)^2 \frac{1}{c_0}. \quad (118)$$

Substituting for  $c_0$  in Eq. (117),

$$\eta_s = \frac{160}{c^4} \frac{1}{\beta^5} \frac{1}{c_{00}} \quad (119)$$

where  $c_{00}$  is taken from Eq. (62). The omega integrals needed are

$$w_{63}^2 = 64 \frac{\pi \sigma_0}{\beta^2 c}, \quad w_{72}^2 = 256 \frac{\pi \sigma_0}{\beta^2 c}, \quad \text{and} \quad w_{81}^2 = 1536 \frac{\pi \sigma_0}{\beta^2 c}. \quad (120)$$

Utilizing these results, the first -order approximation for the shear viscosity is

$$[\eta_s]_1 = 1.2 \frac{k_B T}{\pi a^2 c}, \quad (121)$$

where we have used  $\sigma_0 = a^2/4$ , where  $a$  is the radius of the hard sphere.

### I. Successive approximations

A point worth noting is that successive approximations to the shear viscosity can be obtained in the Chapman-Enskog approximation, a feature that is lacking in the relaxation time approximation. As an example, we calculated up to 16 orders in the case of the ultrarelativistic hard sphere gas. For higher-order calculations, the nested sums in Eqs. (57) and (60) in the calculation of the coefficients  $c_{mn}$  and  $c_{mn,i}$  call for a large number of evaluations (many repeated) of the omega integrals. Fortunately, the omega integrals required in this calculation can be performed analytically (consuming little computer time):

$$\tilde{\omega}_{ij}^k = \frac{\pi a^2}{\beta c} \frac{2^i}{2^6} \left( \frac{i+j-1}{2} \right)! \left( \frac{i-j-1}{2} \right)! \left( \frac{k}{k+1} \right). \quad (122)$$

Our results for shear viscosity are shown in the second column of Table III and in Fig. 1. We note that results up to the third-order approximation exist in the literature in Ref. [26]. Our test of the convergence of higher order approximations here indicates that for all practical purposes the third-order results are adequate. In addition,  $z = mc^2/k_B T$  corrections are also available for the third-order results, which can be gainfully employed to check results of computer simulations in which the mass cannot strictly be set to zero.

In addition to showing the convergence of results, the final result in this case serves as a testbed result that Green-Kubo

TABLE III. Shear viscosity for the ultrarelativistic hard sphere gas up to the 16th order.

Order of approximation	$\eta_s / [k_B T / (\pi a^2 c)]$
1	1.2
2	1.25581395
3	1.2642487
4	1.2663424
5	1.26703133
6	1.26730375
7	1.26742645
8	1.26748735
9	1.26751995
10	1.26753849
11	1.26754958
12	1.26755648
13	1.26756094
14	1.26756391
15	1.26756593
16	1.26756759

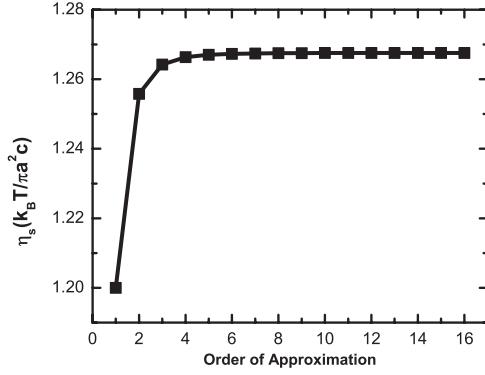


FIG. 1. The shear viscosity calculated up to the 16th-order approximation.

calculations can check. Such calculations are underway and will be reported elsewhere.

#### D. The relaxation time approximation

Here, we start with the collision frequency in Eq. (97). For a constant cross section

$$w_a(E_a) = \frac{1}{16\pi^2} \frac{\sigma_T}{E_a} \int_0^\infty dE_b \int_{-1}^1 dx E_b s e^{-E_b/k_B T}, \quad (123)$$

where  $x = \cos \theta$ . To solve the above integral, we note that

$$s - (p_a + p_b)^2 = 2|p_a||p_b| \left( x - \frac{s}{2|p_a||p_b|} \right). \quad (124)$$

Inserting the identity involving the delta function

$$1 = \int ds \frac{1}{2|p_a||p_b|} \delta \left( x - \frac{s}{2|p_a||p_b|} \right) \quad (125)$$

in the above integral, the collision frequency reads as

$$w_a(E_a) = \frac{\sigma_T (k_B T)^3}{2\pi^2}. \quad (126)$$

Supplying the collision frequency into Eq. (87), the shear viscosity becomes

$$\eta_s = \frac{8}{5} \frac{k_B T}{\sigma_T} \frac{1}{c}. \quad (127)$$

If we use the Bose-Einstein distribution function in the calculation, then the result for the collision frequency is given by

$$w_a(E_a) = \frac{\sigma_T (k_B T)^3}{2\pi^2} \zeta(3) \quad (128)$$

and the shear viscosity is given by

$$\eta_s = \frac{8}{5} \frac{k_B T}{\sigma_T} \frac{1}{c} \frac{\zeta(5)}{\zeta(3)}, \quad (129)$$

where  $\sigma_T = \pi a^2$ .

#### E. The Maxwell gas

Particles in the Maxwell gas are characterized by the differential cross section [30]

$$\sigma(g, \theta) = \frac{m \Gamma(\theta)}{2g}, \quad (130)$$

where  $m$  is the mass,  $g$  is the relative momentum, and  $\Gamma(\theta)$  is an arbitrary function of angle. The unit of  $\Gamma(\theta)$  is  $\text{fm}^3/\text{s}$ . In this calculation, we set  $\Gamma(\theta) = \Gamma$ , where  $\Gamma$  is a constant. Inclusion of  $\theta$  dependence is straightforward.

##### 1. The Chapman-Enskog approximation

We begin by writing the cross section in terms of  $\phi = g/\sqrt{mT}$  as

$$\sigma(g, \theta) = \frac{m}{2g} \Gamma(\theta) = \frac{m \Gamma(\theta)}{2\phi \sqrt{mT}}. \quad (131)$$

The nonrelativistic omega integral for Maxwell particles can be calculated by using Eq. (83):

$$\Omega_2^{(2)} = \frac{5}{4} \pi \Gamma. \quad (132)$$

The quantity  $c_{00}$  can be calculated from Eq. (29) with the result  $c_{00} = 20\pi \Gamma$ . From Eq. (24), the shear viscosity is

$$\eta_s = \frac{k_B T}{2\pi \Gamma}, \quad (133)$$

where we have used  $\gamma_0 = -10\hat{h}$  with  $\hat{h} \rightarrow 1$  in the nonrelativistic limit.

##### 2. The relaxation time approximation

In the relaxation time approximation, the shear viscosity can be calculated using Eqs. (100) and (101). We start by calculating the collision frequency

$$w_a(v_a) = \int d^3 v_b d\Omega \frac{m \Gamma(\theta)}{2m |\vec{v}_a - \vec{v}_b|} f_b |\vec{v}_a - \vec{v}_b|. \quad (134)$$

For angle independent  $\Gamma$ , the collision frequency  $w_a(v_a) = 2\pi \Gamma n$ . From Eq. (101),

$$\eta_s = \frac{1}{30\pi^2} \frac{m^5}{T} \frac{1}{2\pi \Gamma n} z \int_0^\infty dv_a v_a^6 e^{-mv_a^2/(2k_B T)}. \quad (135)$$

The quantity  $n = z(\frac{mT}{2\pi\hbar^2 c^2})^{3/2}$  is defined as in Eq. (110). Setting

$$x = \sqrt{\frac{m}{2k_B T}} v_a \quad (136)$$

and performing the integral, the shear viscosity is

$$\eta_s = \frac{k_B T}{2\pi \Gamma}. \quad (137)$$

### F. Massless pions

In the relativistic regime, we consider massless pions whose elastic differential cross section is given by [25]

$$\sigma(s, \theta) = \frac{s}{64 \pi^2 f_\pi^4} (3 + \cos^2 \theta), \quad (138)$$

where  $s = \sqrt{E_{cm}}$  is a Mandelstam variable and  $f_\pi = 93$  MeV is the pion decay constant.

#### 1. The Chapman-Enskog approximation

In the Chapman-Enskog approximation, the formalism to calculate the shear viscosity is described in Ref. [24] and was summarized in Sec. II B. The shear viscosity is obtained from Eqs. (58) and (60)–(62). The omega integrals that are required for chiral pions are

$$\begin{aligned} \tilde{w}_{63}^2 &= \frac{128}{\pi c^3} \left( \frac{k_B T}{f_\pi} \right)^4, & \tilde{w}_{72}^2 &= \frac{768}{\pi c^3} \left( \frac{k_B T}{f_\pi} \right)^4, \\ \text{and } \tilde{w}_{81}^2 &= \frac{6144}{\pi c^3} \left( \frac{k_B T}{f_\pi} \right)^4. \end{aligned} \quad (139)$$

Using the above omega integrals to calculate  $c_{00}$ , we finally obtain the shear viscosity as

$$\eta_s = \frac{15 \pi}{184} \frac{f_\pi^4}{k_B T} \frac{1}{c}. \quad (140)$$

#### 2. The relaxation time approximation

We first calculate the collision frequency defined in Eq. (97). In order to perform the integral we set  $x = \cos \theta$  and note that

$$s - (k_a + k_b)^2 = 2|k_a| |k_b| \left( x - \frac{s}{2|k_a| |k_b|} \right). \quad (141)$$

Introducing the identity involving the delta function,

$$1 = \int ds \frac{1}{2|p_a| |p_b|} \delta \left( x - \frac{s}{2|p_a| |p_b|} \right), \quad (142)$$

and inserting the above two relations in Eq. (97), we arrive at

$$w_a(E_a) = \frac{\pi}{108} E_a \left( \frac{k_B T}{f_\pi} \right)^4. \quad (143)$$

Inserting the above result into Eq. (87), the shear viscosity for chiral pions reads as

$$\eta = \frac{12 \pi}{25} \frac{f_\pi^4}{T} \frac{1}{\hbar^2 c^3}. \quad (144)$$

### G. Interacting massive pions

We choose the following parametrization for the experimental  $\pi$ - $\pi$  phase shifts adopted by Bertsch *et al.* [31]:

$$\delta_0^0 = \frac{\pi}{2} + \arctan \left( \frac{\epsilon - m_\sigma}{\Gamma_\sigma/2} \right), \quad (145)$$

$$\delta_1^1 = \frac{\pi}{2} + \arctan \left( \frac{\epsilon - m_\rho}{\Gamma_\rho/2} \right), \quad (146)$$

$$\delta_0^2 = -\frac{0.12q}{m_\pi}, \quad (147)$$

where in the symbol  $\delta_l^I$ ,  $I$  is the total isospin of the two pions and  $l$  is the angular momentum. The quantity

$$\epsilon = 2(q^2 + m_\pi^2)^{1/2} \quad \text{with } m_\pi = 140 \text{ MeV}. \quad (148)$$

The phase shift  $\delta_0^0$  corresponds to the  $s$ -wave  $\sigma$  resonance, with the width  $\Gamma_\sigma = 2.06q$  and  $m_\sigma = 5.8m_\pi$ . The phase shift  $\delta_1^1$  is from the  $p$ -wave  $\rho$  resonance, with the width

$$\Gamma_\rho(q) = 0.095q \left( \frac{q/m_\pi}{1 + (q/m_\rho)^2} \right)^2 \quad (149)$$

and  $m_\rho = 5.53m_\pi$ . The phase shift  $\delta_0^2$  accounts for  $s$ -wave repulsive interactions. The isospin averaged cross section for elastic scattering is obtained from

$$\frac{d\sigma(q, \Theta)}{d\Omega} = \frac{4}{q^2} \sum_{l, I} \frac{(2I+1)(2l+1)}{\sum_I (2I+1)} P_l(\cos \Theta) \sin^2 \delta_l^I(q), \quad (150)$$

where the prime denotes that the isospin sum is restricted to values for which  $l+I$  is even, and  $l=0, 1, 2$ , whence

$$\frac{d\sigma(s, \Theta)}{d\Omega} = \frac{4}{q_{cm}^2} \left( \frac{1}{9} \sin^2 \delta_0^0 + \frac{5}{9} \sin^2 \delta_0^2 + \frac{1}{3} \sin^2 \delta_1^1 \cos^2 \Theta \right). \quad (151)$$

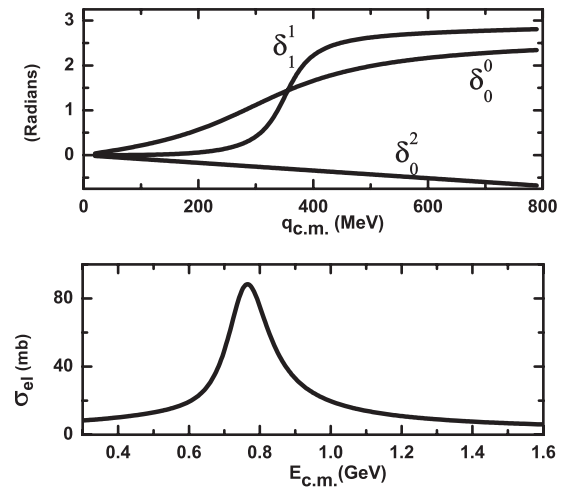


FIG. 2. Top panel: Contributing phase shifts for pion-pion scattering. The phase shift  $\delta_0^0$  corresponds to the  $s$ -wave  $\sigma$  resonance,  $\delta_1^1$  is from the  $p$ -wave  $\rho$  resonance, and  $\delta_0^2$  accounts for  $s$ -wave repulsive interactions. Bottom panel: Total cross section versus the center-of-mass energy. This figure is adapted from Ref. [32].

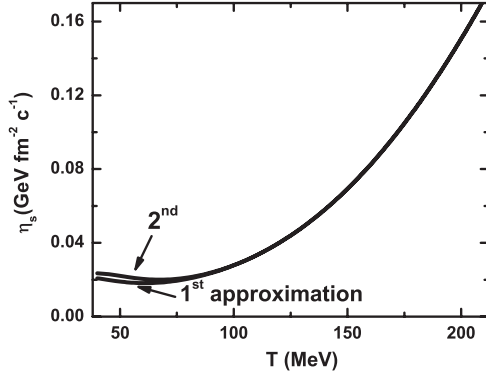


FIG. 3. Shear viscosity versus temperature for a system of interacting pions with experimental cross sections. Results up to the second-order approximation are shown. This figure is adapted from Ref. [25].

In Fig. 2, the phase shifts (top panel) and the total cross section (bottom panel) are shown. The  $s$ -wave  $\sigma$  resonance and the  $p$ -wave  $\rho$  resonance are clearly evident from this figure as the corresponding phase shifts  $\delta_0^0$  and  $\delta_1^1$  both exceed  $\pi$  radians, a signature of resonance formation. Note that the total cross section is dominated by the  $\rho$  resonance with a peak around 770 MeV.

The results for the shear viscosity of interacting pions up to the second-order approximation are shown in Fig. 3 using the experimental differential cross sections. For calculating results beyond the first-order approximation, methods described in Sec. II are employed. The role of the energy dependence of the scattering cross section is evident from this figure. Beyond the  $\rho$ -meson resonance energy of 770 MeV, the experimental cross sections decrease with the center-of-mass-energy which makes the shear viscosity increase with temperature. The results also show the rapid convergence of the Chapman-Enskog approach for the shear viscosity. The first-order results appear quite adequate for all practical purposes in the temperature range of 100–200 MeV of relevance to heavy-ion collisions.

In Fig. 4, the first-order results of shear viscosity from the Chapman-Enskog approach are compared with those from the relaxation time approach (left panel). The right panel shows the ratio that is calculated from the Chapman-Enskog viscosity divided by the relaxation time viscosity.

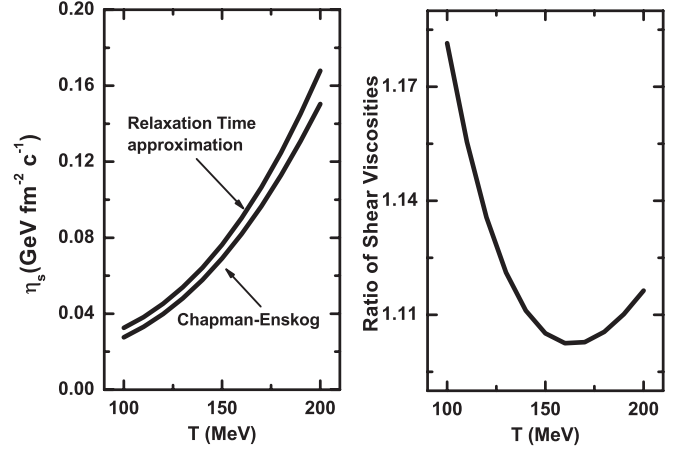


FIG. 4. Left panel: Shear viscosities of pion gas from the relaxation time approximation and the first-order Chapman-Enskog approximation. Right panel: The ratio of the results in the left panel.

#### IV. DISCUSSION OF ANALYTICAL AND NUMERICAL RESULTS

In this section, we collect results of calculations performed using the two different approaches: the Chapman-Enskog approximation and the relaxation time approximation. The nonrelativistic limit ( $z = mc^2/k_B T \gg 1$ ) is examined in the cases of the hard sphere particles (nonrelativistic case) and the Maxwell particles. The ultrarelativistic limit is explored in the cases of the hard sphere gas and massless pions. In the case of massive interacting pions with experimental cross sections, calculations are performed using the general relativistic scheme outlined in Sec. II.

Table IV shows the systems considered along with their corresponding cross sections, and results of  $\eta_s$  from the first-order Chapman-Enskog and the relaxation time approximations. The results in the table and those in the following figures must be viewed bearing in mind one difference that exists in the calculational procedures. The Chapman-Enskog approximation features the transport cross section with an angular weight of  $(1 - \cos^2 \Theta)$  in first-order calculations. The relaxation time approach lacks this angular weighting. The angular integral can be performed analytically for the cases chosen and leads to a factor of  $4/3$  for angle independent cross sections. Even so, it is intriguing that, for the case of Maxwell particles, the two methods give exactly the same result. This is

TABLE IV. Summary of results for shear viscosity. Results for the Chapman-Enskog approach are for the first-order approximation.

Case	Cross section	Chapman-Enskog	Relaxation	Ratio
Shear viscosities of nonrelativistic systems				
Hard sphere (nonrelativistic)	$\sigma = \frac{a^2}{4}$	$0.078 \sqrt{\frac{m k_B T}{\pi}} \frac{1}{a^2}$	$0.049 \sqrt{\frac{m k_B T}{\pi}} \frac{1}{a^2}$	1.59
Maxwell gas	$\sigma_0 = \frac{m \Gamma(\theta)}{2g}$	$\frac{k_B T}{2\pi \Gamma}$	$\frac{k_B T}{2\pi \Gamma}$	1.00
Shear viscosities of ultrarelativistic systems				
Hard sphere (ultrarelativistic)	$\sigma_0 = \frac{a^2}{4}$	$1.2 \frac{k_B T}{\pi a^2} \frac{1}{c}$	$\frac{8}{5} \frac{k_B T}{\pi a^2 c}$	0.75
Chiral pions	$\sigma = \frac{s}{(64\pi^2 f_\pi^4)} (3 + \cos^2 \theta)$	$\frac{15\pi}{184} \frac{f_\pi^4}{T} \frac{1}{\hbar^2 c^3}$	$\frac{12\pi}{25} \frac{f_\pi^4}{T} \frac{1}{\hbar^2 c^3}$	0.169

perhaps because of the fact that the relative velocity appearing in the denominator of the cross section is exactly canceled by a similar factor occurring in the numerator in both methods. In the remaining cases, it is clear from Table IV that the energy dependence of the cross sections plays a crucial role in determining the extent to which results differ between the two approaches.

## V. SUMMARY AND CONCLUSIONS

A quantitative comparison between results from the Chapman-Enskog and relaxation time methods to calculate viscosities was undertaken for the following test cases:

- (1) the nonrelativistic and relativistic hard sphere gas in which particles interact with a constant cross section;
- (2) the Maxwell gas in which the cross section is inversely proportional to the relative velocity of the scattering particles;
- (3) chiral pions for which the cross section is proportional to the squared center-of-mass energy; and
- (4) massive pions for which the differential elastic cross section features resonances is taken from experiments.

The analytical and numerical results of our comparative study reveal that the extent of agreement (or disagreement) between the Chapman-Enskog and the relaxation time approaches depends sensitively on the energy dependence of the differential cross sections employed. Comparisons with the more exact, albeit classical, calculations that employ Green-Kubo techniques will be of much interest in these cases. In the context of the nonrelativistic hard sphere gas, such comparisons exist in the classical literature (see, e.g., Ref. [20], and references therein); the lesson learned here is that higher order approximations of the Chapman-Enskog approach come very close to results of the Green-Kubo approach. In like fashion, our calculations of the shear viscosity of ultrarelativistic hard spheres can be used to check Green-Kubo calculations of the shear viscosity. In the context of relativistic heavy-ion collisions, calculations of the shear viscosity have been performed in Refs. [33–36] for mixtures of hadrons. A detailed comparison of the various approaches with the same energy dependent cross sections is worthwhile, a task that is being undertaken currently and will be published separately.

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## APPENDIX: NONRELATIVISTIC COLLISION FREQUENCY [EQ. (104)]

We start from the expression

$$w_a(v_a) = \int_0^\infty d^3 v_b \sigma_T f_b |\vec{v}_a - \vec{v}_b|, \quad (\text{A1})$$

where

$$f_b = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mv_b^2/(2kT)}. \quad (\text{A2})$$

For a constant differential cross section, the collision frequency is

$$w_a(v_a) = \sigma_T n \left( \frac{m}{2\pi kT} \right)^{3/2} 2\pi \int d v_b v_b^2 e^{-mv_b^2/(2kT)} \times \int_{-1}^1 |\vec{v}_a - \vec{v}_b|. \quad (\text{A3})$$

Setting  $\sqrt{\frac{m}{2kT}} v_b = \zeta_b$ , we have

$$w_a(v_a) = 2\sigma_T n \left( \frac{2kT}{m} \right)^{1/2} \int_0^\infty d\zeta_b \zeta_b^2 e^{-\zeta_b^2} \times \int_{-1}^1 dx |\vec{\zeta}_a - \vec{\zeta}_b|. \quad (\text{A4})$$

The angular integration yields

$$\int_{-1}^1 dx |\vec{\zeta}_a - \vec{\zeta}_b| = \frac{1}{3\zeta_a \zeta_b} ((\zeta_a + \zeta_b)^3 - (\zeta_a - \zeta_b)^3). \quad (\text{A5})$$

The above expression can be further simplified in the cases

$$\begin{aligned} \zeta_a > \zeta_b & \text{ for which } \int_{-1}^1 dx |\vec{\zeta}_a - \vec{\zeta}_b| = 2 \left( \zeta_a + \frac{\zeta_b^2}{3\zeta_a} \right), \\ \zeta_a < \zeta_b & \text{ for which } \int_{-1}^1 dx |\vec{\zeta}_a - \vec{\zeta}_b| = 2 \left( \zeta_b + \frac{\zeta_a^2}{3\zeta_b} \right). \end{aligned} \quad (\text{A6})$$

These two expressions can be inserted into Eq. (A5) so that the integration over  $\zeta_b$  reads as

$$2 \left[ \int_0^{\zeta_a} d\zeta_b \zeta_b^2 e^{-\zeta_b^2} \left( \zeta_a + \frac{\zeta_b^2}{\zeta_a} \right) + \int_{\zeta_a}^\infty d\zeta_b \zeta_b^2 e^{-\zeta_b^2} \left( \zeta_b + \frac{\zeta_a^2}{\zeta_b} \right) \right] \quad (\text{A7})$$

Upon integration by parts, the first term of the above integration results in two terms:

$$\zeta_a \int_0^{\zeta_a} d\zeta_b \zeta_b^2 e^{-\zeta^2} = -\frac{1}{2} \zeta_a e^{-\zeta_a^2} + \frac{1}{2} \int_0^{\zeta_a} d\zeta_b e^{-\zeta_b^2}. \quad (\text{A8})$$

The second term, again using integration by parts, gives

$$\begin{aligned} \frac{1}{\zeta_a} \int_0^{\zeta_a} d\zeta_b \zeta_b^3 (\zeta_b e^{-\zeta_b^2}) &= -\frac{\zeta_a^3}{2} e^{-\zeta_a^2} - \frac{3\zeta_a}{4} e^{-\zeta_a^2} \\ &+ \frac{3}{4} \int_0^{\zeta_a} d\zeta_b e^{-\zeta_b^2}. \end{aligned} \quad (\text{A9})$$

The sum of these two terms is

$$I_1 = -\frac{4}{3} \zeta_a^2 e^{-\zeta_a^2} + \left( \zeta_a + \frac{1}{2\zeta_a} \right) \int_0^{\zeta_a} d\zeta_b e^{-\zeta_b^2} - \frac{1}{2} e^{-\zeta_a^2}. \quad (\text{A10})$$

The last two terms in Eq. (A5) can be integrated to yield

$$I_2 = 2 \left( \int_{\zeta_a}^{\infty} d\zeta_b \zeta_b^3 e^{-\zeta_b^2} + \frac{\zeta_a^2}{3} \int_{\zeta_a}^{\infty} d\zeta_b \zeta_b e^{-\zeta_b^2} \right) = (1 + \zeta_a^2) e^{-\zeta_a^2} + \frac{\zeta_a^2}{3} e^{-\zeta_a^2} = \left( 1 + \frac{4}{3} \zeta_a^2 \right) e^{-\zeta_a^2}. \quad (\text{A11})$$

The sum of  $I_1$  and  $I_2$  is

$$I_T = I_1 + I_2 = e^{-\zeta_a^2} + (2\zeta_a + \zeta_a^{-1}) \int_0^{\zeta_a} dt e^{-t^2}. \quad (\text{A12})$$

Therefore, the collision frequency for the hard sphere gas is given by

$$w_a(v_a) = n \sigma_T \sqrt{\frac{2kT}{\pi m}} \left[ e^{-\zeta_a^2} + (2\zeta_a + \zeta_a^{-1}) \int_0^{\zeta_a} dt e^{-t^2} \right]. \quad (\text{A13})$$

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