Partial-wave expansion for photoproduction of two pseudoscalars on a nucleon

A. Fix[1] and H. Arenhövel[2]

[1]*Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634034 Tomsk, Russia*
[2]*Institut für Kernphysik, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany*

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The amplitudes for photoproduction of two pseudoscalars on a nucleon are expanded in the overall center-of-momentum (c.m.) frame in a model-independent way with respect to the contribution of the final-state partial wave of total angular momentum J and its projection on the normal to the plane spanned by the momenta of the final particles. The expansion coefficients, which are analogs to the multipole amplitudes for single-meson photoproduction, contain the complete information about the reaction dynamics. Results of an explicit evaluation are presented for the moments W_{jm} of the inclusive angular distribution of an incident photon beam with respect to the c.m. coordinate system defined by the final particles, taking photoproduction of $\pi^0\pi^0$ and $\pi^0\eta$ as an example.

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I. INTRODUCTION

The study of multiple-meson production is essential for understanding the properties of baryonic resonances, especially of those having sizable inelasticities and for which only a weak evidence from elastic πN scattering exists. According to the quark model calculation of [1], at least below 2 GeV, some of these resonances must be strongly coupled to $\pi\pi N$ and $\pi\eta N$ channels. Therefore, present experiments on $\pi\pi$ and $\pi\eta$ photoproduction have become a center of attention in programs discussed at various research centers, and a number of new accurate data have already been reported [2–12].

Improvements in the quality of the data have made it possible to perform rather detailed theoretical analyses of photoproduction of two pseudoscalars. The $\pi\pi$ as well as $\pi\eta$ models have already been the object of several studies [13–22]. Mainly, they cover the second and third resonance regions describing with varying degrees of success the existing data and predicting the results of new measurements. A typical analysis is based on an isobar model approach. Its key assumption is that the amplitude is a coherent sum of background and resonances usually parametrized in terms of effective Lagrangians. As a rule, the resonance part contains s-channel resonances via intermediate formation of meson-nucleon and meson-meson isobars decaying into $\pi\pi N$ or $\pi\eta N$. For the adjustable parameters, one usually takes the masses and partial decay widths of the resonances as well as their electromagnetic coupling constants.

Thus, in this approach, one assumes that the disconnected parts of the amplitude, coming from two-body interactions in the final state, may be described via formation and decay of resonances. A typical shape of the Dalitz plot for $\pi\pi$ and $\pi\eta$ production clearly exhibits the presence of $\pi\Delta$ and $\eta\Delta$ components in the corresponding final states, thus lending support to this method. At the same time, within such an isobar model, angular momentum decomposition of the amplitude is ruled by partial-wave transitions of a particular J^P state to quasi-two-body states, such as $\pi\Delta$ or $\eta\Delta$. Therefore, such an approach can hardly be used as a formal basis for a general partial-wave analysis since it crucially depends on the assumptions about the production mechanism.

It is worth to note that one of the main reasons for the lack of a rigorous partial-wave analysis for $\pi\pi$ and $\pi\eta$ photoproduction is that there is no general recipe to deal with reactions involving three particles in the final state. In contrast to single-meson photoproduction, one faces here the technical problems associated with three-body kinematics, where the particle energies and angles are distributed continuously. As a consequence, a conventional partial-wave decomposition of the final state does not provide a multipole representation for practical applications, primarily since there exist a variety of ways to successively couple angular momenta of the participating particles to a total angular momentum.

The issue of a model-independent determination of the amplitude for the production of two pseudoscalars has also been discussed in Ref. [23]. In this work, the various polarization observables are expanded in terms of helicity or transversity amplitudes without making a partial-wave decomposition. For these two representations, the authors obtained general expressions for those observables on which a complete experiment may be based.

In this paper, we present a partial-wave expansion for the photon-induced production of two pseudoscalars on a nucleon, which should be of minimal model dependence. It is based on the correct determination of the partial-wave amplitudes for these reactions with no built-in prejudices concerning the production mechanism. Similar methods have been used to analyze pion production in πN collisions (see, for example Refs. [24,25]). General formal developments may also be found in Ref. [26].

The paper is organized as follows. In the next section, we introduce the partial-wave expansion and construct the transition amplitude for photoproduction of two pseudoscalar mesons. In Sec. III, we use the so far developed formalism to discuss some gross features of $\pi^0\pi^0$ and $\pi^0\eta$ photoproduction. Finally, some general conclusions are drawn in Sec. IV.

II. THE FORMALISM

In this section, we collect the formulas used in the present analysis. As a starting point, the formal results of Ref. [27] are used. There, the formal expressions for the helicity amplitudes as well as for the cross section and the recoil polarization were derived, including various polarization asymmetries with respect to polarized photons and nucleons.

A. *T* matrix

We consider here the photoproduction of two pseudoscalar mesons, denoted m_1 and m_2 with masses M_1 and M_2, respectively. First, we determine the T-matrix elements of the electromagnetic m_1m_2 production current $\vec{J}_{\gamma m_1 m_2}$ between the initial nucleon and the final m_1m_2N state. The four-momenta of incoming photon, outgoing mesons, and initial and final nucleons are denoted by $(\omega_\gamma, \vec{k}\,)$, $(\omega_1, \vec{q}_1\,)$, $(\omega_2, \vec{q}_2\,)$, $(E_i, \vec{p}_i\,)$, and $(E, \vec{p}\,)$, respectively. The helicities of photon and initial and final nucleons are denoted by λ, μ, and ν, respectively. In a general frame, the transition matrix element is given by

$$T_{\nu\lambda\mu} = -^{(-)}\langle \vec{p},\, \vec{q},\, \nu\,|\,\vec{\varepsilon}_\lambda \cdot \vec{J}_{\gamma m_1 m_2}(0)|\,\vec{p}_i, \mu\rangle, \qquad (1)$$

where for the description of the final state we choose the final nucleon momentum $\vec{p} = (p, \theta_p, \phi_p)$ and the relative momentum of the two mesons $\vec{q} = \frac{1}{2}(\vec{q}_1 - \vec{q}_2\,) = (q, \theta_q, \phi_q)$. For the following formal considerations, the knowledge of the specific form of the current $\vec{J}_{\gamma m_1 m_2}$ is not needed.

After separation of the overall center-of-momentum (c.m.) motion, the general form of the T matrix is given by

$$T_{\nu\lambda\mu} = -^{(-)}\langle \vec{p},\, \vec{q},\, \nu\,|J_{\gamma m_1 m_2,\,\lambda}(\vec{k}\,)|\mu\rangle. \qquad (2)$$

It is convenient to introduce a partial-wave decomposition of the outgoing final state according to

$$\begin{aligned} &{}^{(-)}\langle \vec{p},\, \vec{q},\, \nu| \\ &= \frac{1}{4\pi} \sum_{l_p j_p m_p l_q m_q J M} \widehat{l}_p \widehat{l}_q \left(l_p 0 \frac{1}{2}\nu | j_p \nu\right) \\ &\quad \times (j_p m_p l_q m_q | J M)\, D^{j_p}_{\nu m_p}(\phi_p, -\theta_p, -\phi_p) \\ &\quad \times D^{l_q}_{0 m_q}(\phi_q, -\theta_q, -\phi_q)\, {}^{(-)}\!\left\langle q p;\, \left[\left(l_p \frac{1}{2}\right) j_p l_q\right] J M\right|, \end{aligned} \qquad (3)$$

where the "hat" symbol means, for example, $\widehat{l}_q = \sqrt{2l_q + 1}$. Furthermore, l_q and m_q denote total angular momentum and projection, respectively, of the two mesons, l_p, j_p, and m_p orbital and total nucleon angular momentum and its projection, respectively, and J and M the total angular momentum of the partial wave and its projection. All projections refer to a quantization axis to be determined later. For the rotation matrices $D^j_{m'm}$, we follow the convention of Rose [28].

The multipole decomposition of the current reads with $\vec{k} = (k, \theta_\gamma, \phi_\gamma)$ as

$$J_{\gamma m_1 m_2, \lambda}(\vec{k}\,) = -\sqrt{2\pi} \sum_{L M_L} i^L \widehat{L}\, \mathcal{O}^{\lambda L}_{M_L}(k)\, D^L_{M_L \lambda}(\phi_\gamma, \theta_\gamma, -\phi_\gamma), \qquad (4)$$

where $\mathcal{O}^{\lambda L}_{M_L}$ contains the transverse electric and magnetic multipoles

$$\mathcal{O}^{\lambda L}_{M_L} = E^L_{M_L} + \lambda M^L_{M_L}. \qquad (5)$$

For the initial nucleon state, we have

$$\left|\frac{1}{2}\mu\right\rangle = (-1)^{\frac{1}{2}+\mu} \sum_{m=\pm 1/2} \left|\frac{1}{2} m\right\rangle D^{1/2}_{m -\mu}(\phi_\gamma, \theta_\gamma, -\phi_\gamma)\,. \qquad (6)$$

Using the Wigner-Eckart theorem and the sum rule for rotation matrices

$$\begin{aligned} &\sum_{M_L m} \begin{pmatrix} J & L & \frac{1}{2} \\ -M & M_L & m \end{pmatrix} D^{1/2}_{m\,-\mu}(R) D^L_{M_L\,\lambda}(R) \\ &\quad = (-1)^{\lambda-\mu-M} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu-\lambda & \lambda & -\mu \end{pmatrix} D^J_{M\,\lambda-\mu}(R)\,, \end{aligned} \qquad (7)$$

one obtains

$$\begin{aligned} T_{\nu\lambda\mu} = &\frac{(-1)^{\nu+\lambda}}{2\sqrt{2\pi}} \sum_{L l_p j_p m_p l_q m_q J M} (-1)^{l_p + j_p + l_q + J - M}\, i^L\, \widehat{L}\, \widehat{J} \widehat{l}_q\, \widehat{l}_p\, \widehat{j}_p \begin{pmatrix} l_p & \frac{1}{2} & j_p \\ 0 & \nu & -\nu \end{pmatrix} \\ &\times \begin{pmatrix} j_p & l_q & J \\ m_p & m_q & -M \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu-\lambda & \lambda & -\mu \end{pmatrix} \left\langle p\, q;\left[\left(l_p \frac{1}{2}\right) j_p l_q\right] J ||\mathcal{O}^{\lambda L}|| \frac{1}{2}\right\rangle \\ &\times D^{j_p}_{\nu m_p}(\phi_p, -\theta_p, -\phi_p)\, D^{l_q}_{0 m_q}(\phi_q, -\theta_q, -\phi_q)\, D^J_{M\,\lambda-\mu}(\phi_\gamma, \theta_\gamma, -\phi_\gamma). \end{aligned} \qquad (8)$$

Parity conservation results in the following symmetry relation:

$$T_{-\nu-\lambda-\mu}(\Omega_q, \Omega_p, \Omega_\gamma) = (-1)^{\lambda-\mu-\nu} T_{\nu\lambda\mu}(\bar{\Omega}_q, \bar{\Omega}_p, \bar{\Omega}_\gamma), \qquad (9)$$

where for $\Omega = (\theta, \phi)$ we have introduced the notation $\bar{\Omega} = (\theta, -\phi)$.

Now we turn to the choice of our coordinate system in the overall center-of-momentum frame. We use the so-called "rigid body" system K_{fs}, associated with the final-state plane spanned by the final three particles, in which the z axis is taken to be the normal to this plane and parallel to $\vec{p} \times \vec{q}_1$. Thus, the x and y axes are in the final scattering plane (see Fig. 1).

At a given three-particle invariant energy W, the relative orientation of the final particles within the final-state plane is characterized by three independent variables for which we take the angle ϕ_p of the final nucleon momentum and the energies of the two mesons ω_1 and ω_2 (see Fig. 1). After straightforward

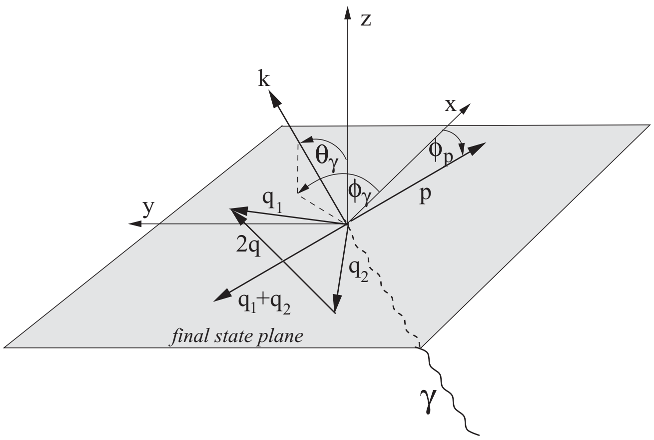

FIG. 1. Definition of the coordinate system in the c.m. system.

algebra, one obtains for the final nucleon momentum p

$$p = |\vec{p}\,| = \sqrt{(W - \omega_1 - \omega_2)^2 - M_N^2}, \tag{10}$$

and for the relative momentum q of the two mesons

$$q^2 = \frac{1}{2}\big(\omega_1^2 + \omega_2^2 - M_1^2 - M_2^2\big) - \frac{p^2}{4}. \tag{11}$$

The orientation of the chosen coordinate system with respect to the beam axis may be specified by $\Omega_\gamma = (\phi_\gamma, \theta_\gamma)$, the spherical angles of the photon momentum $\vec{k}$ with respect to K_{fs}. One readily notes that in this coordinate system, one has $\theta_p = \theta_q = \pi/2$ and therefore

$$D^{j_p}_{\nu m_p}(\phi_p, -\theta_p, -\phi_p) = (-1)^{\nu - m_p} d^{j_p}_{\nu m_p}(\pi/2)\, e^{-i(\nu - m_p)\phi_p}, \tag{12}$$

$$D^{l_q}_{0 m_q}(\phi_q, -\theta_q, -\phi_q) = (-1)^{m_q} d^{l_q}_{0 m_q}(\pi/2)\, e^{i m_q \phi_q}. \tag{13}$$

As will be shown soon, instead of ϕ_q, only $\phi_{qp} = \phi_q - \phi_p$ is needed. It is related to ω_1 and ω_2 by

$$\cos\phi_{qp} = \frac{1}{2qp}\big(\omega_2^2 - \omega_1^2 - M_2^2 + M_1^2\big), \tag{14}$$

with p and q from Eqs. (10) and (11), respectively. Thus, we will take as independent variables besides the photon angles $\Omega_\gamma = (\theta_\gamma, \phi_\gamma)$ and ϕ_p the energies of the two mesons ω_1 and ω_2 instead of p and ϕ_{qp} and obtain the following representation of the T-matrix element making the angular dependence explicit:

$$\begin{aligned} T_{\nu\lambda\mu}(\phi_p, \omega_1, \omega_2, \Omega_\gamma) &= e^{i(\lambda-\mu)\phi_\gamma}\, e^{-i\nu\phi_p} \\ &\quad \times \sum_{JM} t^{JM}_{\nu\lambda\mu}(\omega_1, \omega_2)\, e^{-iM\phi_{\gamma p}} d^J_{M\,\lambda-\mu}(\theta_\gamma), \end{aligned} \tag{15}$$

with the contribution of the final partial wave

$$\begin{aligned} t^{JM}_{\nu\lambda\mu}(\omega_1, \omega_2) &= t^{JM}_{\nu\lambda\mu}(\phi_{qp}) \\ &= \sum_{l_p j_p m_p L} \begin{pmatrix} l_p & \frac{1}{2} & j_p \\ 0 & \nu & -\nu \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu - \lambda & \lambda & -\mu \end{pmatrix} \\ &\quad \times d^{j_p}_{\nu m_p}(\pi/2)\, e^{i(M - m_p)\phi_{qp}}\, \mathcal{O}^{\lambda L J}_M(l_p j_p m_p), \end{aligned} \tag{16}$$

which shows the explicit dependence on ϕ_{qp}. Furthermore, we have introduced for convenience the notation

$$\begin{aligned} \mathcal{O}^{\lambda L J}_M(l_p j_p m_p) &= \frac{(-1)^{1+J}\widehat{J}}{2\sqrt{2\pi}} \sum_{l_q m_q} i^L (-1)^{l_p + j_p + l_q}\, \widehat{l_p}\, \widehat{j_p}\, \widehat{l_q}\, \widehat{L}\, d^{l_q}_{0 m_q}(\pi/2) \\ &\quad \times \begin{pmatrix} j_p & l_q & J \\ m_p & m_q & -M \end{pmatrix} \left\langle p\, q; \left[\left(l_p \frac{1}{2} \right) j_p l_q \right] J \| \mathcal{O}^{\lambda L} \| \frac{1}{2} \right\rangle. \end{aligned} \tag{17}$$

The following symmetry properties hold for the $\mathcal{O}^{\lambda L J}_M(l_p j_p m_p)$:

$$\mathcal{O}^{-\lambda L J}_M(l_p j_p m_p) = (-)^{L + l_p + M - m_p} \mathcal{O}^{\lambda L J}_M(l_p j_p m_p), \tag{18}$$

$$\mathcal{O}^{\lambda L J}_{-M}(l_p j_p - m_p) = (-)^{j_p + J} \mathcal{O}^{\lambda L J}_M(l_p j_p m_p), \tag{19}$$

where the first one is a consequence of parity conservation.

The symmetry relation of Eq. (9) leads to the following symmetry property of the amplitudes $t^{JM}_{\nu\lambda\mu}$:

$$t^{JM}_{-\nu-\lambda-\mu}(\phi_{qp}) = (-1)^{\nu+M} t^{J-M}_{\nu\lambda\mu}(-\phi_{qp}). \tag{20}$$

This means that for each J, the number of independent amplitudes is $4(2J+1)$.

The complex functions $t^{JM}_{\nu\lambda\mu}$, depending on the meson energies ω_1 and ω_2 only, provide a complete description of the process in a manner analogous to the description of a single-meson photoproduction in terms of multipoles. It is worth to point out that, in contrast to the binary reactions, the partial amplitudes are functions of the c.m. energies of the final particles and, therefore, are to be determined for every point of the Dalitz plot.

B. Differential cross section

For the unpolarized differential cross section, one obtains, with the T matrix of Eq. (15),

$$\begin{aligned} \frac{d^4\sigma_0}{d\omega_1 d\omega_2 d\cos\theta_\gamma d\phi_{\gamma p}} &= c(W)\frac{1}{4}\sum_{\nu\lambda\mu} |T_{\nu\lambda\mu}|^2 \\ &= \sum_{jm} S_{jm}(\omega_1, \omega_2) Y_{jm}(\theta_\gamma, \phi_{\gamma p}), \end{aligned} \tag{21}$$

where we have defined

$$\begin{aligned} S_{jm}(\omega_1,\omega_2) &= \frac{\sqrt{\pi}}{2}\, c(W)\,\widehat{j} \sum_{J'M'JM} (-1)^{-M'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \\ &\times \sum_{\nu\lambda\mu} (-1)^{\lambda-\mu} \begin{pmatrix} J' & J & j \\ \lambda-\mu & \mu-\lambda & 0 \end{pmatrix} t^{J'M'}_{\nu\lambda\mu}(\omega_1,\omega_2)^* \\ &\times t^{JM}_{\nu\lambda\mu}(\omega_1,\omega_2) \end{aligned} \tag{22}$$

with

$$c(W) = \frac{M_N^2}{4(2\pi)^4(W^2 - M_N^2)} \tag{23}$$

as a kinematical factor. One should note that the differential cross section depends on the relative angle $\phi_{\gamma p}$ only besides on ω_1, ω_2, and θ_γ as is immediately evident in the absence of polarization effects.

In terms of the electromagnetic multipole contributions, one finds

$$\begin{aligned} S_{jm}(\omega_1,\omega_2) &= \frac{\sqrt{\pi}}{2}\, c(W)\,\widehat{j} \sum_{J_pM_p} \widehat{J}_p^2\, d^{J_p}_{0M_p}(\pi/2)\, e^{i(m+M_p)\phi_{qp}} \\ &\times \sum_{l'_pj'_pm'_pl_pj_pm_p} (-1)^{j'_p-j_p-m_p} \begin{pmatrix} l'_p & l_p & J_p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j'_p & j_p & J_p \\ m'_p & -m_p & -M_p \end{pmatrix} \begin{Bmatrix} l'_p & l_p & J_p \\ j_p & j'_p & \frac{1}{2} \end{Bmatrix} \\ &\times \sum_{J'M'JML'L} (-1)^{J'+J+M'+L'+L} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \begin{Bmatrix} J' & J & j \\ L & L' & \frac{1}{2} \end{Bmatrix} \\ &\times \sum_{\lambda} (-)^{\lambda} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda & 0 \end{pmatrix} \mathcal{O}^{\lambda L'J'}_{M'}(l'_pj'_pm'_p)^*\, \mathcal{O}^{\lambda LJ}_{M}(l_pj_pm_p). \end{aligned} \tag{24}$$

If, with respect to the fixed final-state plane, only the direction of the final nucleon is detected, one obtains a semi-inclusive differential cross section by integrating the expression in Eq. (21) over ω_1 and ω_2 (setting without loss of generality $\phi_p = 0$, which means that ϕ_γ is measured relative to the direction of the nucleon momentum):

$$d\sigma_2/d\Omega_\gamma = \int d\omega_1 d\omega_2 \frac{d^4\sigma_0}{d\omega_1 d\omega_2 d\Omega_\gamma} = \sum_{jm} \widetilde{S}_{jm} Y_{jm}(\Omega_\gamma) \tag{25}$$

as an expansion in terms of spherical harmonics in Ω_γ with

$$\begin{aligned} \widetilde{S}_{jm} &= \int d\omega_1 d\omega_2 S_{jm}(\omega_1,\omega_2) \\ &= \frac{\sqrt{\pi}}{2}\, c(W)\,\widehat{j} \sum_{J'M'JM} (-1)^{-M'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \sum_{\nu\lambda\mu} (-1)^{\lambda-\mu} \begin{pmatrix} J' & J & j \\ \lambda-\mu & \mu-\lambda & 0 \end{pmatrix} \int d\omega_1 d\omega_2 t^{J'M'}_{\nu\lambda\mu}(\omega_1,\omega_2)^*\, t^{JM}_{\nu\lambda\mu}(\omega_1,\omega_2), \end{aligned} \tag{26}$$

or in terms of the multipoles

$$\begin{aligned} \widetilde{S}_{jm} &= \frac{\sqrt{\pi}}{2}\, c(W)\,\widehat{j} \sum_{J_pM_p} \widehat{J}_p^2\, d^{J_p}_{0M_p}(\pi/2) \sum_{l'_pj'_pm'_pl_pj_pm_p} (-1)^{j'_p-j_p-m_p} \begin{pmatrix} l'_p & l_p & J_p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j'_p & j_p & J_p \\ m'_p & -m_p & -M_p \end{pmatrix} \begin{Bmatrix} l'_p & l_p & J_p \\ j_p & j'_p & \frac{1}{2} \end{Bmatrix} \\ &\times \sum_{J'M'JML'L} (-1)^{J'+J+M'+L'+L} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \begin{Bmatrix} J' & J & j \\ L & L' & \frac{1}{2} \end{Bmatrix} \\ &\times \sum_{\lambda} (-)^{\lambda} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda & 0 \end{pmatrix} \int d\omega_1 d\omega_2\, e^{i(m+M_p)\phi_{qp}}\, \mathcal{O}^{\lambda L'J'}_{M'}(l'_pj'_pm'_p)^*\, \mathcal{O}^{\lambda LJ}_{M}(l_pj_pm_p). \end{aligned} \tag{27}$$

Since $d^2\sigma_0/d\Omega_\gamma$ is a real quantity, one has the property

$$\widetilde{S}^*_{jm} = (-)^m \widetilde{S}_{j-m}. \tag{28}$$

Furthermore, the cross section should be invariant under the simultaneous inversion of $\vec{k}$ and $\vec{p}$, i.e., under the transformation $\theta_{\gamma p} \to \pi - \theta_{\gamma p}$. Thus, one finds as additional symmetry property

$$\widetilde{S}_{jm} = (-)^{j+m}\widetilde{S}_{jm}, \tag{29}$$

from which the selection rule $\widetilde{S}_{jm} = 0$ for $j + m =$ odd follows. This property can also be shown straightforwardly using Eq. (26) with the help of Eq. (19). For identical mesons, one finds from Eq. (20) an additional symmetry, namely,

$$\widetilde{S}_{j-m} = (-)^{m}\widetilde{S}_{jm}, \tag{30}$$

which leads in conjunction with Eq. (28) to $\mathrm{Im}\,\widetilde{S}_{jm} = 0$.

It is more convenient to use instead of the differential cross section the corresponding normalized quantity

$$W(\Omega_\gamma) \equiv \frac{1}{\sigma_0}\frac{d^2\sigma_0}{d\Omega_\gamma} = \frac{1}{4\pi} + \sum_{jm,\,j\geqslant 1,\,j+m=\mathrm{even}} \frac{\widehat{j}}{\sqrt{4\pi}}\, W_{jm} Y_{jm}(\Omega_\gamma), \tag{31}$$

where the total cross section σ_0 is given by

$$\sigma_0 = 2\sqrt{\pi}\,\widetilde{S}_{00} = \pi\, c(W) \int d\omega_1 d\omega_2 \sum_{\nu\lambda\mu JM} \frac{1}{2J+1}\left|t^{JM}_{\nu\lambda\mu}(\omega_1,\omega_2)\right|^2, \tag{32}$$

and the expansion coefficients by

$$W_{jm} = \frac{2\sqrt{\pi}}{\sigma_0\,\widehat{j}}\widetilde{S}_{jm} = \frac{\pi}{\sigma_0}c(W)\int d\omega_1 d\omega_2 \times \sum_{\nu\lambda\mu J'JM} (-1)^{\lambda+M+\mu} \begin{pmatrix} J' & J & j \\ M & -M & 0 \end{pmatrix} \times \begin{pmatrix} J' & J & j \\ \lambda-\mu & \mu-\lambda & 0 \end{pmatrix} t^{J'M}_{\nu\lambda\mu}(\omega_1,\omega_2)^*\, t^{JM}_{\nu\lambda\mu}(\omega_1,\omega_2). \tag{33}$$

Using the spherical harmonics expansion (31) should enable one to interpret the experimental results without resorting to a particular model. This expression is an analog to the expansion of the single-meson photoproduction cross section in terms of Legendre polynomials. The coefficients W_{jm} are hermitean forms in the partial amplitudes $t^{JM}_{\nu\lambda\mu}$. They obviously contain the whole information on the dynamics of the reaction with unpolarized particles and their values may in principle be extracted from the measurements and compared with model predictions. The selection rule $W_{jm} = 0$ for $j + m =$ odd may be used for a model-independent partial-wave analysis in the low-energy region of the reaction, where usually only the first few waves contribute.

Otherwise, an integration over the angles θ_γ and ϕ_γ gives the distribution of the events over the Dalitz plot

$$\frac{d^2\sigma}{d\omega_1 d\omega_2} = \pi\, c(W) \sum_{\nu\lambda\mu JM} \frac{1}{2J+1}\left|t^{JM}_{\nu\lambda\mu}(\omega_1,\omega_2)\right|^2. \tag{34}$$

Thus, as is well known, the partial waves of different J do not interfere in the Dalitz plot. In spite of its simplicity, the expression in Eq. (34) can hardly be very useful in reconstructing even the moduli of the amplitudes $t^{JM}_{\nu\lambda\mu}$. Its use implies that one is able to establish a correspondence between variation of the amplitude as function of (ω_1, ω_2) and a specific value of the total angular momentum J. Obviously, for this purpose, a detailed model is needed which relates J to particular decay channels. In this sense, using the moments W_{jm} should be more promising.

It is clear that the information on the unpolarized differential cross section only is insufficient for a model-independent determination of the amplitudes $t^{JM}_{\nu\lambda\mu}$. In the general case of photoproduction of two pseudoscalars, eight independent complex functions are required to fix the spin structure of the amplitudes. Since the overall phase is always arbitrary, one has to measure 15 independent observables at each kinematical point. This issue is discussed in detail in Ref. [23].

Apart from technical difficulties associated with measuring and handling such a large number of observables, the practical realization of the partial-wave analyses is complicated since the number of important amplitudes $t^{JM}_{\nu\mu}$, which should be included into the analysis, grows rapidly as the energy increases. Furthermore, as already discussed above, these amplitudes depend not only on the total c.m. energy but also on the energies of the final particles and therefore should in general be determined at each point (ω_1, ω_2).

In this respect. the complete experiment allowing a model-independent determination of $t^{JM}_{\nu\mu}$ seems very complicated. Therefore, at least in the foreseeable future, different versions of the isobar model will probably remain the major tool to study double-meson photoproduction. In the ideal case, to make a step beyond the limitations of the isobar model approach, one can try to embody unitarity and analyticity of the reaction matrix. The most natural and technically manageable way to treat these issues is to use the multichannel three-body scattering model (in the energy region where the production of three or more mesons may be neglected). In particular, such an approach will allow one to take exactly into account the disconnected parts of the overall amplitude. As already mentioned in the Introduction, the latter come from the interaction in the two-body subsystems in the final state. In the isobar model, they are approximated by the known two-body resonances, such as, e.g., $\Delta(1232)$ for πN pair and $\rho(770)$ or $f_0(600)$ for $\pi\pi$. Once completed, the three-body approach should significantly improve the quality of the phenomenological ansatz, which then may be used to study double-meson production in a more model-independent way. Some important steps toward this goal have already been done in Ref. [29].

Furthermore, in certain cases, e.g., when the reaction is dominated by a single partial wave, using the moments W_{jm} enables one at least to draw a qualitative conclusion with respect to the partial-wave structure. As an illustration, we consider in the next section the theoretically interesting case of $\pi^0\pi^0$ and $\pi^0\eta$ photoproduction on a proton.

III. APPLICATION TO $\gamma p \to \pi^0\pi^0 P$ AND $\gamma p \to \pi^0\eta P$

The measured total cross section for $\gamma p \to \pi^0\pi^0 p$ exhibits a rather steep rise in the energy region below the $D_{13}(1520)$ resonance (see, e.g., [30]). At the same time, the existing models with a dominant contribution from $D_{13}(1520)$ and a moderate role of the Roper resonance predict a cross section which increases rather slowly with increasing energy and is, therefore, far below the data. It is reasonable to assume that the almost linear energy dependence of the data indicates a contribution of a large fraction of s waves in the final state. The main mechanism providing the s-wave part in $\pi\pi$ photoproduction is the Δ Kroll-Ruderman term, appearing after minimal substitution of the electromagnetic interaction into the $\pi N\Delta$ vertex. This term, however, vanishes in the neutral channel. The situation is similar to that in single π^0 photoproduction at low energies. Here, the Kroll-Ruderman does not enter the amplitude, thus leading to a visible suppression of the cross section for $\gamma p \to \pi^0 p$ in comparison to the π^+ or π^- case.

A possible large contribution of the Roper resonance $P_{11}(1440)$ in the region $E_\gamma = 500$–600 MeV as assumed in Ref. [14] seems to be excluded by more recent analyses. Furthermore, this assumption should be in disagreement with the experimental results of Ref. [31] for the helicity-dependent total cross section $\Delta\sigma = \sigma_{3/2} - \sigma_{1/2}$. There, it was found that, in the energy region up to at least $E_\gamma = 800$ MeV, the 3/2 part dominates over the 1/2 part. This means that the P_{11} wave, which contributes only to $\sigma_{1/2}$, should be overwhelmed by the waves with higher spins.

Thus, the question concerning the partial-wave structure of the amplitude for $\gamma N \to \pi^0\pi^0 N$ is still open. In order to reveal in this case the mechanism responsible for an unusually large fraction of the s-wave part in the $\pi^0\pi^0$ amplitude, it is useful to analyze the moments W_{jm} throughout the energy range from threshold up to the $D_{13}(1520)$ peak. In order to keep the number of parameters limited, one can use only the lowest partial waves. Their choice is inspired by the previous isobar model analyses of Refs. [13,15,18], showing that only waves with $J \leqslant 5/2$ are important below $E_\gamma = 1$ GeV.

As an example, we show in Fig. 2 the variation of W_{jm} for $j \leqslant 3$ as predicted by the $\pi\pi$ model of Ref. [18]. The model [18] is based on a traditional phenomenological Lagrangian approach with Born and resonance amplitudes calculated on the tree level. The interaction within the πN and $\pi\pi$ pairs is effectively taken into account via Δ, ρ, and σ. The $\pi\pi N$ state is then produced through intermediate formation of $\pi\Delta$, ρN, and σN channels. The contributions from the resonances are parametrized in the usual way in terms of a Breit-Wigner ansatz with energy-dependent widths. For the parameters of the model, i.e., masses, partial widths, and electromagnetic couplings of resonances, the corresponding average values from the compilation of the Particle Data Group were used.

In the case of $\pi^0\pi^0$ production due to the identity of the two mesons, we have an additional symmetry relation

$$W(\theta_\gamma, \phi_\gamma) = W(\theta_\gamma, 2\pi - \phi_\gamma), \tag{35}$$

which is a consequence of the symmetry property in Eq. (30). The moments for $j = 3, 4$ are small as are those for higher values of j, which are not shown. In the region $E_\gamma = 650$–800 MeV, the moments W_{11} and W_{20} exhibit a crucial energy dependence due to the $D_{13}(1520)$ resonance, dominating the reaction $\gamma p \to \pi^0\pi^0 p$ at this energy. Large

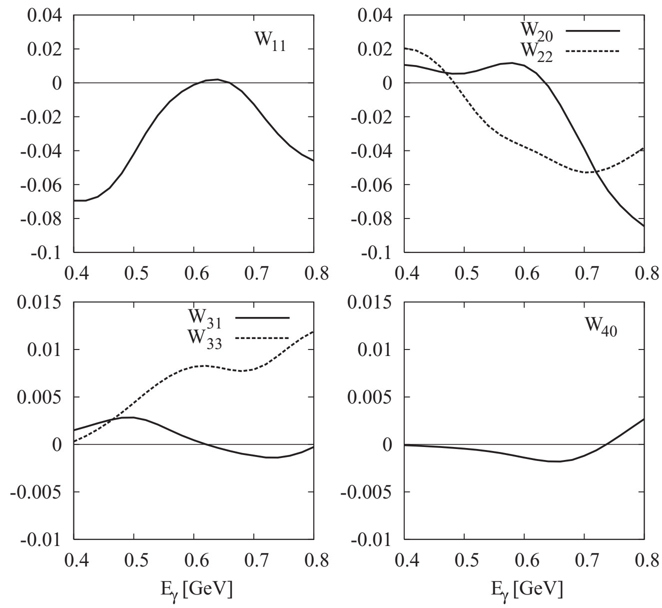

FIG. 2. The moments W_{jm} for $\gamma p \to \pi^0\pi^0 p$ as functions of the photon laboratory energy, normalized such that $W_{00} = 1$.

values of the moments with j odd indicate the presence of waves with opposite parities. In particular, the structure in W_{11} is due to an interference between the wave $J^P = 3/2^-$ dominated by $D_{13}(1520)$ and the waves $J^P = 1/2^+$ and $3/2^+$. The latter are saturated, apart from the Roper resonance, by the Born terms. The contribution of W_{11} becomes minimal in magnitude in the region around $E_\gamma = 650$ MeV, where the real part of the $D_{13}(1520)$ propagator vanishes, and it interferes weakly with the predominantly real Born amplitudes. Thus, if our notion about the $\pi^0\pi^0$ photoproduction mechanism is correct, we expect a rather small value of the moments W_{20} and W_{22} and a relatively large value of W_{11} in the region below the $D_{13}(1520)$ peak.

In this respect, we would like to note that according to the fit in Ref. [20], there must be a large contribution of the resonance $D_{33}(1700)$ to the channel $\pi^0\pi^0 p$ in a wide energy range from the lowest energies up to $E_\gamma = 1.4$ GeV. In particular, inclusion of this resonance into the amplitude explains both the steep rise of the total cross section below $E_\gamma = 700$ MeV and the second peak observed at $E_\gamma = 1.1$ GeV. If the resonance $D_{33}(1700)$ is indeed so important in the $\pi^0\pi^0$ channel, it should increase the values of W_{20} and W_{22}. All in all, a measurement of these moments will help us to understand the role of d-wave resonances with $J = 3/2$ in $\pi^0\pi^0$ photoproduction.

As for $\pi^0\eta$ photoproduction, the partial-wave structure of the corresponding amplitude was investigated in detail in Refs. [3,4,19,21]. There, it was shown that the $J^P = 3/2^-$ wave, containing $D_{33}(1700)$ and probably $D_{33}(1940)$, apparently dominates the reaction in a wide region from threshold to about $E_\gamma = 1.7$ GeV. Other waves, primarily $1/2^+$ and $5/2^+$, manifest themselves in angular distributions of the final particles mostly via interference with the dominant $3/2^-$ wave.

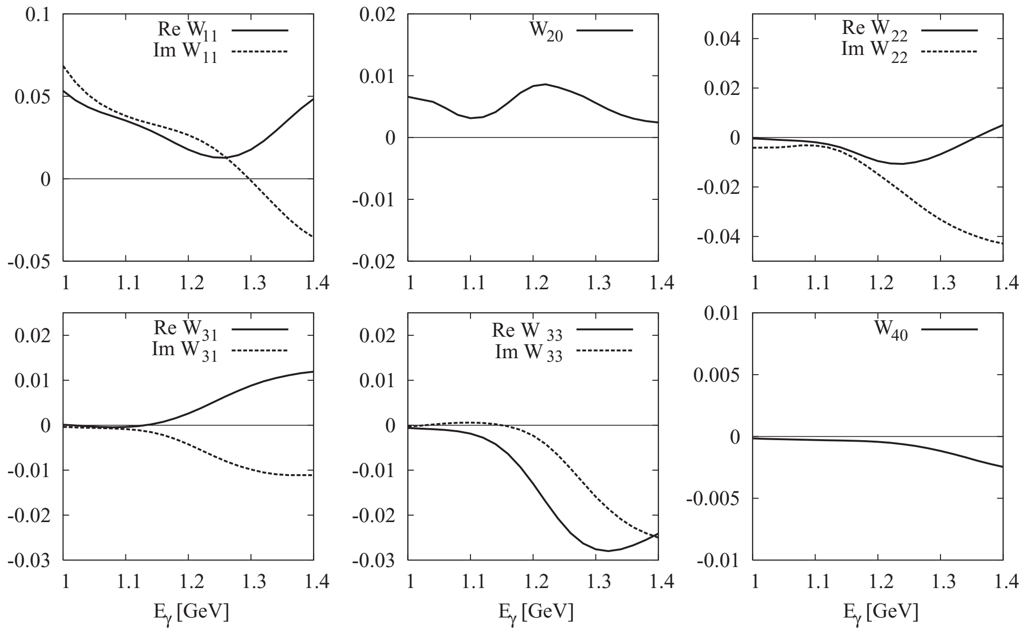

FIG. 3. Same as in Fig. 2 for $\gamma p \to \pi^0 \eta p$. The dashed lines represent the imaginary parts.

In Fig. 3, we present the energy dependence of the expansion coefficients for $\gamma p \to \pi^0 \eta p$ obtained using the isobar model of Ref. [21]. Here, the relation (30) does not hold, so that the moments W_{jm} with $m \neq 0$ have nonvanishing imaginary parts (dashed lines in Fig. 3). The calculation follows the same line as for the $\pi^0\pi^0$ case. Namely, the final $\pi^0 \eta N$ state results from the two-step decay of baryon resonances via the intermediate quasi-two-body channels $\eta\Delta$ and $\pi^0 S_{11}(1535)$. The parameters of the model were fitted to the angular distributions of the final particles measured in Ref. [9]. The fitting procedure is described in [21] and the reader is referred to this work for more details.

First, as one can see in Fig. 3, in spite of the mentioned dominance of the $3/2^-$ wave, the values of W_{20} and W_{22} are small. This is because of the closeness of the 3/2 and 1/2 helicity couplings of the resonance $D_{13}(1700)$ (see, e.g., the discussion in Ref. [21]). As a result, the hermitean forms in $t^{3/2M}_{\nu\lambda\mu}$ entering W_{20} and W_{22} according to (33) almost cancel each other. At the same time, we obtain a rather large value of the coefficient W_{11}, mainly determined by the interference between the resonances $D_{33}(1700)$ and $P_{31}(1750)$. According to these results, we may expect that the data for $\pi^0\eta$ will show relatively small values of all moments except for W_{11}. If this prediction is not confirmed by measurements, one has to critically review the existing conceptions about the dynamics of $\pi^0\eta$ photoproduction, based on the results from Refs. [3,10,12,21,22].

IV. CONCLUSION

Practical methods for the analysis of the partial-wave structure of reactions with three particles in the final state are obviously needed for the study of the dynamical features of two-meson photoproduction. The formalism used in this paper specifies the final $\pi\pi N$ states by means of two c.m. energies and two angles, determining the orientation of the final-state momentum triangle (final-state three-particle plane) with respect to the beam axis. The partial-wave decomposition may then be performed via a transition from the continuum variables (angles) to the set of discrete variables JM being the total angular momentum J and its projection M on the normal to the three-particle plane. The corresponding partial-wave amplitudes $t^{JM}_{\nu\lambda\mu}$ contain the whole information on the production dynamics. We would like to stress the fact that this method does not involve a decomposition with respect to the angular momenta of the final two-body subsystems and is in principle free from any assumptions about the production mechanism.

In this paper, we have considered only the unpolarized differential cross section. Although this quantity does not allow a unique determination of the amplitudes $t^{JM}_{\nu\lambda\mu}$, the information on the angular distribution of the participating particles can serve to place restrictions on contributions of states with definite angular momentum and parity. This in turn is crucial for our understanding of the resonance content of the reaction. For this purpose, the differential cross section has been expanded in terms of spherical harmonics with coefficients or moments W_{jm} in a manner similar to the representation of the binary cross section in terms of Legendre polynomials.

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