Duffin-Kemmer-Petiau equation under a scalar Coulomb interaction

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Approximate analytical solutions of a Duffin-Kemmer-Petiau (DKP) equation are obtained via an elegant ansatz after successive transformations. Apart from the wide application of the DKP equation in both cosmology and theoretical nuclear physics as well as the physical significance of the Coulomb interaction, this is particularly important as we have provided a solution to the corresponding Heun equation.

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I. INTRODUCTION

The outstanding papers of Duffin, Kemmer, and Petiau (DKP) appeared more than 70 yr ago and introduced a basis that enabled theoretical physicists to investigate both spin-zero and spin-one particles on the basis of a single equation in the relativistic regime [1–4]. This equation, under a vector potential, possessed the same mathematical structure of its well-known partner, i.e., the Klein-Gordon (KG) equation, and consequently, the physical community thought the equations completely were equivalent for many decades. Now, however, we know that the equivalence generally violates [5–14], and the former is more appealing for the study of interactions in comparison with its more famous alternatives, i.e., KG and Proca equations [15–20].

As already mentioned, the DKP equation, which finds applications from subatomic to large-scale physics [21–25], due to its similarity with the KG equation, can be investigated by the well-known techniques of quantum mechanics [26–29] under a vector potential [30–40]. A problem just arises as we intend to explore the equation under a scalar potential. Later, we will see that this is because we face a Heun equation [29,40] which is not solvable even for the simplest cases we might think of.

Within the present paper, we briefly review the DKP equation under scalar and vector potentials. Next, by using some transformations, we obtain approximate analytical solutions of the equation under the Coulomb interaction. As a final point, we wish to address the interesting papers of Refs. [41–44], which first worked on the basis of DKP, to investigate related topics in the annals of particle physics. The interested reader might find instructive points about the physical consequences within them.

II. DKP EQUATION

The DKP Hamiltonian for scalar and vector interactions is

$$\left(\beta\cdot\vec{p}c+mc^2+U_s+\beta^0U_v^o\right)\psi(\vec{r})=\beta^0E\psi(\vec{r}), \qquad (1)$$

where

$$\psi(\vec{r})=\begin{pmatrix}\psi_{\text{upper}}\\ i\psi_{\text{lower}}\end{pmatrix}, \qquad (2)$$

the upper and lower components, respectively, are

$$\psi_{\text{upper}}\equiv\begin{pmatrix}\phi\\ \varphi\end{pmatrix},\qquad \psi_{\text{lower}}\equiv\begin{pmatrix}A_1\\ A_2\\ A_3\end{pmatrix}. \qquad (3)$$

β^0 is the usual 5×5 matrix and U_s, U_v^o, respectively, represent the scalar and vector interactions. The equation, in $(3+0)$ dimensions, is written as [1–4]

$$\begin{aligned}(mc^2+U_s)\phi&=\left(E-U_v^o\right)\varphi+\hbar c\vec{\nabla}\cdot\vec{A},\\ \vec{\nabla}\phi&=(mc^2+U_s)\vec{A},\\ (mc^2+U_s)\varphi&=\left(E-U_v^o\right)\phi,\end{aligned} \qquad (4)$$

where $\vec{A}=(A_1,A_2,A_3)$. In Eq. (3), ψ is a simultaneous eigenfunction of J^2 and J_3, i.e.,

$$\begin{aligned}J^2\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}&=\begin{bmatrix}L^2\psi_{\text{upper}}\\ (L+S)^2\psi_{\text{lower}}\end{bmatrix}=J(J+1)\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},\\ J_3\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}&=\begin{bmatrix}L_3\psi_{\text{upper}}\\ (L_3+s_3)\psi_{\text{lower}}\end{bmatrix}=M\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},\end{aligned} \qquad (5)$$

and the general solution is considered as

$$\psi_{JM}(r)=\begin{bmatrix}f_{nJ}(r)Y_{JM}(\Omega)\\ g_{nJ}(r)Y_{JM}(\Omega)\\ i\sum_L h_{nJL}(r)Y_{JL1}^M(\Omega)\end{bmatrix}, \qquad (6)$$

where spherical harmonics $Y_{JM}(\Omega)$ are of order J, $Y_{JL1}^M(\Omega)$ are the normalized vector spherical harmonics, and f_{nJ}, g_{nJ}, and h_{nJL}represent the radial wave functions. The above equations yield the coupled differential equations

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[30–38],

$$\left(E_{n,J}-U_v^0\right)F_{n,J}(r)=(mc^2+U_s)G_{n,J}(r),\quad \left[\frac{dF_{n,J}(r)}{dr}-\frac{J+1}{r}F_{n,J}(r)\right]=-\frac{1}{\alpha_J}(mc^2+U_s)H_{1,n,J}(r),$$

$$\left[\frac{dF_{n,J}(r)}{dr}+\frac{J}{r}F_{n,J}(r)\right]=\frac{1}{\zeta_J}(mc^2+U_s)H_{-1,n,J}(r),$$

$$-\alpha_J\left(\frac{dH_{1,n,J}(r)}{dr}+\frac{J+1}{r}H_{1,n,J}(r)\right)+\zeta\left[\frac{dH_{-1,n,J}(r)}{dr}-\frac{J}{r}H_{-1,n,J}(r)\right]$$

$$=\frac{1}{\hbar c}\left[(mc^2+U_s)F_{n,J}(r)-\left(E_{n,J}-U_v^0\right)G_{n,J}(r)\right], \tag{7}$$

which give [40]

$$\frac{d^2F_{n,J}(r)}{dr^2}\left[1+\frac{\zeta_j^2}{\alpha_j^2}\right]-\frac{dF_{n,J}(r)}{dr}\left[\frac{U_s'}{(m+U_s)}\left(1+\frac{\zeta_j^2}{\alpha_j^2}\right)\right]$$

$$+F_{n,J}(r)\left\{-\frac{J(J+1)}{r^2}\left(1+\frac{\zeta_j^2}{\alpha_j^2}\right)+\frac{U_s'}{(m+U_s)}\left(\frac{J+1}{r}-\frac{\zeta_j^2}{\alpha_j^2}\frac{J}{r}\right)-\frac{1}{\alpha_j^2}\left[(m+U_s)^2-\left(E_{n,J}-U_v^0\right)^2\right]\right\}=0, \tag{8}$$

where $\alpha_J=\sqrt{(J+1)/(2J+1)}$, $f_{nJ}(r)=F(r)/r$, $g_{nJ}=G(r)/r$, $h_{nJJ\pm1}=H_{\pm1}/r$, and $\zeta_J=\sqrt{J/(2J+1)}$. When $U_s=0$, we recover the well-known formula [30–38],

$$\left[\frac{d^2}{dr^2}-\frac{J(J+1)}{r^2}+\left(E_{n,J}-U_v^0\right)^2-m^2\right]F_{n,J}(r)=0. \tag{9}$$

We consider the following Coulomb vector and scalar potentials:

$$U_s=\frac{k}{r}, \tag{10a}$$

$$U_v^0=0, \tag{10b}$$

Substitution of Eq. (10) into Eq. (9) gives

$$\frac{d^2F_{n,J}(r)}{dr^2}+\frac{dF_{n,J}(r)}{dr}\left(\frac{\frac{k}{r^2}}{m+\frac{k}{r}}\right)+F_{n,J}(r)\left\{-\frac{J(J+1)}{r^2}-\frac{kC}{A}\left[\frac{1}{r^3\left(m+\frac{k}{r}\right)}\right]-\frac{1}{\alpha_J^2A}\left(m+\frac{k}{r}\right)^2+\frac{E_{n,J}^2}{\alpha_J^2A}\right\}=0, \tag{11}$$

where

$$A=1+\frac{\zeta_J^2}{\alpha_J^2}, \tag{12a}$$

$$C=J+1-\frac{\zeta_J^2}{\alpha_J^2}J. \tag{12b}$$

The change in variable,

$$F_{n,J}(r)=\sqrt{m+\frac{k}{r}}\phi_{n,J}(r) \tag{13}$$

brings Eq. (11) to the form

$$\frac{d^2\phi_{n,J}(r)}{dr^2}+\left[\frac{k}{r^3\left(m+\frac{k}{r}\right)}-\frac{3k^2}{4r^4\left(m+\frac{k}{r}\right)^2}-\frac{J(J+1)}{r^2}-\frac{kC}{A\left(m+\frac{k}{r}\right)}\frac{1}{r^3}-\frac{m^2}{\alpha_J^2A}+\frac{E_{n,J}^2}{\alpha_J^2A}-\frac{k^2}{\alpha_J^2A}\frac{1}{r^2}-\frac{2mk}{\alpha_J^2A}\frac{1}{r}\right]\phi_{n,J}=0, \tag{14}$$

After decomposition of fractions, we arrive at

$$\frac{d^2\phi_{n,J}(r)}{dr^2}+\left\{\left(\frac{m}{2k}+\frac{Cm}{Ak}-\frac{2mk}{\alpha_J^2A}\right)\frac{1}{r}+\left[\frac{1}{4}-J(J+1)-\frac{C}{A}-\frac{k^2}{\alpha_J^2A}\right]\frac{1}{r^2}\right.$$

$$\left.+\left(-\frac{m^2}{2k}-\frac{Cm^2}{Ak}\right)\frac{1}{k+mr}-\frac{3m^2}{4}\frac{1}{(k+mr)^2}+\left(\frac{E_{n,J}^2}{\alpha_J^2A}-\frac{m^2}{\alpha_J^2A}\right)\right\}\phi_{n,J}(r)=0. \tag{15}$$

Let us now consider an ansatz of the form

$$\phi_{n,J}(r) = f_n(r)\exp[g_J(r)], \tag{16}$$

where

$$f_n(r) = \begin{cases} 1 & n = 0 \\ \prod_{i=1}^{n}\left(r - \alpha_i^n\right) & n > 0, \end{cases} \tag{17a}$$

$$g_J(r) = \gamma\ln(mr + k) + \delta\ln(r) + \xi r \tag{17b}$$

yields

$$\phi''_{0,J}(r) = \left[\left(2\xi\delta + \frac{2\delta\gamma m}{k}\right)\frac{1}{r} + (\delta^2 - \delta)\frac{1}{r^2} + \left(2\xi\gamma m - \frac{2\delta\gamma m^2}{k}\right)\frac{1}{k + mr} + (\gamma^2 m^2 - \gamma m^2)\frac{1}{(k + mr)^2} + \xi^2\right]\phi_{n,J}(r), \tag{18}$$

Equating the coefficients on both sides, we find

$$2\xi\delta + \frac{2\delta\gamma m}{k} = -\left(\frac{m}{2k} + \frac{Cm}{Ak} - \frac{2mk}{\alpha_J^2 A}\right), \tag{19a}$$

$$\delta^2 - \delta = -\left[\frac{1}{4} - J(J+1) - \frac{C}{A} - \frac{k^2}{\alpha_J^2 A}\right], \tag{19b}$$

$$2\xi\gamma m - \frac{2\delta\gamma m^2}{k} = -\left(-\frac{m^2}{2k} - \frac{Cm^2}{Ak}\right), \tag{19c}$$

$$\gamma^2 m^2 - \gamma m^2 = \frac{3}{4}m^2, \tag{19d}$$

$$\xi^2 = -\left(\frac{E_{0,J}^2}{\alpha_J^2 A} - \frac{m^2}{\alpha_J^2 A}\right), \tag{19e}$$

where

$$\delta = \frac{1}{2}\left(1 \pm \left\{1 + 4\left[-\frac{1}{4} + J(J+1) + \frac{C}{A} + \frac{k^2}{\alpha_J^2 A}\right]\right\}^{\frac{1}{2}}\right), \tag{20a}$$

$$\gamma = \frac{1 \pm 2}{2}, \tag{20b}$$

$$\xi = \pm\sqrt{\left(\frac{m^2}{\alpha_J^2 A} - \frac{E_{n,J}^2}{\alpha_J^2 A}\right)}, \tag{20c}$$

$$-2\xi\delta m + 2\xi\gamma m - \frac{m^2}{k} + \frac{2m^2 k}{\alpha_J^2 A} = 0. \tag{20d}$$

Therefore, the spectrum and eigenfunctions of the system are obtained. One more point we wish to make is about the equivalence of KG and DKP equations. We already know that solutions of the KG equation, under a scalar Coulomb term, can be expressed in terms of the Laguerre polynomials. The DKP equation, under the potential, however, appears as a form of the general Heun equation (GHE), which definitely is more complicated and richer [45]. In fact, the GHE only reproduces the hypergeometric functions in very special cases. Nonetheless, our results do not necessarily imply that the DKP equation leads to different predictions from the KG equation when applied to a specific physical process.

III. CONCLUSION

After applying some appropriate transformations and introducing an elegant ansatz, we have obtained approximate analytical solutions of the DKP equation under the Coulomb interaction, which possesses the form of a Heun equation. The results are definitely useful in a wide range of physical problems from meson spectroscopy to cosmology. As another merit of the paper, we wish to state that the solutions of the Heun equation normally have not been reported to be as touchable as we show here.

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[1] N. Kemmer, Proc. R. Soc. London, Ser. A **166**, 127 (1938).
[2] R. J. Duffin, Phys. Rev. **54**, 1114 (1938).
[3] N. Kemmer, Proc. R. Soc. London, Ser. A **173**, 91 (1939).
[4] G. Petiau, University of Paris thesis (1936), Published in Acad. Roy. de Belg., Classe Sci., Mem in 8o 16, No. 2 (1936).
[5] T. R. Cardoso, L. B. Castro, and A. S. de Castro, Int. J. Theor. Phys. **49**, 10 (2010).
[6] T. R. Cardoso and A. S. de Castro, Rev. Bras. Ens. Fis. **29**, 203 (2007).
[7] T. R. Cardoso, L. B. Castro, and A. S. de Castro, Phys. Lett. A **372**, 5964 (2008).
[8] L. Chetouani, M. Merad, T. Boudjedaa, and A. Lecheheb, Int. J. Theor. Phys. **43**, 1147 (2004).
[9] T. R. Cardoso, L. B. Castro, and A. S. de Castro, J. Phys. A: Math. Theor. **43**, 055306 (2010).
[10] A. S. de Castro, J. Phys. A: Math. Theor. **44**, 035201 (2011).
[11] M. Nowakowski, Phys. Lett. A **244**, 329 (1998).
[12] J. T. Lunardi, B. M. Pimentel, R. G. Teixeira, and J. S. Valverde, Phys. Lett. A **268**, 165 (2000).
[13] M. Riedel, *Relativistische Gleichungen fuer Spin-1-Teilchen, Diplomarbeit, Institute for Theoretical Physics* (Johann Wolfgang Goethe-University, Frankfurt/Main, 1979).

[14] E. Fischbach, M. M. Nieto, and C. K. Scott, J. Math. Phys. **14**, 1760 (1973).
[15] B. C. Clark, S. Hama, G. R. Kälbermann, R. L. Mercer, and L. Ray, Phys. Rev. Lett. **55**, 592 (1985).
[16] G. Kalbermann, Phys. Rev. C **34**, 2240 (1986).
[17] R. E. Kozack, B. C. Clark, S. Hama, V. K. Mishra, G. Kälbermann, R. L. Mercer, and L. Ray, Phys. Rev. C **37**, 2898 (1988).
[18] R. E. Kozack, B. C. Clark, S. Hama, V. K. Mishra, R. L. Mercer, and L. Ray, Phys. Rev. C **40**, 2181 (1989).
[19] V. K. Mishra, S. Hama, B. C. Clark, R. E. Kozack, R. L. Mercer, and L. Ray, Phys. Rev. C **43**, 801 (1991).
[20] B. C. Clark *et al.*, Phys. Lett. B **427**, 231 (1998).
[21] V. Gribov, Eur. Phys. J. C **10**, 71 (1999).
[22] I. V. Kanatchikov, Rep. Math. Phys. **46**, 107 (2000).
[23] S. Ait-Tahar, J. S. Al-Khalili, and Y. Nedjadi, Nucl. Phys. A **589**, 307 (1995).
[24] J. T. Lunardi, L. A. Manzoni, B. M. Pimentel, and J. S. Valverde, Int. J. Mod. Phys. A **17**, 205 (2000).
[25] M. de Montigny, F. C. Khanna, A. E. Santana, E. S. Santos, and J. D. M. Vianna, J. Phys. A **33**, L273 (2000).
[26] S. Zarrinkamar, A. A. Rajabi, and H. Hassanabadi, Ann. Phys. (N.Y.) **325**, 2522 (2010).
[27] H. Hassanabadi, H. Rahimov, and S. Zarrinkamar, Ann. Phys. **523**, 566 (2011).
[28] A. D. Alhaidari, H. Bahlouli, and A. Al-Hasan, Phys. Lett. A **349**, 87 (2008).
[29] R. L. Hall, N. Saad, and K. D. Sen, J. Phys. A **51**, 022107 (2010).
[30] Y. Nedjadi and R. C. Barrett, J. Phys. G: Nucl. Part. Phys. **19**, 87 (1993).
[31] Y. Nedjadi, S. Ait-Tahar, and R. C. Barrett, J. Phys. A **31**, 3867 (1998).
[32] A. Boumali, J. Math. Phys. **49**, 022302 (2008).
[33] I. Boztosun, M. Karakoc, and A. Durmus, J. Math. Phys. **47**, 062301 (2006).
[34] B. Boutabia-Cheraitia and T. Boudjedaa, Phys. Lett. A **338**, 97 (2005).
[35] M. Merad, Int. J. Theor. Phys. **46**, 2105 (2007).
[36] Y. Chargui, A. Trabelsi, and L. Chetouani, Phys. Lett. A. **374**, 2907 (2010).
[37] K. Sogut and A. Havare, J. Phys. A: Math. Theor. **43**, 225204 (2010).
[38] F. Yasuk, C. Berkdemir, A. Berkdemir, and C. Önem, Phys. Scr. **71**, 340 (2005)
[39] A. Okninski, Int. J. Theor. Phys. **50**, 729 (2011).
[40] S. Zarrinkamar, A. A. Rajabi, H. Hassanabadi, and H. Rahimov, Mod. Phys. Lett. A **26**(22), 1621 (2011).
[41] E. Fischbach *et al.*, Phys. Rev. D **9**, 2183 (1974).
[42] E. Fischbach *et al.*, Prog. Theor. Phys. **51**, 1585 (1974).
[43] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **10**, 4049 (1974).
[44] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **11**, 1442 (1975).
[45] E. S. Cheb-Terrab, J. Phys. A **37**, 9923 (2004).