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Fluid dynamical equations and transport coefficients of relativistic gases with non-extensive statistics

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We derive equations for fluid dynamics from a non-extensive Boltzmann transport equation consistent with Tsallis' non-extensive entropy formula. We evaluate transport coefficients employing the relaxation time approximation and investigate non-extensive effects in leading order dissipative phenomena at relativistic energies, like heat conductivity, shear, and bulk viscosity.

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I. INTRODUCTION

The transport properties of matter at extreme high temperature and energy density, well proved in high-energy accelerator experiments at the Relativistic Heavy Ion Collider (RHIC) and at the Large Hadron Collider (LHC), raised some fundamental questions related to the smallest distance scale in physics. Theories, motivated by conformal field theory and higher dimensional dual gravity, in fact predicted a lower limit for the shear viscosity over entropy density ratio being around $1/4\pi$ in natural units [1]. Although this limitation can be overcome in more sophisticated, nonlinear dual gravity models [2] or other approaches [3–5], the actual calculation [6–11] and measurement of the viscosity and related properties of elementary matter remained in the focus of research interest.

Moreover, the scientific evaluation of experimental signals, extracted from analyses of particle spectra and correlations, can be interpreted in terms of thermal concepts only if our knowledge about the material quality of a (strongly) interacting quark-gluon plasma is well established [12]. Besides the interpretation of temperature [13,14], the whole hydrodynamical and statistical approach relies on our basic assumptions about the equation of state (EoS) and the dissipative capacities of this new stage of strongly interacting matter [15,16].

Modern statistical physics methods for the theoretical description of high-energy phenomena include a consideration of non-extensive thermodynamics [17–25]. Besides giving a more general framework than Boltzmann did for the entropy formula, also the mesoscopic background including Langevin, Fokker-Planck, or Boltzmann type equations can be extended to involve deviations from the classical 19th century picture [26–30]. Most of these generalizations depend on a single parameter, denoted by q, which also influences the canonical equilibrium state leading from an exponential Gibbs distribution in energy to a power-law tailed one. These are observable in experimental transverse momentum spectra and multiplicity fluctuations [31–34]; therefore the important question arises, how are the kinetic and hydrodynamical approaches affected by the value of this parameter. Are there qualitative or only quantitative minor deviations to be expected when $q \neq 1$? Fits to the RHIC and the LHC spectra make a hadronic matter value of $q_H \approx 1.08..1.2$ [35–37] and a quark matter value of $q_Q \approx 1.22$ [38] probable.

These fits point out a deviation from the standard Boltzmann-Gibbs (BG) statistics for $q = 1$. The degree of non-equilibrium in a system is defined relative to a fixed local q equilibrium. Assuming $q \neq 1$ means that the local equilibrium differs from the standard BG equilibrium. On the other hand it may be regarded as non-equilibrium compared to the classical $q = 1$ if one considers a time evolution of this parameter. However, we do not consider this possibility in the present paper.

We investigate a general non-equilibrium system which—for any given q—dissipates energy and produces entropy. This we call q non-equilibrium. Similarly to classical BG systems this q non-equilibrium is related to the response of the system to gradients of different thermodynamical intensives. This q non-equilibrium state does not relax to a standard BG equilibrium, but to a local q equilibrium.

In this paper we derive the relativistic fluid dynamical equations of motion from a non-extensive relativistic Boltzmann transport equation (NEBE) and calculate the transport coefficients employing the relaxation time approximation for the collision integral. Such calculations were done in the nonrelativistic limit [39] with a non-extensive Boltzmann equation [40,41] slightly different from ours. The q-generalized first-order relativistic Navier-Stokes-Fourier equations of relativistic dissipative fluid dynamics are also derived within this framework. We use the metric of flat space-time $g^{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$ and natural units throughout this work, $\hbar = k_B = c = 1$.

II. NON-EXTENSIVE BOLTZMANN EQUATION AND RELATIVISTIC HYDRODYNAMICS

We follow Lavagno [42] and recall the q-modified non-extensive Boltzmann equation (NEBE) compatible with Tsallis' [17–19] suggestion for a generalized non-extensive entropy formula. Extensions of the classical Boltzmann equation, like the one we adopt here from Ref. [42], are put forward without any derivation from microscopic quantum field dynamics, with the purpose to study classical statistical effects due to correlations possibly induced by finite system size compared to the characteristic interaction range [18].

For the sake of simplicity we neglect external field effects which simplifies the Vlasov operator in the following general

Boltzmann-type equation:

$$k^\mu \partial_\mu \tilde{f}_k = C[f], \tag{1}$$

where $f_k \equiv f(x^\mu, k^\mu)$ is the single-particle phase-space distribution function and $\tilde{f}_k = (f_k)^q$, with $q \in (0, 2)$. $C[f]$ is the collision integral. The space-time coordinates are $x^\mu = (t, \mathbf{x})$ and the particle four-momentum $k^\mu = (k^0, \mathbf{k})$ is normalized to the mass $m = \sqrt{k^\mu k_\mu}$. The NEBE is a closed equation assuming that the number of binary collisions around space-time coordinates x^μ is proportional to an expression $H_q[f_k, f_{k'}]$, where the explicit form of the collision integral involving binary collision with incoming momenta p^μ and p'^μ and outgoing momenta k^μ and k'^μ is given as

$$C[f] = \frac{1}{2}\int dK' dP dP' W_{kk'\to pp'} \times (H_q[f_p, f_{p'}] - H_q[f_k, f_{k'}]). \tag{2}$$

Here the factor 1/2 takes into account the symmetry of the initial or final momentum configuration in the case of identical particles if we interchange the labels, i.e., $(k^\mu, k'^\mu) \leftrightarrow (k'^\mu, k^\mu)$ or $(p^\mu, p'^\mu) \leftrightarrow (p'^\mu, p^\mu)$. Furthermore, $dK \equiv g d^3\mathbf{k}/[(2\pi\hbar)^3 k^0]$ is the Lorentz-invariant momentum-space volume, with g being the number of internal degrees of freedom, e.g., spin degeneracy, while $W_{kk'\to pp'}$ is the Lorentz-invariant transition rate. The transition rate is symmetric with respect to the sequence of final states $W_{kk'\to pp'} = W_{kk'\to p'p}$, and it is symmetric for time-reversed processes $W_{kk'\to pp'} = W_{pp'\to kk'}$. In this way it fulfils the detailed balance property. The collision integral is positive for any $f \geqslant 0$ and it does not change sign under time reversal. The q-generalized version of the so-called Stosszahlansatz, the assumption of molecular chaos,

$$H_q[f_k, f_{k'}] \equiv \exp_q[\ln_q(f_k) + \ln_q(f_{k'})], \tag{3}$$

uses the q-deformed exponential and logarithm functions,

$$\exp_q(x) \equiv [1 + (1-q)x]^{1/(1-q)}, \tag{4}$$

$$\ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q}. \tag{5}$$

For $q \to 1$ the above expressions are the standard exponential and natural logarithm functions. Here we stress that for $q = 1$ the detailed balance solution of the above equations obeys the classical Boltzmann-Gibbs statistics, not the usual quantum statistics.

The entropy four-current has to be properly defined as

$$S^\mu \equiv -\int dK k^\mu [\tilde{f}_k \ln_q(f_k) - f_k], \tag{6}$$

demanding positive entropy production according to the second law of thermodynamics:

$$\partial_\mu S^\mu \equiv -\int dK \ln_q(f_k)[k^\mu \partial_\mu \tilde{f}_k] \geqslant 0. \tag{7}$$

Note that Lavagno defined the entropy four-current as $S_L^\mu \equiv -\int dK\, k^\mu \tilde{f}_k[\ln_q(f_k) - 1]$, while the current form was introduced, correcting Lavagno's definition, by Osada and Wilk [43–46]. We shall return to this point later.

To elucidate the relation to a fluid dynamical description, let us define now the following momentum-space integrals of the NEBE,

$$\partial_\mu N^\mu \equiv \int dK\, k^\mu \partial_\mu \tilde{f}_k = \int dK C[f], \tag{8}$$

$$\partial_\mu T^{\mu\nu} \equiv \int dK\, k^\nu k^\mu \partial_\mu \tilde{f}_k = \int dK k^\nu C[f], \tag{9}$$

where we introduced the q-modified particle four-current and symmetric energy-momentum tensor,

$$N^\mu \equiv \int dK\, k^\mu \tilde{f}_k, \tag{10}$$

$$T^{\mu\nu} \equiv \int dK\, k^\mu k^\nu \tilde{f}_k. \tag{11}$$

It is straightforward to show that the left-hand side of Eqs. (8) and (9) vanishes when the particle number, the energy, and the momentum in individual collisions are conserved, that is, if $p^\mu + p'^\mu = k^\mu + k'^\mu$ holds. This property is expressed by specifying the so-called collision invariants by using $\psi = \alpha + \beta_\mu k^\mu$, such that

$$\int dK \alpha C[f] = 0, \tag{12}$$

$$\int dK \beta_\mu k^\mu C[f] = 0. \tag{13}$$

It implies the macroscopic conservation laws of the particle four-current and energy-momentum tensor for any solution of the NEBE,

$$\partial_\mu N^\mu = 0, \tag{14}$$

$$\partial_\mu T^{\mu\nu} = 0, \tag{15}$$

besides a non-negative entropy production from Eq. (7).

The entropy production vanishes in local q equilibrium and it leads to the collision invariant $\psi = \ln_q(f_{0k})$. The distribution function which satisfies $\partial_\mu S^\mu(f_{0k}) = 0$ is the canonical equilibrium one. This result is equivalent with the requirement that $\ln_q(f_{0k}) = \alpha_0 + \beta_0^\mu k_\mu$, whence the q-equilibrium distribution is given by

$$f_{0k} = \exp_q(\alpha_0 - \beta_0 k^\mu u_\mu). \tag{16}$$

Here $\alpha = \alpha_0$ and $\beta^\mu = \beta_0 u^\mu$ are the collisional invariants given previously, and u^μ is a four-vector normalized to 1, $u^\mu u_\mu = 1$. These quantities will be identified with the inverse temperature, $\beta_0 = 1/T$; the chemical potential over temperature, $\alpha_0 = \mu/T$; and the fluid dynamical four-velocity of matter u^μ. The formula in Eq. (16) reduces to the well-known Jüttner distribution or relativistic Maxwell-Boltzmann distribution for $q = 1$, that is, $f_J = \exp(\alpha_0 - \beta_0 k^\mu u_\mu)$. Therefore it is clear that in local q equilibrium there are only five fields that completely characterize the system for any given q.

For the purpose of a quick reference let us introduce $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, used to project an arbitrary four-vector into another four-vector orthogonal to u^μ. Any four-vector, A^μ, and in particular the four-momenta of particles can be decomposed into two parts using an arbitrary fluid dynamic flow velocity, $u^\mu : k^\mu = E_k u^\mu + k^{\langle\mu\rangle}$, where $E_k = k^\mu u_\mu$ is the local rest frame (LRF) energy of the particle and $k^{\langle\mu\rangle} = \Delta^{\mu\nu} k_\nu$

contains the LRF momenta [47]. The LRF or co-moving frame is defined with $u^{\mu}_{\mathrm{LRF}} = (1, 0, 0, 0)$. Moreover, the space-time gradient or four-divergence, $\partial_\mu \equiv \partial/\partial x^\mu = u_\mu d/d\tau + \nabla_\mu$, should also be decomposed into parts parallel and orthogonal to the flow. Here, the co-moving or proper time derivative, $d/d\tau \equiv u^\mu \partial_\mu$, is also denoted by an overdot, $\dot{A}^\mu \equiv dA^\mu/d\tau$, and $\nabla_\mu \equiv \Delta^\nu_\mu \partial_\nu$ denotes the gradient. For second-rank tensors the orthogonal and traceless projection is defined as $A^{\langle\mu\nu\rangle} = \Delta^{\mu\nu\alpha\beta} A_{\alpha\beta}$, where $\Delta^{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\beta\nu} + \Delta^{\nu\alpha}\Delta^{\beta\mu}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$.

Now, using the local q-equilibrium distribution function we are prepared to introduce the following generalized thermodynamic integrals:

$$\mathcal{I}_{q(i,j)} \equiv \frac{1}{(2j+1)!!}\int dK (E_k)^{i-2j}(\Delta^{\mu\nu}k_\mu k_\nu)^j f_{0k}, \tag{17}$$

$$\mathcal{J}_{q(i,j)} \equiv \frac{1}{(2j+1)!!}\int dK (E_k)^{i-2j}(\Delta^{\mu\nu}k_\mu k_\nu)^j (f_{0k})^q, \tag{18}$$

$$\mathcal{K}_{q(i,j)} \equiv \frac{q}{(2j+1)!!}\int dK (E_k)^{i-2j}(\Delta^{\mu\nu}k_\mu k_\nu)^j (f_{0k})^{2q-1}, \tag{19}$$

where $i, j \geqslant 0$ are natural numbers and $(2j+1)!! = (2j+1)!/(j!2^j)$ denotes the double factorial. At a given fixed temperature $(f_{0k})^q = (\partial f_{0k}/\partial \alpha_0)|_{\beta_0}$ and $q(f_{0k})^{2q-1} = [\partial (f_{0k})^q/\partial\alpha_0]|_{\beta_0}$, whence it follows that

$$\mathcal{J}_{q(i,j)} = \left(\frac{\partial \mathcal{I}_{q(i,j)}}{\partial \alpha_0}\right)_{\beta_0}, \quad \mathcal{K}_{q(i,j)} = \left(\frac{\partial \mathcal{J}_{q(i,j)}}{\partial \alpha_0}\right)_{\beta_0}. \tag{20}$$

The above integrals for $q < 1$ run over a finite range in energy, $E_k \leqslant T/(1-q)$, and are always convergent. For $q > 1$ the integrands have power-law tails satisfying a convergence criterion: $(i+2) < 1/(q-1)$ is required for the $\mathcal{I}_q(i,j)$ integrals to be finite. Correspondingly the maximal index i is 1 higher for the $\mathcal{J}_q$ and 2 higher for the $\mathcal{K}_q$ integrals. Finally, for $q = 1$ the familiar Jüttner distribution makes all integrals convergent by its exponential tail for any finite indices. In the case $\mathcal{I}_{q=1}(i,j) = \mathcal{J}_{q=1}(i,j) = \mathcal{K}_{q=1}(i,j) = I(i,j)$ we retrieve the familiar relativistic thermodynamic integrals (see Refs. [48–50]). Note that in the papers by Lavagno [42] and Lavagno *et al.* [51], the q-modified Bessel function of the second kind was introduced, $K_n(q,z)$, where $z = m\beta_0$, which in our notation corresponds to $\mathcal{J}_{q(i,j)}$ in the LRF.

It is also straightforward to show by partial integration that the following recursive relations hold in equilibrium:

$$\mathcal{J}_{q(i,j)} = -\frac{1}{\beta_0}\mathcal{I}_{q(i-1,j-1)} + \frac{i-2j}{\beta_0}\mathcal{I}_{q(i-1,j)}, \tag{21}$$

$$\mathcal{K}_{q(i,j)} = -\frac{1}{\beta_0}\mathcal{J}_{q(i-1,j-1)} + \frac{i-2j}{\beta_0}\mathcal{J}_{q(i-1,j)}. \tag{22}$$

Furthermore, one can also show that

$$\mathcal{I}_{q(i+2,j)} = m^2\mathcal{I}_{q(i,j)} - (2j+3)\mathcal{I}_{q(i+2,j+1)}, \tag{23}$$

$$\mathcal{J}_{q(i+2,j)} = m^2\mathcal{J}_{q(i,j)} - (2j+3)\mathcal{J}_{q(i+2,j+1)}, \tag{24}$$

$$\mathcal{K}_{q(i+2,j)} = m^2\mathcal{K}_{q(i,j)} - (2j+3)\mathcal{K}_{q(i+2,j+1)}, \tag{25}$$

as well as

$$\begin{aligned}\dot{\mathcal{I}}_{q(i,j)} &\equiv \frac{\partial \mathcal{I}_{q(i,j)}}{\partial \alpha_0}\dot{\alpha}_0 + \frac{\partial \mathcal{I}_{q(i,j)}}{\partial \beta_0}\dot{\beta}_0 \\ &= \mathcal{J}_{q(i,j)}\dot{\alpha}_0 - \mathcal{J}_{q(i+1,j)}\dot{\beta}_0,\end{aligned} \tag{26}$$

and similarly

$$\begin{aligned}\dot{\mathcal{J}}_{q(i,j)} &\equiv \frac{\partial \mathcal{J}_{q(i,j)}}{\partial \alpha_0}\dot{\alpha}_0 + \frac{\partial \mathcal{J}_{q(i,j)}}{\partial \beta_0}\dot{\beta}_0 \\ &= \mathcal{K}_{q(i,j)}\dot{\alpha}_0 - \mathcal{K}_{q(i+1,j)}\dot{\beta}_0.\end{aligned} \tag{27}$$

Let us recall the definition of the q-modified entropy four-current and calculate the equilibrium entropy density, $s_0 = S^\mu_0 u_\mu$, with $S^\mu_0 = S^\mu(f_{0k})$ for the q-equilibrium distribution function from Eq. (16). Making use of the previously introduced q-generalized thermodynamic integrals we obtain

$$\begin{aligned}s_0 &\equiv -\alpha_0\int dK E_k \tilde{f}_{0k} + \beta_0\int dK E_k^2 \tilde{f}_{0k} + \int dK E_k f_{0k} \\ &= -\alpha_0\mathcal{J}_{q(1,0)} + \beta_0\mathcal{J}_{q(2,0)} + \mathcal{I}_{q(1,0)}.\end{aligned} \tag{28}$$

Identifying the momentum integrals, or for short "the moments" as the particle density $n_0 = \mathcal{J}_{q(1,0)}$, the energy density $e_0 = \mathcal{J}_{q(2,0)}$, and the pressure $p_0 = -\mathcal{J}_{q(2,1)}$. Here we have used Eq. (21) to obtain $\mathcal{J}_{q(2,1)} = -\beta_0^{-1}\mathcal{I}_{q(1,0)}$, which translates into the familiar ideal gas EoS, $p_0 = \beta_0^{-1}\mathcal{I}_{q(1,0)}$. This is the reason why the definition of the entropy four-current has to be given in the specific form of Eq. (6). Hence the fundamental thermodynamic equation follows directly from Eq. (28):

$$s_0 = -\alpha_0 n_0 + \beta_0(e_0 + p_0). \tag{29}$$

Note that $\beta_0 p_0 \neq n_0$, because $n_0 = \mathcal{J}_{q(1,0)} \neq \mathcal{I}_{q(1,0)}$. However, if the number of particles is conserved, the integrals of $\mathcal{I}_q$ and $\mathcal{J}_q$ should lead to the same number of particles per unit volume, although the normalizations of the distribution functions differ.

The definition of entropy leads to the fundamental thermodynamic relation and the well-known Gibbs-Duhem relations:

$$ds_0 = -\alpha_0 dn_0 + \beta_0 de_0, \tag{30}$$

$$dp_0 = \frac{n_0}{\beta_0}d\alpha_0 - \frac{(e_0+p_0)}{\beta_0}d\beta_0. \tag{31}$$

The last equation follows from Eqs. (22) and (27). Note that the above thermostatic relations resemble those of classical BG thermodynamics. Kinetic theory does not specify the zeroth law of thermodynamics; therefore, whether the thermodynamic temperature is different from the temperature in kinetic theory is an open question [52]. However, choosing a single parameter for the temperature, the equations of q fluid dynamics become formally identical with those of classical fluid dynamics for $q = 1$.

Before going further we discuss the fluid dynamical equations in q equilibrium. The particle four-current and energy-momentum tensor are calculated from Eqs. (10) and (11) using the local q-equilibrium distribution function from Eq. (16); hence

$$N^\mu_0 = n_0 u^\mu, \tag{32}$$

$$T^{\mu\nu}_0 = e_0 u^\mu u^\nu - p_0\Delta^{\mu\nu}. \tag{33}$$

These decompositions are formally identical with those of a perfect fluid; hence some authors call it a perfect q fluid [45,46]. Moreover, the conservation laws from Eqs. (14) and (15) lead, at least formally, to the well-known Euler equations of perfect fluid dynamics,

$$\partial_\mu N_0^\mu = 0, \tag{34}$$

$$\partial_\mu T_0^{\mu\nu} = 0, \tag{35}$$

making a closed system of equations by supplementing with an EoS. These equations imply a vanishing local entropy production in q local equilibrium, $\partial_\mu S_0^\mu = 0$, for smooth initial conditions, i.e., without discontinuities, and EoS's without first-order phase transitions. Note that in general the EoS is not restricted to be an ideal gas EoS. In our case the EoS is given as $p_0 = \frac{1}{3}(e_0 - m^2\mathcal{J}_{q(0,0)})$, which in the massless limit returns the familiar $e_0 = 3p_0$, and hence the speed of sound is $c_s = \sqrt{1/3}$, irrespective of the q parameter [42].

On this account the general formula can be derived by rewriting the above equations with the help of the thermodynamic integrals and relations $d\alpha_0 = (de_0 + \mathcal{K}_{q(3,0)}d\beta_0)/\mathcal{K}_{q(2,0)}$ and $d\beta_0 = -(de_0 - \mathcal{K}_{q(2,0)}d\alpha_0)/\mathcal{K}_{q(3,0)}$. After some algebra we get that the speed of sound squared, $c_s^2 = (\frac{\partial p_0}{\partial e_0})$, at fixed entropy per particle, s_0/n_0, is given as

$$c_s^2 = \frac{1}{3} + \frac{m^2}{3}\frac{\mathcal{D}_{q(1,0)}}{\mathcal{D}_{q(2,0)}} - \frac{m^2}{3}\frac{(\mathcal{K}_{q(3,0)}\mathcal{K}_{q(0,0)} - \mathcal{K}_{q(2,0)}\mathcal{K}_{q(1,0)})}{h_0\mathcal{D}_{q(2,0)}}, \tag{36}$$

where $h_0 \equiv \mathcal{K}_{q(3,1)}/\mathcal{K}_{q(2,1)} = (e_0 + p_0)/n_0$ is the enthalpy per particle and $\mathcal{D}_{q(1,0)} = \mathcal{K}_{q(2,0)}\mathcal{K}_{q(0,0)} - \mathcal{K}_{q(1,0)}^2$ and $\mathcal{D}_{q(2,0)} = \mathcal{K}_{q(3,0)}\mathcal{K}_{q(1,0)} - \mathcal{K}_{q(2,0)}^2$.

III. BEYOND *q* EQUILIBRIUM

If a system is out of equilibrium the distribution function is different from the local q-equilibrium distribution, $\tilde{f}_k \neq \tilde{f}_{0k}$. This will lead to additional terms in the thermodynamic quantities and increase the entropy until the system relaxes to local q equilibrium according to the microscopic dynamics described by a NEBE. Once local q equilibrium is reached these additional dissipative quantities vanish.

Making use of the previously introduced notations, the macroscopic fields, such as the particle four-current and the energy-momentum tensor, can be decomposed in a general frame in the following way:

$$N^\mu \equiv u^\mu \int dK E_k \tilde{f}_k + \int dK k^{\langle\mu\rangle}\tilde{f}_k, \tag{37}$$

$$T^{\mu\nu} \equiv u^\mu u^\nu \int dK E_k^2 \tilde{f}_k + \frac{1}{3}\Delta^{\mu\nu}\int dK(\Delta^{\alpha\beta}k_\alpha k_\beta)\tilde{f}_k + 2u^{(\mu}\int dK E_k k^{\langle\nu\rangle)}\tilde{f}_k + \int dK k^{\langle\mu}k^{\nu\rangle}\tilde{f}_k, \tag{38}$$

where the round brackets around the greek indices denote symmetrization: $A^{(\mu\nu)} = (A^{\mu\nu} + A^{\nu\mu})/2$. Therefore, we can uniquely identify the fundamental fluid dynamical quantities such as the particle density, the energy density, and the isotropic pressure:

$$n \equiv u_\mu N^\mu = \int dK E_k \tilde{f}_k, \tag{39}$$

$$e \equiv u_\mu u_\nu T^{\mu\nu} = \int dK E_k^2 \tilde{f}_k, \tag{40}$$

$$p \equiv -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} = -\frac{1}{3}\int dK(\Delta^{\alpha\beta}k_\alpha k_\beta)\tilde{f}_k. \tag{41}$$

Similarly the particle diffusion current, the energy-momentum current, and the stress tensor are given by

$$V^\mu \equiv \Delta^\mu_\alpha N^\alpha = \int dK k^{\langle\mu\rangle}\tilde{f}_k, \tag{42}$$

$$W^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta} = \int dK E_k k^{\langle\mu\rangle}\tilde{f}_k, \tag{43}$$

$$\pi^{\mu\nu} \equiv T^{\langle\mu\nu\rangle} = \int dK k^{\langle\mu}k^{\nu\rangle}\tilde{f}_k. \tag{44}$$

These macroscopic fields, expressed above as momentum integrals over the non-equilibrium distribution function, $\tilde{f}_k$, are part of the particle four-current and energy-momentum tensor of a non-equilibrated fluid. Some of these quantities are related to their equilibrium thermodynamical pendants through the so-called matching conditions. It is standard practice to assume that the particle density and the energy density are unchanged from their equilibrium values,

$$n = n_0, \quad e = e_0, \tag{45}$$

while the isotropic pressure, p, separates into two parts: the thermodynamical pressure,

$$p_0 = -\frac{1}{3}\int dK(\Delta^{\alpha\beta}k_\alpha k_\beta)\tilde{f}_{0k}, \tag{46}$$

and the bulk viscous pressure,

$$\Pi = -\frac{1}{3}\int dK(\Delta^{\alpha\beta}k_\alpha k_\beta)\delta\tilde{f}_k, \tag{47}$$

such that $p = p_0 + \Pi$. Here we introduce the deviation from equilibrium, $\delta\tilde{f}_k = \tilde{f}_k - \tilde{f}_{0k}$. The matching conditions require that $\int dK E_k\delta\tilde{f}_k = \int dK E_k^2\delta\tilde{f}_k = 0$, while $V^\mu(\tilde{f}_k) \equiv V^\mu(\delta\tilde{f}_k)$, $W^\mu(\tilde{f}_k) \equiv W^\mu(\delta\tilde{f}_k)$, and $\pi^{\mu\nu}(\tilde{f}_k) \equiv \pi^{\mu\nu}(\delta\tilde{f}_k)$. Now, it is clear that only the non-equilibrium deviations lead to dissipation, since the moments of the equilibrium distribution function lead to vanishing dissipative quantities $\Pi(\tilde{f}_{0k}) = V^\mu(\tilde{f}_{0k}) = W^\mu(\tilde{f}_{0k}) = \pi^{\mu\nu}(\tilde{f}_{0k}) = 0$.

For completeness, here we calculate the entropy four-current from Eq. (6) up to first order in deviations from equilibrium $\delta\tilde{f}_k$ by expanding the entropy four-current around $\tilde{f}_{0k}$. This expression is given as

$$S^\mu(\tilde{f}_k) = S_0^\mu - \int dK k^\mu(\alpha_0 - \beta_0 k^\mu u_\mu)\delta\tilde{f}_k + O(\delta\tilde{f}_k^2) = S_0^\mu - \alpha_0 V^\mu + \beta_0 W^\mu + O(\delta\tilde{f}_k^2), \tag{48}$$

where $S_0^\mu = -\alpha_0 N_0^\mu + \beta_0 T_0^{\mu\nu}u_\nu + \beta_0 p_0 u^\mu$ and we make use of the definitions from Eqs. (42) and (43). Note that the higher-order deviations from q equilibrium can only be calculated once $\delta\tilde{f}_k$ is specified. The first-order result is independent of such details.

We recall that Eckart [53] was the first to discuss relativistic dissipative fluids in a proper way. The energy-momentum tensor decomposition proposed by him contained no bulk viscous pressure, $\Pi = 0$, and no particle diffusion current, $V^\mu = 0$; hence the fluid dynamical flow velocity was fixed to the conserved particles, $u^\mu = N^\mu/\sqrt{N^\mu N_\mu}$. Moreover the energy-momentum current in this case was identified with the heat flow, $q^\mu = W^\mu$. Later Landau and Lifshitz [54] introduced a different decomposition where the flow is tied to the flow of the energy-momentum $u^\mu = T^{\mu\nu}u_\nu/\sqrt{T^{\mu\alpha}u_\alpha T_{\mu\beta}u^\beta}$. This defines the flow as the only timelike eigenvector of the energy-momentum tensor. Therefore the energy-momentum diffusion current vanishes, $W^\mu = 0$, in this frame. These physically different choices are related by the general expression for the heat flow, $q^\mu \equiv W^\mu - h_0 V^\mu$.

Here we show that using the NEBE we obtain formally the same results as in the case of the classical Boltzmann equation, but they are generalized to include the non-extensivity of entropy.

IV. TRANSPORT COEFFICIENTS IN THE RELAXATION TIME APPROXIMATION

Because the equations of dissipative fluid dynamics are not closed due to the fact that $\tilde{f}_k$ is unknown, we need to find additional equations and relations that close the system of equations. Let us assume, following Chapman and Cowling [55], that the non-equilibrium contributions from the streaming term are negligible near the equilibrium $\delta\tilde{f}_k \ll \tilde{f}_{0k}$, that is, $k^\mu\partial_\mu\delta\tilde{f}_k \to 0$. In this way $k^\mu\partial_\mu\tilde{f}_k \simeq k^\mu\partial_\mu\tilde{f}_{0k}$ and therefore all non-equilibrium moments vanish from the left-hand side of the transport equation. Hence we obtain the gradients of thermodynamic quantities multiplied by transport coefficients. We calculate the streaming term in q equilibrium as

$$k^\mu\partial_\mu\tilde{f}_{0k} = q f_{0k}^{2q-1}\left(k^\mu\partial_\mu\alpha_0 - E_k k^\mu\partial_\mu\beta_0 - \beta_0 k^\nu k^\mu\partial_\mu u_\nu\right). \tag{49}$$

From here we follow the pioneering works of Anderson and Witting [56] and employ the relaxation time approximation. This simplifies the collision integral of the NEBE,

$$C[f_k] \simeq -E_k\frac{(\tilde{f}_k - \tilde{f}_{0k})}{\tau_C}, \tag{50}$$

where τ_C is the relaxation time. The relaxation time can be interpreted as a mean time between collisions. Furthermore, this approximation fixes the local rest frame so that the energy flow vanishes, as is shown below. Note that these approximations are made to simplify the collision term, which in the NEBE is even more complicated than in the classical case. Nevertheless, these model equations and solutions can be used to make very reasonable first estimates for the transport coefficients.

Replacing $\tilde{f}_k = \tilde{f}_{0k} + \delta\tilde{f}_k$ in the NEBE and using the previous assumptions we get the following transport equation,

$$k^\mu\partial_\mu\tilde{f}_{0k} = -\frac{E_k}{\tau_C}\delta\tilde{f}_k. \tag{51}$$

Utilizing Eqs. (49) and (50) we arrive at

$$\begin{aligned}\delta\tilde{f}_k = \tau_C q f_{0k}^{2q-1}\Bigg[&\left(\frac{\beta_0}{E_k}\frac{\theta}{3}(k^\alpha k^\beta\Delta_{\alpha\beta}) - \dot{\alpha}_0 + E_k\dot{\beta}_0\right)\\ &+ \left(h_0^{-1} - E_k^{-1}\right)k^\mu\nabla_\mu\alpha_0 + \frac{\beta_0}{E_k}k^{\langle\mu}k^{\nu\rangle}\sigma_{\mu\nu}\Bigg].\end{aligned} \tag{52}$$

Here we replaced $\partial_\mu u_\nu = u_\mu\dot{u}_\nu + \frac{1}{3}\theta\Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$, where $\theta = \nabla_\mu u^\mu$ is the expansion scalar, $\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$ is the shear stress tensor, and $\omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity. In the last term $\beta_0 k^{\langle\mu}k^{\nu\rangle}\omega_{\mu\nu} = 0$, due to the fact that the vorticity is antisymmetric while $k^{\langle\mu}k^{\nu\rangle}$ is symmetric. Note that we made use of the conservation laws of perfect fluids to calculate the proper time derivatives and hence we express them in terms of gradients. Thus, applying Eq. (27) for different i and j values we obtain

$$\dot{\alpha}_0 = n_0\mathcal{D}^{-1}_{q(2,0)}[h_0\mathcal{K}_{q(2,0)} - \mathcal{K}_{q(3,0)}]\theta, \tag{53}$$

$$\dot{\beta}_0 = n_0\mathcal{D}^{-1}_{q(2,0)}[h_0\mathcal{K}_{q(1,0)} - \mathcal{K}_{q(2,0)}]\theta, \tag{54}$$

$$\dot{u}^\mu = \beta_0^{-1}\left[h_0^{-1}\nabla^\mu\alpha_0 - \nabla^\mu\beta_0\right]. \tag{55}$$

Knowing the non-equilibrium distribution function, we can calculate now dissipative quantities. The stress tensor can be obtained in a straightforward manner from Eq. (44), together with the definitions of the integral from Eq. (19), as being

$$\begin{aligned}\pi^{\mu\nu} &= \tau_C\beta_0\sigma^{\alpha\beta}\left(q\int dK f_{0k}^{2q-1}E_k^{-1}k^{\langle\mu}k^{\nu\rangle}k_{\langle\alpha}k_{\beta\rangle}\right)\\ &= 2(\tau_C\beta_0\mathcal{K}_{q(3,2)})\sigma^{\mu\nu},\end{aligned} \tag{56}$$

whence the traditional Newton-Navier-Stokes relation between the stress tensor and shear tensor emerges:

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{57}$$

The shear viscosity coefficient is given by

$$\eta = \tau_C\beta_0\mathcal{K}_{q(3,2)}. \tag{58}$$

The relativistic generalization of the Newtonian shear $\sigma^{\mu\nu}$ and the Navier-Stokes relation between the stress and the shear have the same form for the q-modified statistics as the ones obtained using the classical BG statistics.

Similarly to the shear viscosity the heat-conductivity coefficient can be calculated. One can show that $W^\mu \sim (\mathcal{K}_{q(2,1)} - h_0\mathcal{K}_{q(3,1)}) = 0$ in the Anderson-Witting relaxation time approximation. Using Eq. (42) and $q^\mu = -h_0V^\mu$, the result is

$$\begin{aligned}q^\mu &= \tau_C\left(\nabla^\mu\alpha_0\right)\Bigg[h_0\left(\frac{q}{3}\int dK f_{0k}^{2q-1}E_k^{-1}(k^\mu k^\nu\Delta_{\mu\nu})\right)\\ &\quad - \left(\frac{q}{3}\int dK f_{0k}^{2q-1}(k^\mu k^\nu\Delta_{\mu\nu})\right)\Bigg]\\ &= \tau_C(\nabla^\mu\alpha_0)[h_0\mathcal{K}_{q(1,1)} - \mathcal{K}_{q(2,1)}].\end{aligned} \tag{59}$$

The Fourier-Navier-Stokes law becomes

$$q^\mu = -\kappa\left(h_0^{-1}\beta_0^{-2}\nabla^\mu\alpha_0\right), \tag{60}$$

where $\nabla^\mu\alpha_0 = -h_0T^{-2}\left(\nabla^\mu T - T\dot{u}^\mu\right)$ and the coefficient for heat conductivity or thermal conductivity is given by

$$\kappa = \tau_C h_0\beta_0^2(\mathcal{K}_{q(2,1)} - h_0\mathcal{K}_{q(1,1)}). \tag{61}$$

Finally the bulk viscous pressure, also known as the volume or second viscosity, can be calculated from Eq. (47) where we used Eqs. (53) and (54) to express the proper time derivatives in Eq. (52); hence

$$\begin{aligned}\Pi = & -\tau_C \frac{\theta}{3}\left[\beta_0 \frac{q}{3}\int dK f_{0k}^{2q-1} E_k^{-1}(\Delta^{\alpha\beta}k_\alpha k_\beta)^2\right.\\ & + n_0\left(\frac{\mathcal{K}_{30}-h_0\mathcal{K}_{20}}{\mathcal{D}_{20}}\right) q\int dK f_{0k}^{2q-1}(\Delta^{\alpha\beta}k_\alpha k_\beta)\\ & \left. - n_0 \left(\frac{\mathcal{K}_{20}-h_0\mathcal{K}_{10}}{\mathcal{D}_{20}}\right) q \int dK f_{0k}^{2q-1}E_k(\Delta^{\alpha\beta}k_\alpha k_\beta)\right]\\ = & -\tau_C\frac{\theta}{3}\left[5\beta_0\mathcal{K}_{q(3,2)} + 3n_0\left(\frac{\mathcal{K}_{30}-h_0\mathcal{K}_{20}}{\mathcal{D}_{20}}\right)\mathcal{K}_{q(2,1)}\right.\\ & \left. - 3n_0\left(\frac{\mathcal{K}_{20}-h_0\mathcal{K}_{10}}{\mathcal{D}_{20}}\right)\mathcal{K}_{q(3,1)}\right]. \end{aligned} \tag{62}$$

From this the relativistic version of the classical Stokes result follows immediately

$$\Pi = -\zeta\theta, \tag{63}$$

with the bulk viscosity coefficient

$$\begin{aligned}\zeta = & \tau_C \frac{n_0}{3\mathcal{D}_{20}}\left[5\frac{\beta_0}{n_0}\mathcal{D}_{20}\mathcal{K}_{q(3,2)} + 3(\mathcal{K}_{30}-h_0\mathcal{K}_{20})\mathcal{K}_{q(2,1)}\right.\\ & \left. - 3(\mathcal{K}_{20}-h_0\mathcal{K}_{10})\mathcal{K}_{q(3,1)}\right]. \end{aligned} \tag{64}$$

Once again we note that the above results formally resemble those of the BG statistics; the only explicit q dependence occurs in the definitions of the q-modified thermodynamic integrals. All formulas show that the transport coefficients are directly proportional to the mean free time between collisions and so inversely proportional to the cross section.

Having defined the dissipative quantities, the deviation from the q equilibrium Eq. (52) can be written as

$$\delta \tilde{f}_k = q f_{0k}^{2q-1}(C_\Pi \Pi + C_q k^\mu q_\mu + C_\pi k^\mu k^\nu \pi_{\mu\nu}), \tag{65}$$

where C_Π, C_q, and C_π are tedious expressions inversely proportional to the respective transport coefficients and may be calculated from the above formulas.

V. NON-EXTENSIVE EFFECTS IN THE TRANSPORT COEFFICIENTS

In this section we investigate the previously obtained results in the LRF where $E_k|_{\text{LRF}} = \sqrt{m^2+k^2}$. The thermodynamic integrals are calculated in hyperspherical coordinates: $k^\mu k_\mu = m^2$, $k^\mu u_\mu \equiv k^0 = m\cosh\chi$, $k^2 = m^2\sinh^2\chi$, $\Delta^{\mu\nu}k_\mu k_\nu = -m^2\sinh^2\chi$, and $dK = \frac{g}{(2\pi)^3}4\pi m^2 \sinh^2\chi\, d\chi$. Note that the integrals for $q \geqslant 1$ are calculated up to infinity while for $q < 1$ the upper limit of integration is $\chi_{\max} = \cosh^{-1}(\frac{T}{m(1-q)})$. Furthermore, the chemical potential is set to zero, $\mu = 0$, while the spin degeneracy is $g = 1$.

In Fig. 1 we plot the natural logarithm of the qth power of the q-equilibrium distribution function from Eq. (16) in the LRF, with $m = 140$ MeV and $T = 140$ MeV for longitudinal momenta $k_L \in [0, 10]$ GeV and transverse momenta $k_T = 0$. The functions are plotted for $q = 0.9$ (thick line), $q = 1$ (thin line), $q = 1.1$ (dashed line), and $q = 1.2$ (dotted line). Note that for $q = 0.9$ the function is practically cut off at $k_L = m\sinh\chi_{\max}$.

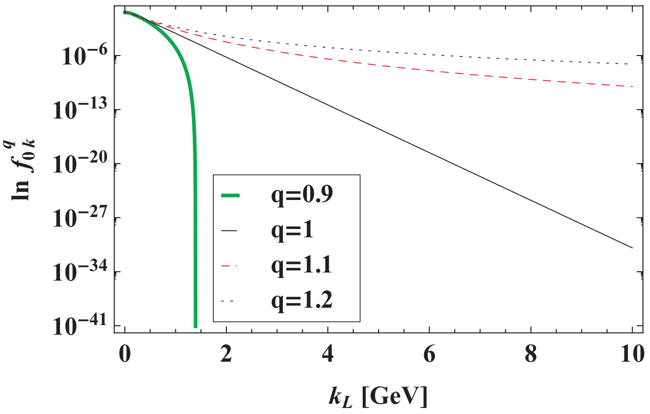

FIG. 1. (Color online) The $\tilde{f}_{0k}$ distribution as a function of longitudinal momenta for different q parameters: $q = 0.9$ (thick line), $q = 1$ (thin line), $q = 1.1$ (dashed line), and $q = 1.2$ (dotted line). The mass of particles and the temperature are $m = T = 140$ MeV.

Next we plot the speed of sound squared c_s^2 as a function of temperature and q parameter from Eq. (36) (Fig. 2). The mass of particles is fixed at $m = 140$ MeV, while the temperature is $T \in [2, 500]$ MeV. We see that for any value of q the speed of sound is increasing with temperature while it saturates for $T \gg m$ to the ultrarelativistic limit, $c_s^2 = 1/3$, independently of q. This is a straightforward result from Eq. (36). The speed of sound is larger for larger q at any given temperature.

The shear viscosity divided by the relaxation time from Eq. (58), $\eta\tau_C^{-1} = \beta_0\mathcal{K}_{q(3,2)}$, is shown in Fig. 3(a). On the

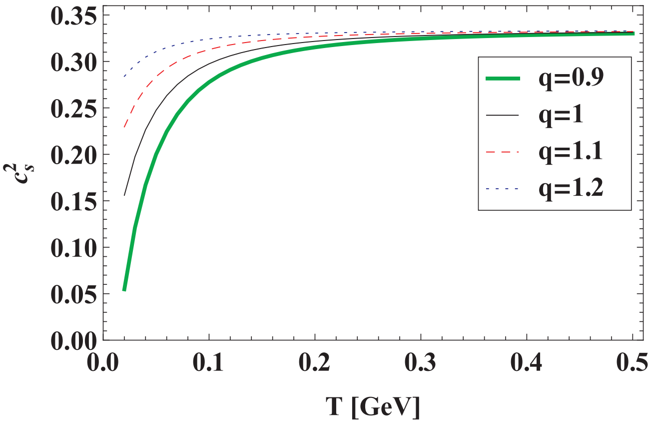

FIG. 2. (Color online) The c_s^2 as a function of temperature for different q parameters: $q = 0.9$ (thick line), $q = 1$ (thin line), $q = 1.1$ (dashed line), and $q = 1.2$ (dotted line).

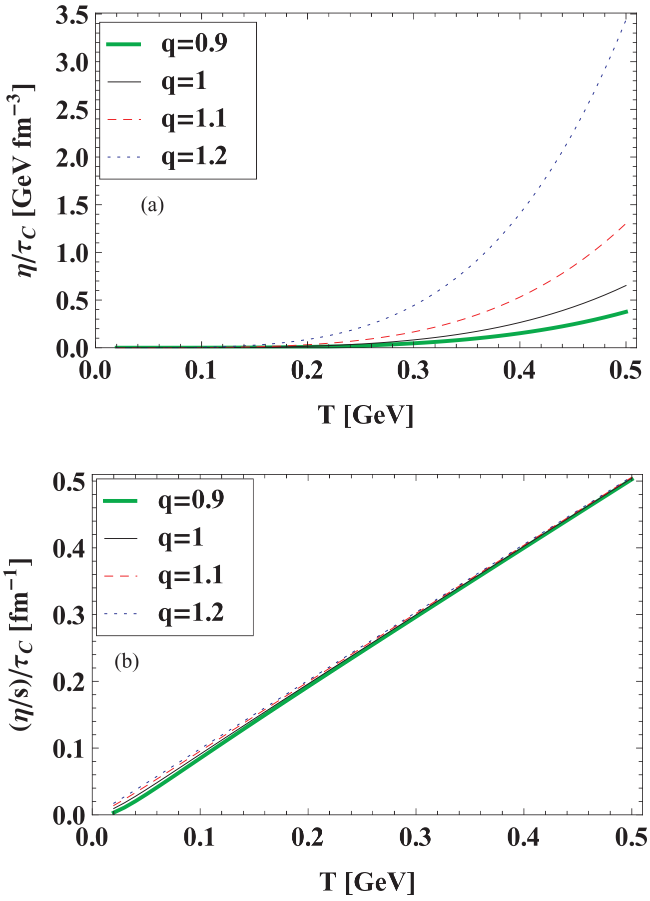

FIG. 3. (Color online) The shear viscosity coefficient divided by the relaxation time $\eta\tau_C^{-1}$ (a) and the shear viscosity coefficient over entropy density divided by the relaxation time $(\eta/s)\tau_C^{-1}$ (b) as a function of temperature for different q values.

other hand the shear viscosity to entropy ratio divided by the relaxation time $\frac{\eta}{s}\tau_C^{-1} = \frac{\mathcal{K}_{q(3,2)}}{\mathcal{J}_{q(2,0)}-\mathcal{J}_{q(2,1)}}$ can be inspected for different values of q in Fig. 3(b).

We observe that the viscosity increases with increasing temperature for all q values. This behavior is more pronounced for larger q values; hence there is a monotonic increase in the viscosity coefficient with increasing q at a given temperature. At the same time it turns out that the shear viscosity to entropy ratio at a given temperature is almost insensitive to different values of q. Figure 3(b) demonstrates that for a given relaxation time τ_C the η/s ratio does not change significantly and at high temperatures the q dependence diminishes. This also means that the ratio of dissipative to equilibrium quantities such as $\pi^{\mu\nu}/(e_0 + p_0) \simeq (\eta/s)/T$ does not change at high temperatures. This is due to the fact that the entropy density also increases with increasing q values by about the same factor. In the classical Maxwell-Boltzmann massless limit $(\eta\tau_C^{-1})_{q=1} = \beta_0 I_{(3,2)} = 4/(15e_0)$ and $s_{(q=1)} = \beta_0(I_{(2,0)} - I_{(2,1)}) = 4\beta_0 e_0/3$, we recover the familiar results [57].

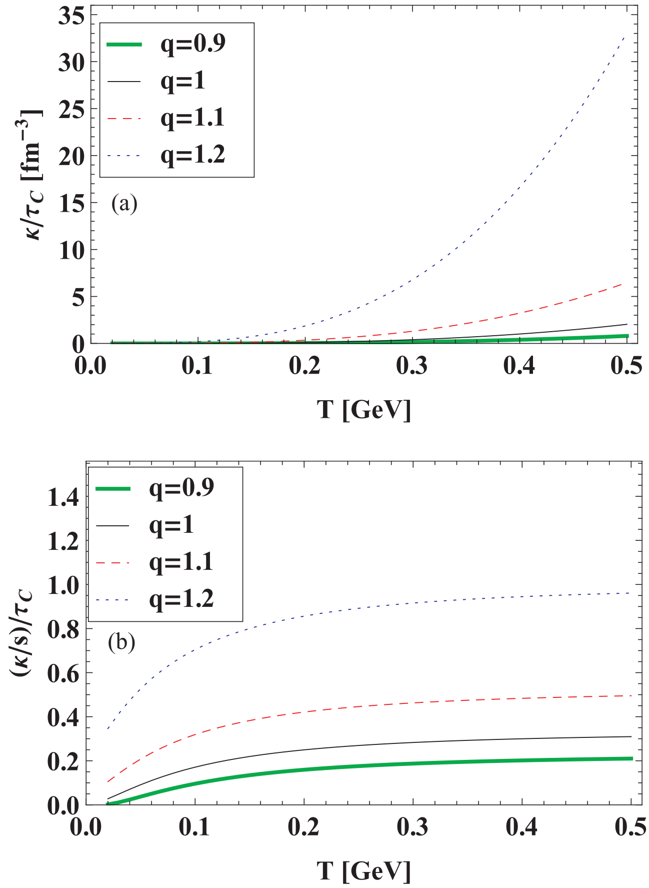

FIG. 4. (Color online) The heat-conductivity coefficient divided by the relaxation time (a) and the heat-conductivity coefficient to entropy density ratio divided by the relaxation time (b) as a function of temperature for different q values.

Similarly to the viscosity coefficient, we also plot the coefficient for the heat-conductivity from Eq. (61), as well as the ratio of heat conductivity to entropy density divided by the relaxation time. Both of these quantities, shown in Figs. 4(a) and 4(b), are sensitive to the value of the parameter q.

Furthermore, we observe that the heat conductivity increases with increasing q values, and the ratio $\frac{\kappa}{s}\tau_C^{-1}$ increases in a noticeable fashion.

Figures 5(a) and 5(b) show the bulk viscosity coefficient divided by the relaxation time and the bulk viscosity coefficient to entropy density ratio divided by the relaxation time.

We observe that the value of the bulk viscosity coefficient is very small. It is about 10^{-3} times smaller than the coefficient of shear viscosity $\zeta \simeq 10^{-3}\eta$, which corroborates the classical result that the bulk viscosity in the nonrelativistic or ultrarelativistic massless limit vanishes, while it is much smaller than the shear viscosity in between for massive particles [57,58]. On the other hand the bulk viscosity to entropy density ratio does show a monotonic increase with increasing q but only for $T \ll m$. For $T > m$ the behavior is reversed due to the fact

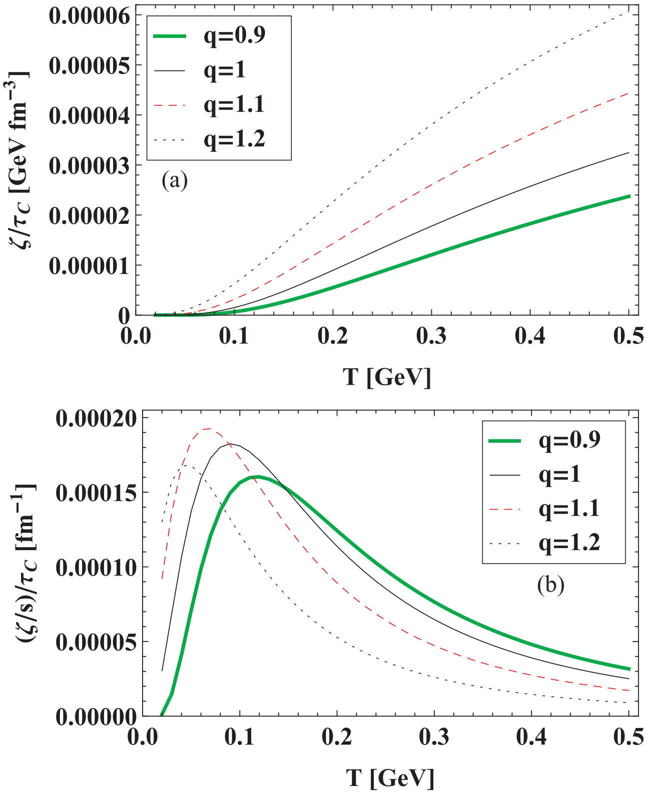

FIG. 5. (Color online) The bulk viscosity coefficient divided by the relaxation time (a) and the bulk viscosity coefficient to entropy density ratio divided by the relaxation time (b) as a function of temperature for different q values.

that the entropy density increases with temperature at a slower rate compared to the viscosity.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have calculated the transport coefficients from a q-modified Boltzmann equation. The results show that all transport coefficients increase monotonically with increasing q values for relativistic ideal gases. The speed of sound is also increasing with increasing q but it saturates quickly to the well-known ultrarelativistic limit independent of q.

Furthermore, we derived the q-generalized versions of the classical Navier-Stokes-Fourier equations of relativistic dissipative fluid dynamics from a q-generalized Boltzmann transport equation. These equations relate the dissipative quantities to the thermodynamical forces linearly with positive transport coefficients that dependent on the parameter q.

The resulting equations are nevertheless parabolic, hence acausal and unstable [59]. Hyperbolicity related problems are not solved by introducing the parameter q. This manqué can be resolved by an appropriate choice of the relaxation time in the evolving fluid [60]. Superior but more complicated equations of motion may be derived from the NEBE as in the classical theory, with methods different from the one presented in this work (see, for example, Refs. [61–65]).

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