Chiral symmetry in the low-energy limit of QCD at finite temperature

Marco Frasca*
Via Erasmo Gattamelata 3, 00176 Roma, Italy
(Received 5 July 2011; revised manuscript received 5 November 2011; published 28 November 2011)

We derive a nonlocal Nambu-Jona-Lasinio model from QCD with a form factor exactly obtained in the infrared limit. Using this model, with all parameters properly fixed via QCD, we consider the case of finite temperature and compute the solution of the gap equation at low temperature, small momentum, and zero chemical potential. Taking the quark masses to be zero, we prove that the theory undergoes a phase transition with a critical temperature that is exactly determined. These results prove unequivocally that the picture of the vacuum of QCD as a liquid of instantons is a very good approximation.

DOI: 10.1103/PhysRevC.84.055208 PACS number(s): 12.39.Fe, 11.30.Rd, 12.38.Mh, 24.85.+p

I. INTRODUCTION

Currently, evidence for the existence of a phase transition in QCD, at finite temperature and chemical potential, relies on lattice computations. This was first realized in a pioneering work by Fodor and Katz [1], and further backed up by more recent studies [2–4], notwithstanding some criticisms that were cast due to the infamous sign problem [5]. Studies of the behavior of QCD at finite temperature and density, from a theoretical standpoint, are generally performed using phenomenological models such as the Nambu-Jona-Lasinio model or a sigma model. The appearance of the infamous sign problem in lattice analysis has prompted some authors to introduce an imaginary chemical potential [6–8], and the need for a consistent agreement between lattice computations and theoretical models has prompted the introduction of more general models using the Polyakov loop. In this framework some authors were able to prove the existence of a statistical confinement of quarks and to describe the phase diagram of QCD [9,10]. The relevant point to note for our purposes is that, when quark masses are taken into account, instead of a real phase transition there is a crossover between a confined and a deconfined phase. In the same range of temperatures, broken chiral symmetry is also seen. But when the chemical potential and the masses of the quarks are taken to be zero, a first-order phase transition is indeed expected at a given critical temperature.

It is clear from this situation that a proof based on first principles of at least a chiral symmetry breaking, starting from the equations of QCD, does not exist yet. Efforts in this direction date back to the 1980s, when chiral perturbation theory was used initially [11–13] but did not produce a value for the critical temperature. The difficulty lies in our inability to obtain a model for the low-energy behavior of QCD directly from theory. Quite recently, we were able to prove that a nonlocal Nambu-Jona-Lasinio model describes such a low-energy limit for QCD [14–16], and this result has been obtained also by Kei-Ichi Kondo [17]. The crucial point in our derivation has been an analytical closed form for the gluon propagator in the limit of very low energies [18,19].

The form of the gluon propagator is an essential cornerstone result that permits one to perform explicitly numerous low-energy computations directly from the equations of QCD. This is clearly shown in a recent paper by Hell *et al.* [20]. The authors were able to give a complete account of a nonlocal Nambu-Jona-Lasinio model, at both zero and finite temperatures, but they only guessed the form of the propagator by using the idea that the ground state of QCD is that of a liquid of instantons [21]. We will show below that our scenario is perfectly consistent with this view. We show that the theory, at zero chemical potential and zero quark mass, indeed undergoes a symmetry breaking at low temperature, and we obtain the critical temperature computed on the lattice. We note, however, that the value of the critical temperature obtained from lattice computation is uncertain, with two groups obtaining different values. But for our purposes it is enough to be in the right range.

This article is structured as follows. In Sec. II we present results in the infrared limit of QCD. In Sec. III we show a derivation of the nonlocal Nambu-Jona-Lasinio model from QCD. In Sec. IV we solve the gap equation at low temperature and low momentum, giving the main result. In Sec. V we give our conclusions.

II. QCD IN THE INFRARED LIMIT

As usual, our starting point will be the generating functional of QCD. We take

$$S_{\rm QCD} = -\frac{1}{4}\int d^4x\,{\rm Tr}F^2 + \int d^4x \sum_q \bar{q}(x)\left(i\partial\!\!\!/ - g\frac{\lambda^a}{2}A\!\!\!/^a\right)\times q(x) - \int d^4x(\bar{c}^a\partial_\mu\partial^\mu c^a + g\bar{c}^a f^{abc}\partial_\mu A^{b\mu}c^c), \tag{1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, g is the coupling, which in this case is dimensionless, $q(x)$ are the quark fields, $A^a_\mu(x)$ are the vector potentials of the Yang-Mills field, and c^a is the ghost field. So, it straightforward to write

*marcofrasca@mclink.it

down

$$Z_{\rm QCD}[j,\bar\eta,\eta,\bar\epsilon,\epsilon]=\mathcal{N}\int[dA][d\bar q][dq][d\bar c][dc]\\ \times e^{iS_{\rm QCD}}e^{i\int d^4x\sum_q[\bar\eta_q(x)q(x)+\bar q(x)\eta_q(x)]}\\ \times e^{i\int d^4x\, j^a_\mu(x)A^{\mu a}(x)}e^{i\int d^4x(\bar\epsilon^a c^a+\bar c^a\epsilon^a)}. \quad (2)$$

Our aim is to find a proper approximation in the low-energy limit. We will perform an expansion in the inverse of the 't Hooft coupling. In order to manage this functional, it is essential to find a way to reduce this theory to a simpler one. We seek set of classical solutions, in the proper infrared limit of the coupling going to infinity, to start a perturbation series for a quantum field theory that holds in the same approximation of strong coupling. *A posteriori*, we will verify the soundness of our choice of classical solutions to obtain a quantum field theory by comparison with numerical solutions on the lattice and with Dyson-Schwinger equations.

A. Gluon propagator

With this aim in mind, we have recently proved the following theorem, which holds just for *classical solutions* and produces an asymptotic mapping between the scalar field and the Yang-Mills theory in the limit of the coupling going to infinity:

Mapping Theorem. An extremum of the action

$$S=\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4\right)$$

is also an extremum of the SU(N) Yang-Mills Lagrangian when one properly chooses A^a_μ with some components being zero and all others being equal, and when $\lambda=Ng^2$, where g is the coupling constant of the Yang-Mills field when only time dependence is retained. In the most general case the following mapping holds:

$$A^a_\mu(x)=\eta^a_\mu\phi(x)+O(1/\sqrt{N}g),$$

where η^a_μ is a constant that becomes exact for the Lorenz gauge.

A first proof of this theorem was given in Ref. [18] and, after a criticism by Terence Tao, a final proof was presented in Ref. [19], with which Tao agreed [22]. In the following we give a cursory proof for the sake of completeness, but it should be kept in mind that here we are still working with classical solutions. So, let us consider the equation of motion of the scalar field,

$$\partial^2\phi+\lambda\phi^3=0. \quad (3)$$

Now we consider a gradient expansion for this equation in the following way. Let us rescale the time variable as $t\to\sqrt{\lambda}t$. The above equation becomes

$$\partial_t^2\phi+\phi^3=\frac{1}{\lambda}\Delta\phi, \quad (4)$$

and we are in a position to apply perturbation theory to this equation in the limit $\lambda\to\infty$, setting $\phi=\sum_{n=0}^\infty\lambda^{-n}\phi_n$. We note at this point a peculiarity of perturbation expansions for nonlinear differential equations. Let us consider the small-perturbation case and just rescale the field as $\phi\to\xi\phi$, where ξ a function of the coupling λ. Applying this rescaling to Eq. (3), we get

$$\partial^2\phi+\lambda\xi^2\phi^3=0. \quad (5)$$

Now we take $\lambda'=\lambda\xi^2$ and our perturbation expansion is now for λ'. So, our weak perturbation series appears to be arbitrary, and—because of the time rescaling—this also applies to the strong perturbation series we are considering. Indeed, our strong perturbation series is just dual to the weak one and must share the same properties (see Ref. [23]). What makes this arbitrariness irrelevant is that the perturbation series must be mathematically consistent and one must impose $\lambda\to 0$ for a weak perturbation and $\lambda\to\infty$ for the strong one. But these are formally an infinitesimal quantity and an infinite one, and multiplying them by a constant is irrelevant: Our expansion will just get an overall multiplying constant ξ. So, in the leading-order equation, one has the equation $\partial_t^2\phi_0+\phi_0^3=0$, which yields the solution

$$\phi_0=\mu 2^{1/4}{\rm sn}\left(\frac{1}{2^{1/4}}\mu t+\theta\right), \quad (6)$$

where μ and θ are two integration constants that can be applied depending on space variables. But if these are taken to be exactly constant, we have discovered a set of exact solutions of the equation we started with. Now one can do a Lorentz boost and transform this into a covariant set of massive exact solutions [24]. In this case, doing a Lorentz boost yields a resummation of all the perturbation series in the inverse of λ.

Now let us consider Yang-Mills equations for a generic SU(N) group and a generic gauge:

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\alpha}\right)\partial_\nu\left(\partial^\mu A^a_\mu\right)+gf^{abc}A^{b\mu}\left(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu\right)\\ +gf^{abc}\partial^\mu\left(A^b_\mu A^c_\nu\right)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=0. \quad (7)$$

As for the scalar field, we implement a strong coupling expansion with the rescaling $t\to\sqrt{N}gt$ and impose the expansion $A^a_\mu=\sum_{n=0}^\infty(\sqrt{N}g)^{-n}A^{a(n)}_\mu$. At the leading order of the expansion one gets the equation

$$\partial_t^2A^{a(0)}_\nu-\left(1-\frac{1}{\alpha}\right)\partial_t^2A^{a(0)}_0\delta_{\nu 0}+f^{abc}f^{cde}A^{b\mu(0)}A^{d(0)}_\mu A^{e(0)}_\nu\\ =0. \quad (8)$$

Now we find a set of components of the Yang-Mills field, chosen to be all equal, that reduces this leading-order equation to the one of the scalar field $\partial_t^2\phi_0+\phi_0^3=0$, provided we do the identification $\lambda=Ng^2$. We can write

$$A^a_\mu(t,0)=\eta^a_\mu\phi(t,0)+O\left(\frac{1}{\sqrt{N}g}\right). \quad (9)$$

A Lorentz boost restores covariance and creates η coefficients depending on momentum. The η coefficients can be fixed through a gauge choice. For example, in the Landau gauge one can have

$$\eta_\mu^a \eta_\nu^b = \delta_{ab}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \quad (10)$$

We now share couple of important considerations. First, we see that gauge invariance is not lost due to the mapping and that this mapping is an asymptotic one, holding in the limit of the coupling going to infinity. So, these classical solutions can be used in the infrared limit of a quantum field theory, preserving the substantial physical behavior in the ultraviolet limit of the quantum theories for the scalar and the Yang-Mills fields (triviality and asymptotic freedom, respectively).

Now let us evaluate the two-point function for the Yang-Mills field. One gets immediately

$$\begin{aligned} D_{\mu\nu}^{ab}(x-y) &= \langle \mathcal{T} A_\mu^a(x) A_\nu^b(y)\rangle \\ &= \eta_\mu^a \eta_\nu^b \langle \mathcal{T}\phi(x)\phi(y)\rangle + O(1/\sqrt{N}g) \\ &= \eta_\mu^a \eta_\nu^b \Delta(x-y) + O(1/\sqrt{N}g), \end{aligned} \quad (11)$$

where we have set $\Delta(x-y) = \langle \mathcal{T}\phi(x)\phi(y)\rangle$ for the two-point function of the scalar field. So, we need to identify the two-point function for the scalar field in the proper limit. Indeed, we have recently proved [25,26] that in the limit of the coupling going to infinity, the scalar field reaches a *trivial* infrared fixed point, and the two-point function is exactly determined as

$$\Delta(p) = \sum_{n=0}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon}, \quad (12)$$

where

$$B_n = (2n+1)\frac{\pi^2}{K^2(i)} \frac{(-1)^{n+1} e^{-(n+1/2)\pi}}{1 + e^{-(2n+1)\pi}}, \quad (13)$$

and $K(i) = \int_0^{\pi/2} d\theta/\sqrt{1+\sin^2\theta} \approx 1.3111028777$. The spectrum of the theory, in the strong coupling limit, is given by

$$m_n = \left(n + \frac{1}{2}\right)\frac{\pi}{K(i)}\left(\frac{Ng^2}{2}\right)^{1/4}\Lambda. \quad (14)$$

From the mass spectrum we can identify a string tension that will be useful in the following. We set, using the mapping theorem,

$$\sqrt{\sigma} = \left(\frac{Ng^2}{2}\right)^{1/4}\Lambda = (2\pi N\alpha_s)^{1/4}\Lambda. \quad (15)$$

Here Λ is an arbitrary parameter arising from integration of the equations of the theory. So, being an integration constant, it should be obtained from experiment. Finally, we note the following functional expansion for the generating functional of the Yang-Mills theory [27,28] that holds in a strong coupling limit:

$$A_\mu^a = \Lambda \int d^4y\, D_{\mu\nu}^{ab}(x-y) j^{b\nu}(y) + O(j^3). \quad (16)$$

The mapping theorem grants that the propagator in this equation is the same as that given in Eq. (12) with a proper choice of the η parameters. The form of the propagator shared by the two theories in the infrared limit is evidence that both theories are trivial in the infrared case, provided that a kind of Källen-Lehman representation with a nonpositive definite spectral function holds (see, e.g., Ref. [29]). This can also be seen by the form of the spectrum that has only free quasiparticle states but no bounded interacting states. This by no means implies that QCD is trivial; rather this theory is infrared safe due to the presence of quarks.

B. Consistency of the choice of the classical solutions

So far, we have chosen a set of classical solutions and built upon them a quantum field theory without any support or proof of this choice being the right one. Indeed, we have seen that this construction is self consistent provided that we consider an increasingly large coupling, but we cannot claim that other solutions exist that would providing a proper description in the same limit or that the gauge configurations would grant an optimal saddle point for the path integral of the theory. So, the only way we have to be sure that this picture is the proper one is to compare it with numerical data. On the lattice, very large volumes were considered for the gluon propagator in the Landau gauge in Refs. [30–32], while in Refs. [33,34] a numerical solution for Dyson-Schwinger equations was provided.

Our aim will be to show how, with increasing volume, the numerical data and our analytical results tend to coincide. Numerical Dyson-Schwinger equations represent our infinite-volume limit and we expect a very near coincidence of results in this case.

We consider two kind of lattice computations: A set of volumes up to 80^4 directly obtained with measurements on the lattice for SU(3), and measurements at 128^4 from Fig. 2 in Ref. [31] for SU(2). We are able to show in this way that, by increasing the volume, our propagator describes even more accurately the one measured on the lattice in the deep infrared. We would like to point out that the mass gap is different for these two cases as it depends on the value of β that, just for this section, has nothing to do with temperature but is the coupling on the lattice. Figures 1–3 depict the situation with volumes

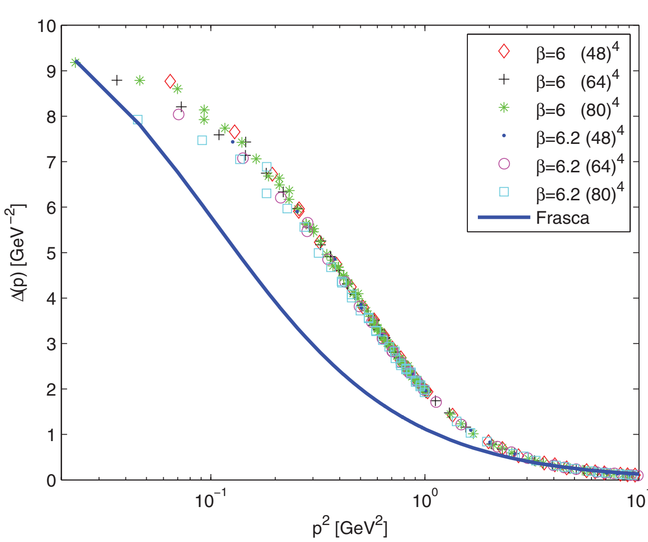

FIG. 1. (Color online) Gluon propagator in the Landau gauge for SU(3), 80^4 with a mass gap of $m_0 = 321$ MeV.

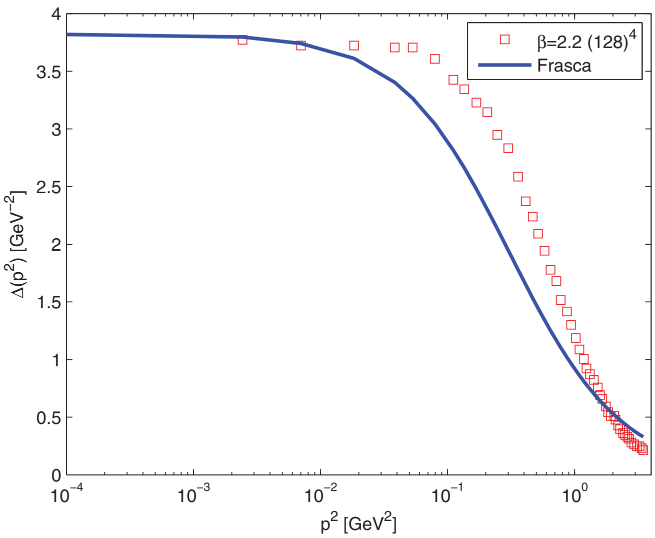

FIG. 2. (Color online) Gluon propagator in the Landau gauge for SU(2), 128^4 with a mass gap of $m_0 = 555$ MeV.

taken to be increasingly large on the lattice, to match even better our gluon propagator. Note that we consider a weak dependence on the gauge group as shown in Ref. [35], which is fully consistent with our discussion above.

This agreement between lattice computations at increasing volume and the perfect match for the numerical Dyson-Schwinger equations with our propagator give strong support to our picture, and to the view that the our choice of classical solutions provides a correct starting point for a perturbative quantum field theory in the infrared limit.

C. QCD in the infrared limit

Now we apply the Landau gauge through Eq. (10) and change the potential through Eq. (16), obtaining at the leading order, after noting that the ghost field decouples at this order,

$$S_{\rm QCD} = \frac{1}{2}\int d^4x\, d^4y j^{a\mu}(x)D^{ab}_{\mu\nu}(x-y)j^{b\nu}(y) + \int d^4x \times \sum_q \bar{q}(x)\Big(i\partial\!\!\!/ - g\frac{\lambda^a}{2}\gamma^\mu\Lambda\int d^4y\, D^{ab}_{\mu\nu}(x-y)j^{b\nu}(y) - g^2\Lambda\frac{\lambda^a}{2}\gamma^\mu\int d^4y'\, D^{ab}_{\mu\nu}(x-y') \times \sum_{q'}\bar{q}'(y')\frac{\lambda^b}{2}\gamma^\nu q'(y')\Big)q(x) + O(1/\sqrt{N}g). \tag{17}$$

Now we use the propagator (11), obtaining in the end

$$S_{\rm QCD} = \frac{1}{2}\int d^4x\, d^4y\, j^{a\mu}(x)\Delta(x-y)j^a_\mu(y) + \int d^4x\sum_q \bar{q}(x) \times \left(i\partial\!\!\!/ - g\frac{\lambda^a}{2}\gamma^\mu\Lambda\int d^4y\,\Delta(x-y)j^a_\mu(y)\right)q(x) - g^2\Lambda\int d^4x\, d^4y'\,\Delta(x-y')\sum_q\bar{q}(x)\frac{\lambda^a}{2}\gamma^\mu q(x) \times \sum_{q'}\bar{q}'(y')\frac{\lambda^a}{2}\gamma_\mu q'(y') + O(1/\sqrt{N}g). \tag{18}$$

So, we see that the existence of the infrared trivial fixed point in a pure Yang-Mills theory has the effect of recovering, directly from QCD, a nonlocal Nambu-Jona-Lasinio model always reducible to a local one [14,17]. The physics of this model, both at zero and at finite temperature, has been fully exploited by Hell, Cristoforetti, Roessner, and Weise [20], with the substantial difference that they are forced to guess the form of the form factor (the gluon propagator) using a model of a liquid of instantons. Here the form factor is directly obtained from QCD, but we will see below that their guess is excellent.

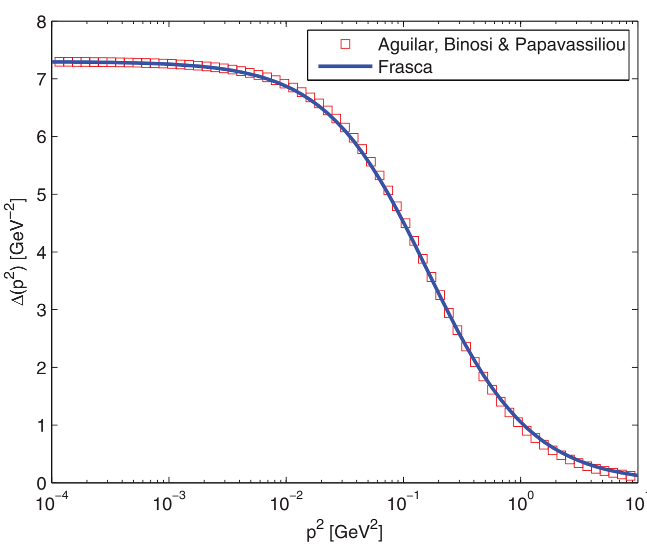

FIG. 3. (Color online) Gluon propagator in the Landau gauge for SU(3) obtained by numerically solving Dyson-Schwinger equations, with a mass gap $m_0 = 399$ MeV.

III. NONLOCAL NAMBU-JONA-LASINIO MODEL AND THE GAP EQUATION

In order to completely define the model we must try to better analyze the propagator. We realize from Eq. (12) that higher excited states are exponentially damped, and so we can limit our analysis to a single scalar field interacting with quarks. So, we approximate the propagator as $\Delta(p) \approx B_0/(p^2 - m_0^2 + i\epsilon)$, where $m_0 \approx 1.19\sqrt{\sigma}$ and $\sigma = (0.44\ \text{GeV})^2$ is the string tension, and we neglect the other contributions coming from higher excited states. This means that the Gaussian term $\frac{1}{2}\int d^4x\, d^4y\, j^{a\mu}(x)\Delta(x-y)j^a_\mu(y)$ can be rewritten in this approximation using an arbitrary scalar field σ over which we integrate as $\frac{1}{2}\int d^4x[(\partial\sigma)^2 - m_0^2\sigma^2]$, provided we take

$$\sigma = \sqrt{3(N^2-1)/B_0}\Lambda\int d^4y\,\Delta(x-y)j(y) \tag{19}$$

and use the currents as $j^a_\mu = \eta^a_\mu j$. So, one finally has

$$S_{\rm QCD} = \frac{1}{2}\int d^4x\left(\frac{1}{2}(\partial\sigma) - \frac{1}{2}m_0^2\sigma^2\right) + \int d^4x \sum_q \bar{q}(x) \times \left(i\partial\!\!\!/ - g\sqrt{\frac{B_0}{3(N^2-1)}}\frac{\lambda^a}{2}\gamma^\mu\eta^a_\mu\sigma(x)\right)q(x) - g^2\Lambda\int d^4x\, d^4y'\,\Delta(x-y')\sum_q \bar{q}(x)\frac{\lambda^a}{2}\gamma^\mu q(x) \times \sum_{q'}\bar{q}'(y')\frac{\lambda^a}{2}\gamma_\mu q'(y') + O(1/\sqrt{N}g). \quad (20)$$

We get a coupling for the σ field that can be ignored for our purposes. In order to recover in full the nonlocal model of Ref. [20], we have to identify the form factor depending on the gluon propagator. We get immediately

$$\mathcal{G}(p) = -\frac{1}{2}g^2\sum_{n=0}^{\infty}\frac{B_n}{p^2 - (2n+1)^2[\pi/2K(i)]^2\sigma + i\epsilon} = \frac{G}{2}\mathcal{C}(p), \quad (21)$$

where G is the Nambu-Jona-Lasinio constant that in our case is given by $G = 2\mathcal{G}(0) = (g^2/\sigma)\sum_{n=0}^{\infty}B_n/(2n+1)^2[\pi/2K(i)]^2 \approx 0.7854(g^2/\sigma)$ so that $\mathcal{C}(0) = 1$, which is definitely fixed according to QCD. In Ref. [20], a guess was put forward for $\mathcal{C}(p)$ using the model of a liquid of instantons. In Ref. [21] the form factor for this case takes the form

$$\mathcal{C}_I(p) = p^2\left(\pi d^2\frac{d}{d\xi}[I_0(\xi)K_0(\xi) - I_1(\xi)K_1(\xi)]\right)^2, \quad \xi = \frac{|p|d}{2}, \quad (22)$$

where I_n and K_n are Bessel functions. In the following we normalize this function to be 1 at zero momentum by dividing it by $\mathcal{C}_I(0)$. Weise *et al.* fix the functional form to $\mathcal{C}(p) = \exp(-p^2d^2/2)$ with $d^{-1} \approx 0.56$ GeV in order to avoid too much computational weight. We compared our $\mathcal{C}(p)$ with that given in Ref. [20], fixing $\sigma = (0.44\ \text{GeV})^2$ for the string tension and using $d^{-1} \approx 0.58$ GeV, similar to the guess by Weiss *et al.* The result is presented in Fig. 4. The agreement is so strikingly good with the instanton form factor that our conclusions strongly support a description of the ground state of QCD as an instanton liquid. This result was already pointed out in Ref. [28] by comparison with lattice results [36] for the running coupling in the infrared limit.

With the given expression for the form factor, which represents one of the most important results given in this paper, we are able to write the gap equation for massless quarks as obtained from Ref. [20]:

$$M(p) = \mathcal{C}(p)v \quad (23)$$

and

$$v = \frac{4NN_f}{m_0^2 + 1/G}\int\frac{d^4p}{(2\pi)^4}\mathcal{C}(p)\frac{M(p)}{p^2 + M^2(p)}, \quad (24)$$

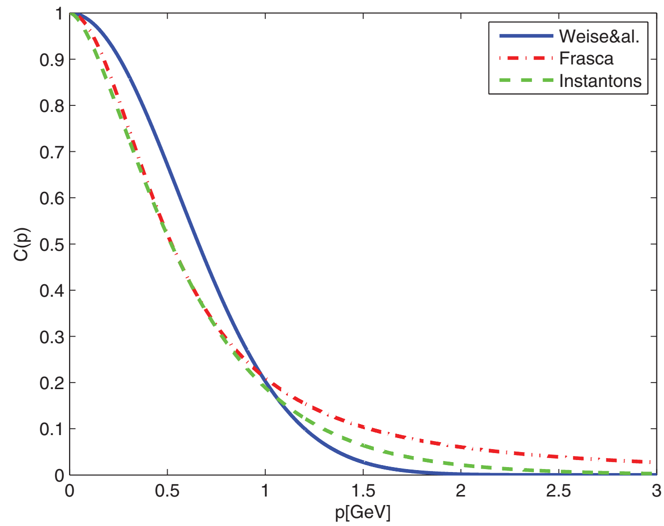

FIG. 4. (Color online) Comparison between our form factor with that used in Ref. [20] and the instanton liquid in Ref. [21] used as a model in Ref. [20].

where v is the vacuum expectation value of the σ field, N is the number of colors, and N_f is the number of flavors. Our aim is to prove the existence of a phase transition at finite temperature.

IV. FINITE-TEMPERATURE GAP EQUATION

We can evaluate the above results at finite temperature by using Matsubara sums. So, finally we can write down the gap equation as

$$M(\omega_k, \boldsymbol{p}) = \mathcal{C}(\omega_k, \boldsymbol{p})v \quad (25)$$

and

$$v = \frac{4NN_f}{m_0^2 + 1/G}\beta^{-1}\sum_{k=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\mathcal{C}(\omega_k, \boldsymbol{p}) \times \frac{M(\omega_k, \boldsymbol{p})}{\omega_k^2 + \boldsymbol{p}^2 + M^2(\omega_k, \boldsymbol{p})}, \quad (26)$$

where the Matsubara frequencies are $\omega_k = (2n+1)\pi T$ and n is an integer. The limits we are interested in are those at small momentum and temperature. The first one is needed for consistency with the Nambu-Jona-Lasinio model, while the second is needed to identify the existence of a phase transition. Now we are in a position to prove the existence of a critical point for which $v = 0$ and chiral symmetry is restored. Setting $v = 0$ in Eq. (26), we have to solve

$$\frac{4NN_f}{m_0^2 + 1/G}\beta^{-1}\sum_{k=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\mathcal{C}^2(\omega_k, \boldsymbol{p})\frac{1}{\omega_k^2 + \boldsymbol{p}^2} = 1. \quad (27)$$

A possible study of this equation is through numerical techniques. But taking a look at Fig. 4, after a simple numerical evaluation we note that the form factor is about 0.8 for a momentum of 260 GeV. Lattice computations such as those in Refs. [1,3] estimate the critical temperature to be about 170 MeV, well below the limit where the form factor is approximated by unity. This means that, for our aims, the form factor can be reduced to a step function that drops to zero at about 300 MeV and is unity for lower energies. With this crude approximation we are able to get an analytic expression for the critical temperature. Indeed, in this case the integral can be exactly evaluated and we get

$$T_c^2 \approx \frac{3}{\pi^2}\left[\Lambda^2 - \frac{\pi^2}{NN_f}\left(m_0^2 + \frac{1}{G}\right)\right], \tag{28}$$

which proves, starting directly from QCD, that a critical point indeed exists for which chiral symmetry is broken. This formula is in close agreement with the one in a recent work by Scoccola and Gómez Dumm [37]. The main difference is that we have fixed the proper value of the mass gap m_0 due to the form factor.

Now we can get an estimation of Λ, a parameter otherwise fixed by experiment for the Nambu-Jona-Lasinio model, by fixing $T_c = 0.17$ GeV as given by lattice computations. Taking $\sigma = (0.44\ \text{GeV})^2$, $g \approx 3$, $N = 3$, and $N_f = 2$ we get $\Lambda = 0.77$ GeV, a perfectly reasonable value for the Nambu-Jona-Lasinio model. This value decreases by increasing the number of flavors. So, from this computation we can conclude that both groups in Refs. [3,4] get a perfectly reasonable value for the crossover temperature, provided that it can be maintained at zero quark mass and chemical potential.

V. CONCLUSIONS

We have successfully shown how the quantum field theory of QCD reduces, in the low-energy limit, to a nonlocal Nambu-Jona-Lasinio model with all parameters properly fixed to physical values. In addition, the form of the gluon propagator in such a low-energy limit is exactly known. This implies that, when the computation is extended to a finite-temperature case, there exists a phase transition with chemical potential and quark masses set to zero.

This result should only be considered a starting point for future analysis. The most important limitation of this work is that we have not been able yet to accomodate the Polyakov loop in this approach. Presently, this is an important tool for understanding the phase diagram of QCD. It will be interesting to see how the crossover emerges when one assumes a quark mass different from zero and makes the gap equation more complex with the introduction of a chemical potential. This is our working program for the near future.

ACKNOWLEDGMENTS

Numerous discussions of this subject occurred with Marco Ruggieri. I thank him very much for clarifying several points in this matter that otherwise would have remained obscure. I would like to thank Thomas Hell for pointing out the right equation to use for an instanton liquid, which is also the one that was used in his paper with Weise, Roessner, and Cristoforetti [20]. Finally, I would like to thank Orlando Oliveira for providing me his measurements of the gluon propagator on the lattice, and Arlene Aguilar and Daniele Binosi for providing me the numerical results of their work on Dyson-Schwinger equations.

[1] Z. Fodor and S. D. Katz, J. High Energy Phys. 03 (2002) 014.
[2] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, Nature (London) **443**, 675 (2006).
[3] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg, and K. K. Szabo, J. High Energy Phys. 06 (2009) 088.
[4] A. Bazavov *et al.*, Phys. Rev. D **80**, 014504 (2009).
[5] P. de Forcrand, in *Proceedings of the XXV International Symposium on Lattice Field Theory, Regensburg, 2007*, PoS(LATTICE2007)010.
[6] M. G. Alford, A. Kapustin, and F. Wilczek, Phys. Rev. D **59**, 054502 (1999).
[7] P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002).
[8] M. D'Elia and M. P. Lombardo, Phys. Rev. D **67**, 014505 (2003).
[9] K. Fukushima, Phys. Rev. D **77**, 114028 (2008).
[10] H. Abuki, R. Anglani, R. Gatto, G. Nardulli, and M. Ruggieri, Phys. Rev. D **78**, 034034 (2008).
[11] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).
[12] J. Gasser and H. Leutwyler, Phys. Lett. B **188**, 477 (1987).
[13] K. Rajagopal and F. Wilczek, Nucl. Phys. B **399**, 395 (1993).
[14] M. Frasca, Int. J. Mod. Phys. E **18**, 693 (2009).
[15] M. Frasca, Nucl. Phys. Proc. Suppl. **186**, 260 (2009).
[16] M. Frasca, Nucl. Phys. Proc. Suppl. **207-208**, 196 (2010).
[17] K. I. Kondo, Phys. Rev. D **82**, 065024 (2010).
[18] M. Frasca, Phys. Lett. B **670**, 73 (2008).
[19] M. Frasca, Mod. Phys. Lett. A **24**, 2425 (2009).
[20] T. Hell, S. Rössner, M. Cristoforetti, and W. Weise, Phys. Rev. D **79**, 014022 (2009).
[21] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).
[22] T. Tao (private communication); http://wiki.math.toronto.edu/DispersiveWiki/index.php/Talk:Yang-Mills_equations.
[23] M. Frasca, Phys. Rev. A **58**, 3439 (1998).
[24] M. Frasca, J. Nonlin. Math. Phys. **18**, 291 (2011).
[25] M. Frasca, Phys. Rev. D **73**, 027701 (2006).
[26] M. Frasca, Int. J. Mod. Phys. A **22**, 2433 (2007).
[27] R. T. Cahill and C. D. Roberts, Phys. Rev. D **32**, 2419 (1985).
[28] M. Frasca, arXiv:0802.1183 [hep-th].
[29] F. Strocchi, *Selected Topics on the General Properties of Quantum Field Theory*, Lecture Notes in Physics Vol. 51 (World Scientific, Singapore, 1993), pp 1–173.
[30] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, in *Proceedings of the XXV International Symposium on Lattice Field Theory, Regensburg, 2007*, PoS(LATTICE2007)290.
[31] A. Cucchieri and T. Mendes, in *Proceedings of the XXV International Symposium on Lattice Field Theory, Regensburg, 2007*, PoS(LATTICE2007)297.
[32] O. Oliveira, P. J. Silva, E. M. Ilgenfritz, and A. Sternbeck, in *Proceedings of the XXV International Symposium on Lattice Field Theory, Regensburg, 2007*, PoS(LATTICE2007)323.

[33] A. C. Aguilar and A. A. Natale, J. High Energy Phys. 08 (2004) 057.
[34] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **78**, 025010 (2008).
[35] A. Maas, J. High Energy Phys. 02 (2011) 076.
[36] P. Boucaud, F. De Soto, A. Le Yaouanc, J. P. Leroy, J. Micheli, H. Moutarde, O. Pene, and J. Rodriguez-Quintero, J. High Energy Phys. 04 (2003) 005.
[37] D. Gómez Dumm, N. N. Scoccola, Phys. Rev. C **72**, 014909 (2005).