# Influence of Coulomb distortion on polarization observables in elastic electromagnetic hadron-lepton scattering at low energies

Hartmuth Arenhövel*

*Institut für Kernphysik, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany*

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Polarization observables in elastic electromagnetic hadron-lepton scattering at low energies are studied, with special emphasis on the influence of Coulomb distortion. The spin-dependent contributions to the scattering matrix, i.e., the hyperfine and spin-orbit interactions of leptons and hadrons, are calculated in a distorted-wave Born approximation based on nonrelativistic Coulomb wave functions. For like charges the Coulomb repulsion greatly reduces the size of polarization observables compared to the plane-wave Born approximation, whereas for opposite charges the Coulomb attraction leads to a substantial increase in these observables for hadron laboratory kinetic energies below about 20 keV.

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## I. INTRODUCTION

Recently, Coulomb effects on polarization transfer from polarized electrons or positrons to initially unpolarized protons or antiprotons in elastic electromagnetic scattering have been studied in a distorted-wave (DW) approximation at low energies [1]. These studies were motivated by the idea of polarizing hadrons by their scattering on polarized electrons or positrons in a storage ring [2]. However, in view of this design, it turned out that the considered observable, i.e., the total cross section for the scattering of initially unpolarized hadrons off polarized leptons to polarized final hadrons, the polarization transfer $P_{z00z}$ cannot contribute to a net polarization of the hadrons in the storage ring. The reason for that is that this polarization observable does not contain a genuine hadronic spin-flip process [3,4], which is necessary for a net polarization change. Moreover, our previous numerical results were criticized by Milstein *et al.* [4], who had taken a partial-wave expansion of the Coulomb scattering wave function instead of the integral representation used in Ref. [1]. Indeed, it turned out that, besides a minor error, the main reason for the gross overestimation of the polarization transfer cross section was an accuracy problem in the numerical evaluation, namely, the relevant quantity was calculated as a difference of two almost-equal numbers multiplied by a huge factor [5,6]. Besides correcting this problem, I have extended the previous study to the formal consideration of all possible polarization observables in this scattering reaction including such spin-flip transitions using the DW Born approximation. In addition to the previously considered hyperfine interaction, I also include the spin-orbit interactions of lepton and hadron in the present work.

As a starting point, the most general scattering cross section, allowing for the polarization of all initial and final particles described by corresponding spin density matrices, is introduced in the next section, defining the various polarization observables in terms of bilinear Hermitean forms of the $T$ matrix elements. The nonrelativistic $T$ matrix with the lowest order relativistic contributions from spin-orbit and hyperfine interactions is presented. Then I specialize to the case where only the polarization of the final lepton is not measured, the so-called triple polarization cross section.

The structure functions are evaluated in plane-wave (PWA) and DW Born approximation (DWA) using nonrelativistic Coulomb scattering wave functions. For numerical evaluation in the DWA two methods were applied: (i) a partial-wave expansion as in Ref. [4] and (ii) an integral representation of the Coulomb wave function according to Ref. [7]. Results for the structure functions and spin-flip triple cross sections are presented for the case where the initial lepton and the initial and final hadrons are polarized along the incoming hadron momentum in Sec. III and a summary is given in Sec. IV. Some details are presented in the appendixes: in Appendix A, the general scattering cross section in terms of the various contributions to the $T$ matrix and, in Appendix B, the two methods for numerical evaluation of hyperfine and spin-orbit interactions. A more detailed account is presented in Ref. [8].

## II. FORMAL DEVELOPMENTS

Reviews on polarization phenomena may be found for lepton hadron scattering in Ref. [9], for nuclear physics in Ref. [10], and for nucleon-nucleon scattering in Ref. [11]. I consider hadron-lepton scattering in the c.m. system, where hadron stands for a proton or antiproton and lepton for an electron or a positron,

$$h(\vec{p}\,) + l(-\vec{p}\,) \longrightarrow h(\vec{p}\,') + l(-\vec{p}\,'), \tag{1}$$

allowing for initial and final hadron and lepton polarization. The hadron initial and final three momenta are denoted $\vec{p}$ and $\vec{p}\,'$, respectively. All possible observables of this reaction can be obtained from the "quadruple-polarization" cross section, for which the spin states of all initial and final particles are described by the corresponding general spin density matrices $\rho^{l/h}(\vec{P}^{i/f}_{l/h})$, where the initial density matrices characterize the

*arenhoevel@kph.uni-mainz.de

 

spin properties of the and beam and the final ones those of the detected particles. This cross section is given by the general trace

$$\frac{d\sigma^{\text{quadruple}}_{\vec{P}_h^f,\vec{P}_h^i,\vec{P}_l^f,\vec{P}_l^i}(\theta,\phi)}{d\Omega} = \mathcal{O}\big(\vec{P}_h^f,\vec{P}_l^f,\vec{P}_h^i\vec{P}_l^i;\theta,\phi\big) = \frac{M_l^2M_h^2}{\pi^2W^2\big(1+|\vec{P}_l^f|\big)\big(1+|\vec{P}_h^f|\big)} \times \text{Trace}\big[\widehat{T}^\dagger\widehat{\rho}^h\big(\vec{P}_h^f\big)\widehat{\rho}^l\big(\vec{P}_l^f\big)\widehat{T}\widehat{\rho}^h\big(\vec{P}_h^i\big)\widehat{\rho}^l\big(\vec{P}_l^i\big)\big], \quad (2)$$

where $\widehat{T}=\widehat{T}(\theta,\phi)$ denotes the $T$ matrix of the scattering process, with $(\theta,\phi)$ the scattering angles, $\rho(\vec{P})$ the spin density matrix for a spin-1/2 particle, and $\vec{P}$ characterizing the polarization of the corresponding particle in the initial and final states, respectively. The trace refers to the hadron and lepton spin degrees of freedom. The factor in front takes into account the final phase space, the incoming flux, and a normalization factor for the case of partially polarized final states. The invariant energy of the hadron-lepton system is denoted $W=E_h+E_l$, and the masses of hadron and lepton $M_h$ and $M_l$, respectively. In the c.m. frame I use as the reference system the $z$ axis along the incoming hadron momentum $\vec{p}$. The $x$ and $y$ axes are chosen to form a right-handed orthogonal system.

In view of the fact that in this work I am interested in the low-energy regime, a nonrelativistic framework is adopted. The nonrelativistic density matrices for possible polarization of initial and final states of a spin-1/2 particle have the standard form,

$$\widehat{\rho}(\vec{P}) = \tfrac{1}{2}(1+\vec{P}\cdot\vec{\sigma}), \quad (3)$$

with the vector $\vec{P}$ describing the polarization of the particle. One should note that, in general, $|\vec{P}^{i/f}_{h/l}|\leqslant 1$.

As already mentioned, all possible polarization observables can be obtained from Eq. (2). In detail they are as follows.

(i) The unpolarized differential cross section:

$$\frac{d\sigma_0(\theta,\phi)}{d\Omega} = \mathcal{O}(\vec{0},\vec{0},\vec{0},\vec{0};\theta,\phi) = S^0(\theta,\phi). \quad (4)$$

(ii) Beam, target, and beam-target asymmetries of the differential cross section for unpolarized final states in the notation of Bystricky *et al.* [11],

$$\frac{d\sigma_{\vec{P}_h^i,\vec{P}_l^i}(\theta,\phi)}{d\Omega} = \mathcal{O}\big(\vec{0},\vec{0},\vec{P}_h^i,\vec{P}_l^i;\theta,\phi\big) = \frac{d\sigma_0(\theta,\phi)}{d\Omega}\Bigg(1+\sum_j P^i_{h,j}A_{00j0}(\theta,\phi) + \sum_k P^i_{l,k}A_{000k}(\theta,\phi) + \sum_{j,k}P^i_{h,j}P^i_{l,k}A_{00jk}(\theta,\phi)\Bigg), \quad (5)$$

with beam and target asymmetries, respectively,

$$A_{00j0}(\theta,\phi) = \frac{1}{S^0}\frac{\partial}{\partial P^i_{h,j}}\mathcal{O}\big(\vec{0},\vec{0},\vec{P}^i_{h/l},\vec{0};\theta,\phi\big), \quad A_{000k}(\theta,\phi) = \frac{1}{S^0}\frac{\partial}{\partial P^i_{l,k}}\mathcal{O}\big(\vec{0},\vec{0},\vec{P}^i_{h/l},\vec{0};\theta,\phi\big) \quad (6)$$

and the beam-target asymmetry tensor

$$A_{00jk}(\theta,\phi) = \frac{1}{S^0}\frac{\partial^2}{\partial P^i_{h,j}\partial P^i_{l,k}}\mathcal{O}\big(\vec{0},\vec{0},\vec{P}^i_h,\vec{P}^i_l;\theta,\phi\big). \quad (7)$$

(iii) Polarization of the final lepton or hadron for unpolarized beam and target (again, notation of Ref. [11]):

$$P_{0j00}(\theta,\phi) = \frac{1}{S^0}\frac{\partial}{\partial P^f_{l,j}}\mathcal{O}\big(\vec{0},\vec{P}_l^f,\vec{0},\vec{0};\theta,\phi\big), \quad P_{j000}(\theta,\phi) = \frac{1}{S^0}\frac{\partial}{\partial P^f_{h,j}}\mathcal{O}\big(\vec{P}_h^f,\vec{0},\vec{0},\vec{0};\theta,\phi\big). \quad (8)$$

(iv) Various correlations between the polarization of one outgoing particle and the beam and/or target polarizations. For example, in the case of outgoing hadron polarization for initial lepton polarization but an unpolarized incoming hadron, the lepton-hadron polarization transfer is given by

$$P_{j00k}(\theta,\phi) = \frac{1}{S^0}\frac{\partial^2}{\partial P^f_{h,j}\partial P^i_{l,k}}\mathcal{O}\big(\vec{P}_h^f,\vec{0},\vec{0},\vec{P}_l^i;\theta,\phi\big). \quad (9)$$

(v) Another interesting example is the hadron spin flip of a hadron initially polarized by the scattering on an initially polarized lepton. This is the special case of the so-called "triple polarization" cross section, with all particles polarized except for the final lepton,

$$\frac{d\sigma^{\text{triple}}_{\vec{P}_h^f,\vec{P}_h^i,\vec{P}_l^i}(\theta,\phi)}{d\Omega} = \mathcal{O}\big(\vec{P}_h^f,\vec{0},\vec{P}_h^i\vec{P}_l^i;\theta,\phi\big), \quad (10)$$

for the case $\vec{P}_h^f=-\vec{P}_h^i$. This is the relevant quantity for the method of polarizing hadrons by electromagnetic scattering on polarized leptons in a storage ring.

### A. The nonrelativistic *T* matrix

For explicit evaluation of the trace in Eq. (2), one needs to know the spin dependence of the $T$ matrix. In a nonrelativistic approach but including contributions of the order $M^{-2}$, the $T$ matrix contains the Coulomb, the lepton and hadron spin-orbit, and the lepton-hadron hyperfine interactions. Separating the various contributions, the $T$ matrix is given in obvious notation by

$$\widehat{T} = \widehat{T}_C + \widehat{T}_{LS_l} + \widehat{T}_{LS_h} + \widehat{T}_{SS}, \quad (11)$$

where the Coulomb interaction $\widehat{T}_C$, the spin-orbit interactions of the lepton and hadron, $\widehat{T}_{LS_{l/h}}$, respectively, and the hyperfine interaction $\widehat{T}_{SS}$ are given by

$$\widehat{T}_C = 4\pi\alpha a_C, \quad \widehat{T}_{LS_{l/h}} = 4\pi\alpha\vec{b}_{l/h}\cdot\vec{\sigma}^{l/h}, \quad \widehat{T}_{SS} = 4\pi\alpha\vec{\sigma}^h\cdot\overleftrightarrow{d}\cdot\vec{\sigma}^l. \quad (12)$$

Here $\alpha$ denotes the Sommerfeld fine structure constant, $b_{l/h}$ a vector, and $\overleftrightarrow{d}$ a symmetric rank 2 tensor. The tensor $\overleftrightarrow{d}$ can be decomposed into a scalar and a spherical tensor of rank 2, i.e., a symmetric Cartesian tensor with vanishing trace (notation of Ref. [12]),

$$\overleftrightarrow{d} = d^{[0]} + d^{[2]}, \quad \text{with} \quad d^{[0]}_{ij} = d_0\delta_{ij},$$
$$d_0 = \tfrac{1}{3}\mathrm{Trace}(\overleftrightarrow{d}) \quad \text{and} \quad d^{[2]}_{ij} = d_{ij} - d_0\delta_{ij}. \tag{13}$$

Furthermore, the parameters $a_C$, $\vec{b}_{l/h}$, and $\overleftrightarrow{d}$ depend on what kind of approximation is used. These are as follows.

(i) The plane-wave approximation, corresponding to a pure one-photon exchange. The nonrelativistic reduction of the $T$ matrix including the lowest order relativistic contribution reads

$$\begin{aligned}\widehat{T}^{\rm PW} = \frac{4\pi\alpha}{q^2}\bigg\{ & Z_l Z_h\left(1 + \frac{\vec{P}^2}{4M_lM_h}\right) \\ & - \frac{1}{8}\left(Z_h\frac{2\mu_l - 1}{M_l^2} + Z_l\frac{2\mu_h - 1}{8M_h^2}\right)q^2 \\ & - \frac{Z_h}{8M_l}\left(\frac{2\mu_l - 1}{M_l} + \frac{2\mu_l}{M_h}\right)i(\vec{\sigma}_l \times \vec{q}\,)\cdot\vec{P} \\ & - \frac{Z_l}{8M_h}\left(\frac{2\mu_h - 1}{M_h} + \frac{2\mu_h}{M_l}\right)i(\vec{\sigma}_h \times \vec{q}\,)\cdot\vec{P} \\ & + \frac{\mu_l\mu_h}{4M_lM_h}(\vec{\sigma}_l\cdot\vec{q}\,\vec{\sigma}_h\cdot\vec{q} - q^2\,\vec{\sigma}_l\cdot\vec{\sigma}_h\,)\bigg\},\end{aligned} \tag{14}$$

with the three-momentum transfer $\vec{q} = \vec{p}\,' - \vec{p}$, $\vec{P} = \vec{p} + \vec{p}\,'$, $Z_l$ and $Z_h$ as the lepton and hadron charges, and $\mu_l$ and $\mu_h$ as their magnetic moments, respectively. From this expression, one reads off the parameters, keeping in the spin-independent term the lowest order only,

$$a_C^{\rm PW} = \frac{Z_lZ_h}{q^2}, \quad \vec{b}^{\rm PW}_{l/h} = i c^{LS}_{l/h}\frac{\vec{p}\,'\times\vec{p}}{q^2},$$
$$d^{\rm PW}_{ij} = c^{SS}(\widehat{q}_i\widehat{q}_j - \delta_{ij}), \tag{15}$$

where $\widehat{q}$ denotes the unit vector along the three-momentum transfer $\vec{q}$ and $q = |\vec{q}\,|$. The separation into a scalar and a traceless tensor according to Eq. (13) reads

$$d_0^{\rm PW} = -\tfrac{2}{3}c^{SS}, \quad d^{[2]\rm PW}_{ij} = c^{SS}\big(\widehat{q}_i\widehat{q}_j - \tfrac{1}{3}\delta_{ij}\big). \tag{16}$$

Furthermore, the strength parameters are

$$c_l^{LS} = \frac{Z_h}{4M_l}\left(\frac{2\mu_l - 1}{M_l} + 2\frac{\mu_l}{M_h}\right),$$
$$c_h^{LS} = \frac{Z_l}{4M_h}\left(\frac{2\mu_h - 1}{M_h} + 2\frac{\mu_h}{M_l}\right), \quad c^{SS} = \frac{\mu_l\mu_h}{4M_lM_h}. \tag{17}$$

One should note that the strength parameter of the hadronic spin-orbit interaction is about 3 orders of magnitude smaller than the parameter of the leptonic one, because their ratio is approximately $c_h^{LS}/c_l^{LS} \approx 2\mu_h M_l/M_h \approx 3\times 10^{-3}$.

(ii) The DWA using nonrelativistic Coulomb scattering wave functions $\psi^{C(\pm)}_{\vec{p}}$. The relevant quantity for Coulomb effects is the Sommerfeld Coulomb parameter $\eta_C = \alpha Z_l Z_h/v$, with $v$ denoting the relative hadron-lepton velocity. Within this approach one finds

$$a_C^{\rm DW} = e^{i\phi_C}a_C^{\rm PW}, \quad \text{with} \quad \phi_C(\theta) = -\eta_C\ln[\sin^2(\theta/2)], \tag{18}$$

$$\vec{b}^{\rm DW}_{l/h} = i\frac{c^{LS}_{l/h}}{4\pi}\int\frac{d^3r}{r^3}\psi^{C(-)}_{\vec{p}\,'}(\vec{r}\,)^*\,(\vec{r}\times\vec{\nabla})\,\psi^{C(+)}_{\vec{p}}(\vec{r}), \tag{19}$$

$$\begin{aligned}d^{\rm DW}_{ij} = & -\frac{c^{SS}}{4\pi}\int d^3r\,\psi^{C(-)}_{\vec{p}\,'}(\vec{r}\,)^* \\ & \times\left[\frac{1}{r^3}(3\hat{r}_i\,\hat{r}_j - \delta_{ij}) + \frac{8\pi}{3}\delta_{ij}\delta(\vec{r}\,)\right]\psi^{C(+)}_{\vec{p}}(\vec{r}\,).\end{aligned} \tag{20}$$

Again separating the hyperfine contribution into a scalar and a traceless tensor, one obtains

$$d_0^{\rm DW} = -\frac{2}{3}\,c^{SS}N(\eta_C)^2,$$
$$d^{[2]\rm DW}_{ij} = \frac{c^{SS}}{4\pi}\int\frac{d^3r}{r^3}\psi^{C(-)}_{\vec{p}\,'}(\vec{r}\,)^*\,(3\hat{r}_i\,\hat{r}_j - \delta_{ij})\,\psi^{C(+)}_{\vec{p}}(\vec{r}), \tag{21}$$

with $N(\eta_C)$ given in Appendix B. One should note that the tensor $d^{\rm DW}_{ij}$ is symmetric as well as $d^{[2]\rm DW}_{ij}$. The two methods for the evaluation are outlined in Appendix B.

### B. The triple-polarization cross section

The evaluation of the general trace in Eq. (2) is presented in Appendix A, where also the case is considered that the final lepton polarization is not analyzed but all other particles may be polarized, the so-called triple polarization cross. Here I consider this triple-polarization cross section given in Eq. (A17) of Appendix A for the special case where the initial and final hadrons and the initial lepton are polarized along the incoming hadron direction. Then with $\vec{P}_h^{i/f} = \lambda_h^{i/f}\hat{z}$, $\vec{Q}_h = \lambda_h^+\hat{z}$, $\vec{P}_l^i = \lambda_l^i\hat{z}$, and $\lambda_h^\pm = \lambda_h^i \pm \lambda_h^f$, the cross section simplifies considerably (for more details see Ref. [8]), and one finds that the remaining structure functions become $\phi$ independent, which is easy to understand, as all polarizations are assumed to be along the $z$ axis, ruling out any $\phi$ dependence. Thus the cross section becomes

$$\begin{aligned}\frac{d\sigma^{\rm triple}_{\lambda_h^f,\lambda_h^i,\lambda_l^i}(\theta)}{d\Omega} = & \big(1 + \lambda_h^i\lambda_h^f\big)\big[S_C(\theta) + S_0(\theta) + L_0^l(\theta)\big] \\ & + \big(1 - \lambda_h^i\lambda_h^f\big)L_0^h(\theta) + \lambda_l^i\big[\lambda_h^-\big(S_2^-(\theta,\phi) + L_2^h(\theta)\big) \\ & + \lambda_h^+\big(S_2^+(\theta) + L_2^l(\theta)\big)\big] + \lambda_h^i\lambda_h^f S_2(\theta),\end{aligned} \tag{22}$$

where the structure functions are given by

$$L_0^{l/h}(\theta) = V\big|b_0^{l/h}(\theta)\big|^2, \tag{23}$$
$$L_2^{l/h}(\theta) = 2V\,\mathrm{Re}\big[d_{13}^{[2],0}(\theta)b_0^{l/h}(\theta)^*\big], \tag{24}$$

$$S_0(\theta) = V\left(3|d_0|^2 + \sum_{j=1}^{3}\left|d_{jj}^{[2],0}(\theta)\right|^2 + 2\left|d_{13}^{[2],0}(\theta)\right|^2\right), \quad (25)$$

$$S_2^+(\theta) = 2V\mathrm{Re}\big[a_C^*\big(d_0 + d_{33}^{[2],0}(\theta)\big)\big], \quad (26)$$

$$S_2^-(\theta) = 2V\big[\mathrm{Re}\big(d_0^* d_{33}^{[2],0}(\theta) - d_{11}^{[2],0}(\theta)^* d_{22}^{[2],0}(\theta)\big) - |d_0|^2\big], \quad (27)$$

$$S_2(\theta) = 2V\big[2\mathrm{Re}\big(d_0^* d_{33}^{[2],0}(\theta)\big) - 2|d_0|^2 - \left|d_{11}^{[2],0}(\theta)\right|^2 - \left|d_{22}^{[2],0}(\theta)\right|^2 - \left|d_{13}^{[2],0}(\theta)\right|^2\big]. \quad (28)$$

The expression in Eq. (22) is an extension of the triple-polarization cross section given in Eq. (16) of Ref. [3] by including the contributions from the hadron and lepton spin-orbit interactions. As a special case, one considers the so-called hadronic spin-flip cross section for complete hadron polarization, i.e., $\lambda_h^i = -\lambda_h^f = \lambda_h = \pm 1$, and the non-spin-flip cross section with $\lambda_h^i = \lambda_h^f = \lambda_h = \pm 1$. For the spin-flip cross section one finds

$$\frac{d\sigma^{\mathrm{sf}}_{-\lambda_h,\lambda_h,\lambda_l^i}(\theta,\phi)}{d\Omega} = 2L_0^h(\theta) - S_2(\theta) + 2\lambda_l^i\lambda_h\big[S_2^-(\theta,\phi) + L_2^h(\theta)\big]. \quad (29)$$

It is governed by the hyperfine terms $S_2$ and $S_2^-$ and the hadronic spin-orbit interaction via $L_0^h$ and $L_2^h$. On the other hand, the non-spin-flip cross section is given by

$$\frac{d\sigma^{\mathrm{nsf}}_{\lambda_h,\lambda_h,\lambda_l^i}(\theta,\phi)}{d\Omega} = 2\big[S_C(\theta) + S_0(\theta) + L_0^l(\theta)\big] + S_2(\theta) + 2\lambda_l^i\lambda_h\big[S_2^+(\theta,\phi) + L_2^l(\theta)\big]. \quad (30)$$

Its polarization-independent part is overwhelmingly dominated by the Coulomb term $S_C$, with additional tiny contributions from the hyperfine and the leptonic spin-orbit interactions. The difference of the non-spin-flip cross section for $\lambda_h = \pm 1$,

$$\frac{1}{2}\left(\frac{d\sigma^{\mathrm{nsf}}_{1,1,\lambda}(\theta,\phi)}{d\Omega} - \frac{d\sigma^{\mathrm{nsf}}_{-1,-1,\lambda}(\theta,\phi)}{d\Omega}\right) = 2\lambda\big[S_2^+(\theta,\phi) + L_2^l(\theta)\big], \quad (31)$$

is considered lepton-hadron polarization transfer in Ref. [2]. This polarization transfer is dominated by the hyperfine structure function $S_2^+$ because the additional spin-orbit contribution $L_2^l$ is comparably small, as shown in the next section. It differs by a factor of 2 and the presence of $L_2^l$ from the lepton-hadron polarization transfer $P_{z00z}$ for the scattering of unpolarized hadrons on polarized leptons as considered in Ref. [1], where I considered only the leading term $S_2^+$, the interference between Coulomb and hyperfine amplitudes, neglecting higher order contributions. The more complete expression reads

$$P_{z00z}\frac{d\sigma_0(\theta,\phi)}{d\Omega} = \frac{\partial^2}{\partial\lambda_h^f\partial\lambda_l^i}\frac{d\sigma^{\mathrm{triple}}_{\lambda_h^f,\lambda_h^i,\lambda_l^i}(\theta,\phi)}{d\Omega}\bigg|_{\lambda_h^i=0} = S_2^+(\theta,\phi) - S_2^-(\theta,\phi) + L_2^l(\theta,\phi) - L_2^h(\theta,\phi). \quad (32)$$

In addition to $S_2^+$, it includes $S_2^-$, which is quadratic in the hyperfine amplitude, and $L_2^l$ and $L_2^h$, the contributions from the interference of hyperfine and leptonic and hadronic spin-orbit amplitudes, respectively. However, the largest of these additional terms, $L_2^l$, is still quite small, if not negligible, compared to $S_2^+$.

## III. RESULTS FOR STRUCTURE FUNCTIONS AND POLARIZATION CROSS SECTIONS

For evaluation of the structure functions in Eq. (22) and the corresponding cross section, I have used two methods for the calculation of Coulomb distortion, the integral representation as well as the partial-wave expansion. The integral representation has been used mainly to check the convergence of the partial-wave expansion as described in detail in Ref. [8]. Thus all results presented in this section are based on the partial-wave expansion. For the hyperfine amplitude it was found that an expansion up to a partial wave with $l_{\max} = 2000$ was sufficient, but for the spin-orbit interaction, being much more slowly convergent, $l_{\max} = 4000$ was taken.

### A. The structure functions

First, I discuss the various structure functions which determine the triple-polarization cross section of Eq. (22). For the partial-wave expansion, one easily finds the following expressions in the c.m. frame:

$$L_0^{l/h}(\theta) = -\tfrac{1}{4}\,V\left(c_{l/h}^{LS}\right)^2\cot^2(\theta/2), \quad (33)$$

$$L_2^{l/h}(\theta) = -V c_{l/h}^{LS} c^{SS}\cos^2(\theta/2), \quad (34)$$

$$S_0(\theta) = 2\,V(c^{SS})^2, \quad (35)$$

$$S_2^+(\theta) = -\frac{V c^{SS}}{2p^2}\cot^2(\theta/2), \quad (36)$$

$$S_2^-(\theta) = -2\,V(c^{SS})^2\sin^2(\theta/2), \quad (37)$$

$$S_2(\theta) = -2\,V(c^{SS})^2\big(1 + \sin^2(\theta/2)\big). \quad (38)$$

The structure functions, evaluated in the c.m. frame for a low laboratory kinetic energy of $T_h = 1$ keV corresponding to $\eta_C \approx 5$, are shown in Fig. 1 for the various approximations, i.e., plane-wave approximation (PW) and Coulomb distortion for like ($\mathrm{DW}^+$) and opposite ($\mathrm{DW}^-$) charges. More results at higher energies are presented in Ref. [8].

The diagonal structure function $S_0$, induced by the hyperfine interaction, shows a rather flat, almost constant angular behavior. Its size scales roughly proportional to the inverse of the kinetic energy $T_h$. Compared to the plane-wave approximation, the DWA is strongly enhanced for opposite charges ($\mathrm{DW}^-$) and strongly suppressed for like charges ($\mathrm{DW}^+$), by several orders of magnitude. This enhancement or suppression is increasingly reduced with growing kinetic energy $T_h$ and approaches the plane-wave result above $T_h = 10$ MeV (for details see Ref. [8]). The pure Coulomb contribution $S_C$, however, is much larger, by more than 10 orders of magnitude.

With respect to the other two diagonal structure functions from the leptonic and hadronic spin-orbit interactions $L_0^l$

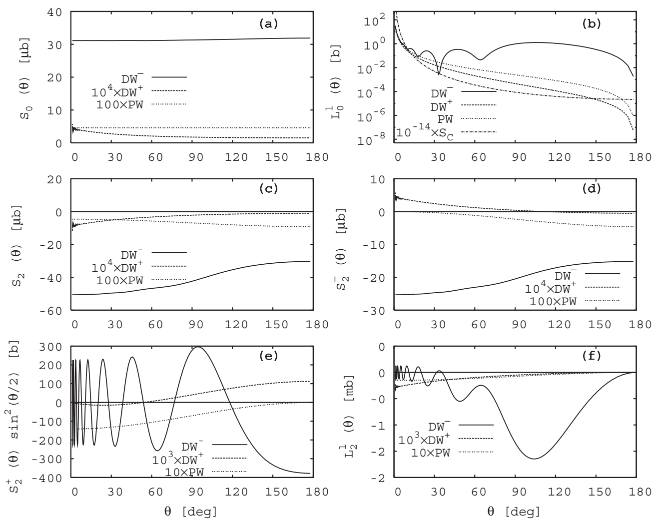


FIG. 1. Structure functions of the triple-polarization differential cross section in the c.m. system in plane-wave (PW) and distorted-wave (DW) approximations for like charges (DW$^+$) and opposite charges (DW$^-$) for a proton laboratory kinetic energy $T_h = 1$ keV.

and $L_0^h$, respectively, it suffices to show only the former one, because $L_0^h$ differs in magnitude only by the factor $(c_{LS}^h/c_{LS}^l)^2 \approx 0.9 \times 10^{-5}$. One readily notes that $L_0^l$ exhibits a strong peaking in the forward direction only and tends to oscillate at small angles for the lowest $T_h$ considered here. Over the whole angular range, especially in the forward direction, $L_0^l$ is much larger, by several orders of magnitude, than $S_0$ but it is still almost negligible compared to the size of $S_C$. The effect of Coulomb distortion is qualitatively similar to what one observes in $S_0$.

Only these diagonal structure functions contribute to the unpolarized cross section. However, as already mentioned, their relative contribution is extremely small as can be seen by comparison with the pure Coulomb structure function $S_C$, which is also shown in the upper right panel in Fig. 1, indicated by the large reduction factor applied to $S_C$.

The two hyperfine-hyperfine interference structure functions, $S_2$ and $S_2^-$, which both are negative throughout, exhibit a similar pattern, a smooth angular distribution with a slight decrease in size at backward angles. Again, one notes sizable enhancements for opposite charges by Coulomb distortion and suppression for like charges. Also, these two structure functions are quite small like $S_0$ because they are quadratic in the hyperfine amplitudes.

Much larger is the third interference structure function $S_2^+$ because it is an interference between the Coulomb and the hyperfine amplitudes. Thus it is strongly forward peaked. For this reason, it is displayed in Fig. 1 multiplied by $\sin^2(\theta/2)$. Moreover, Coulomb distortion induces a strong oscillatory behavior, again in conjunction with a large enhancement for opposite charges and strong suppression for like charges.

Finally, the spin-orbit-hyperfine interference structure function $L_2^l$ is comparable in size to $S_2$ and $S_2^-$ but exhibits quite a different pattern. It is strongly enhanced by distortion for opposite charges and possesses a pronounced broad minimum around 100°. It falls off at forward and backward angles, with many oscillations in the forward direction. With increasing kinetic energy the minimum moves toward smaller angles with fewer oscillations. Like the diagonal structure function $L_0^h$, the hadronic interference structure function $L_2^h$ is quite small.

### B. The triple-polarization cross section

Now I discuss the c.m. triple-polarization cross section of Eq. (22). Previously, in Ref. [3] only the hyperfine amplitude besides the Coulomb one was considered, whereas the hadron spin-orbit interaction was already included in Ref. [4]. However, as mentioned above, its contribution to the helicity-dependent part of the spin-flip cross section in Eq. (29) is negligible, whereas in the helicity-independent part the diagonal contribution $L_0^h$ is comparable in size to $S_2$ in the forward direction. Much more important is the leptonic spin-orbit contribution, which, however, appears in the non-spin-flip cross section only [see Eq. (30)], where it is buried completely by the Coulomb contribution $S_C$.

The results for the c.m. spin-flip cross section for parallel and opposite initial spin orientations of hadron and lepton is shown in Fig. 2. One notes again the strong influence of Coulomb distortion. Furthermore, the leptonic spin-orbit interaction plays a relatively important role in the region of the minimum as can also be seen in Fig. 2 by a comparison with the curves labeled “hfs” for which the spin-orbit interaction is switched off. One readily notes a substantial increase when the spin-orbit part is included compared to the pure hyperfine case. Furthermore, the spin-orbit interaction induces oscillations, in particular, in the forward direction. The difference in the two spin-flip cross sections determines the net hadron polarization in a storage ring of initially unpolarized hadrons scattered at polarized leptons.

The non-spin-flip cross section in Eq. (30) is overwhelmingly dominated by the Coulomb contribution $S_C$. The small dependence on $\lambda_l^i$ and $\lambda_h$ leads to different scattering strengths for hadron polarization parallel or antiparallel to lepton polarization [see Eq. (31)].

### C. The integrated structure functions and cross sections

Finally, I present results for the integrated c.m. structure functions and spin-flip cross sections, which are the relevant quantities for the polarization buildup in a storage ring. They are defined by the integration over the solid angle except for the small cone in the forward direction with $\theta < \theta_{\min}$, where the minimal scattering angle is defined by the requirement that the impact parameter should not exceed a given value $b$,

$$\theta_{\min} = 2\arctan(\eta_C/l), \tag{39}$$

with $l = bp$ as the classical angular momentum. In the present work I have chosen $b = 10^{10}$ fm. The choice of this value has been justified in Ref. [3]. The dependence on this parameter is discussed below. Thus for any structure function or cross section $\mathcal{O}(\theta)$, I define as an integrated quantity

$$\langle \mathcal{O} \rangle = 2\pi \int_{\theta_{\min}}^{\pi} d(\cos\theta)\mathcal{O}(\theta)\,. \tag{40}$$

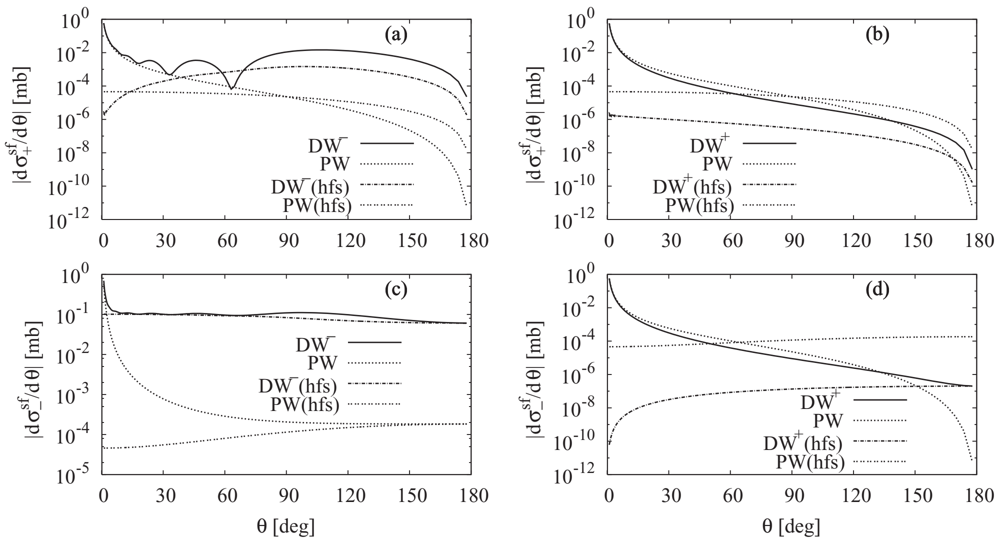


FIG. 2. Absolute value of the c.m. spin-flip cross sections $d\sigma^{\rm sf}_\pm/d\Omega$ for initial hadron polarization parallel (upper panels) and opposite (lower panels) to lepton polarization along the initial relative momentum in the plane-wave (PW) and distorted-wave (DW) approximation for like charges (DW$^+$; right panels) and opposite charges (DW$^-$; left panels) for a proton laboratory kinetic energy $T_h = 1$ keV. Curves labeled "hfs" were obtained without the spin-orbit contribution.

For the plane-wave approximation, one finds easily the following expressions:

$$\big\langle L_0^{l/h}\big\rangle = -2\pi\ V\big(c^{LS}_{l/h}\big)^2\big(\ln(\sin(\theta_{\rm min}/2)) + \tfrac{1}{2}\cos^2(\theta_{\rm min}/2)\big), \tag{41}$$

$$\big\langle L_2^{l/h}\big\rangle = -2\pi\ V c^{LS}_{l/h} c^{SS}(1-\sin^2(\theta_{\rm min}/2)(1+\cos^2(\theta_{\rm min}/2))), \tag{42}$$

$$\langle S_0\rangle = 8\pi\ V(c^{SS})^2, \tag{43}$$

$$\langle S_2^+\rangle = 4\pi\ V\frac{c^{SS}}{p^2}\big(\ln(\sin(\theta_{\rm min}/2)) - \tfrac{1}{2}\cos^2(\theta_{\rm min}/2)\big), \tag{44}$$

$$\langle S_2^-\rangle = -4\pi\ V(c^{SS})^2\cos^2(\theta_{\rm min}/2)\big(1+\sin^2(\theta_{\rm min}/2)\big), \tag{45}$$

$$\langle S_2\rangle = -4\pi\ V(c^{SS})^2\big(3-\sin^4(\theta_{\rm min}/2)\big). \tag{46}$$

Thus in PWA all of the integrated structure functions except for $\langle S_2^+\rangle$ are almost independent of $T_h$ except for a very weak dependence via the minimal scattering angle. One should note the logarithmic divergence for $\theta_{\rm min}\to 0$ in $\langle L_0^{l/h}\rangle$ and $\langle S_2^+\rangle$. It corresponds to the logarithmic divergence in the angular momentum $l$ of the partial-wave expansion noted in Ref. [4] which appears when integrating over the whole range of scattering angles.

The results for those structure functions which determine the integrated spin-flip cross section

$$\langle\sigma^{\rm sf}_\pm\rangle = 2\big\langle L_0^h\big\rangle - \langle S_2\rangle + 2 \pm \big[\langle S_2^-\rangle + \big\langle L_2^h\big\rangle\big], \tag{47}$$

are exhibited in Fig. 3. They show a strong increasing influence of Coulomb distortion with decreasing hadron kinetic energy

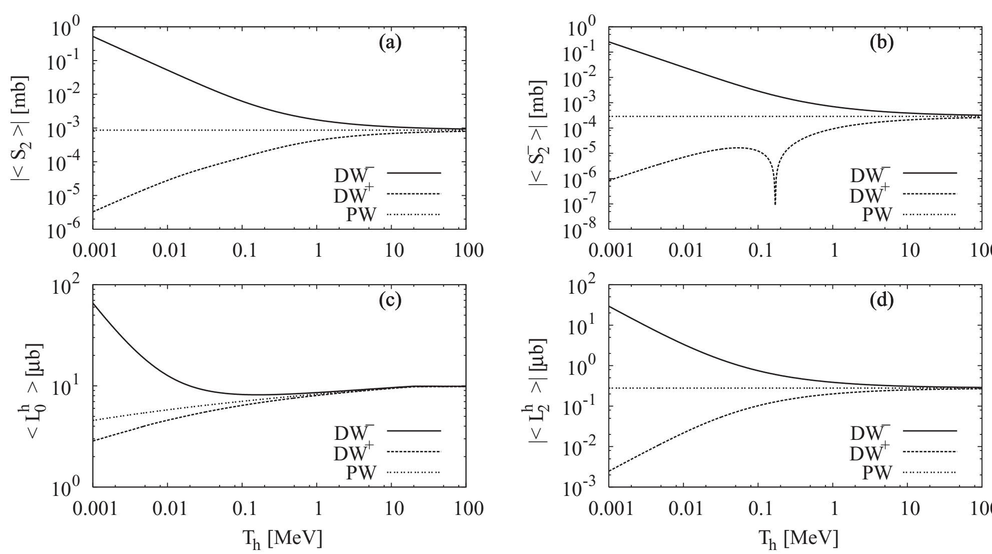


FIG. 3. Integrated structure functions $\langle S_2\rangle$ (a), $\langle S_2^-\rangle$ (b), $\langle L_0^h\rangle$ (c), and $\langle L_2^h\rangle$ (d) as a function of the proton laboratory kinetic energy $T_h$ for the plane-wave approximation (PW) and with Coulomb distortion for like charges (DW$^+$) and opposite charges (DW$^-$).

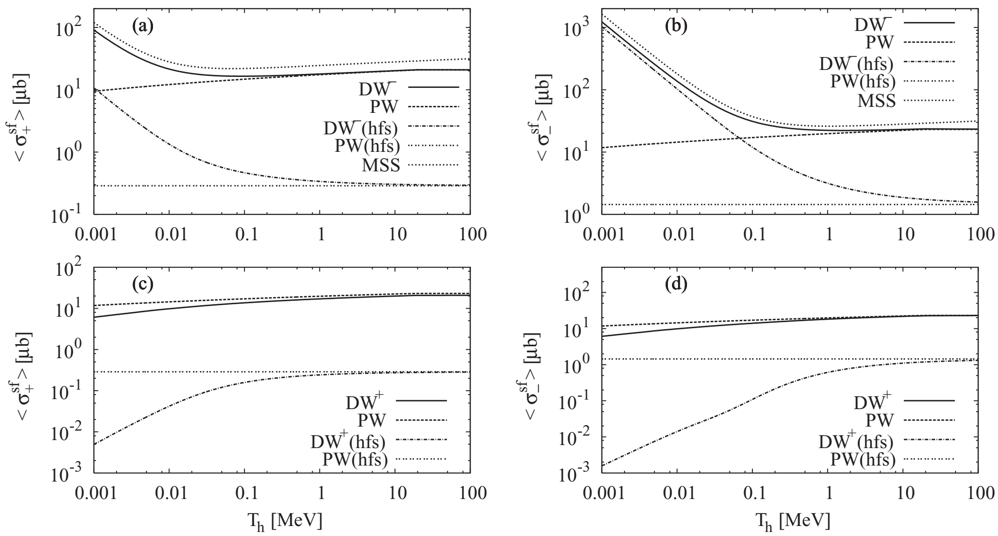


FIG. 4. Integrated spin-flip cross section $\langle\sigma_+^{\rm sf}\rangle$ (a, c) and $\langle\sigma_-^{\rm sf}\rangle$ (b, d) as a function of the proton laboratory kinetic energy $T_h$ for the plane-wave approximation (PW) and with Coulomb distortion for like charges [DW$^+$ (c, d)] and opposite charges [DW$^-$(a, b)]. Curves labeled "hfs" include the hyperfine amplitude only, and in (a) and (c) the curves labeled MSS represent the results of Ref. [4].

$T_h$ leading to large enhancements for opposite charges and strong suppression for like charges compared to the plane-wave case. The corresponding integrated spin-flip cross section of Milstein *et al.* [4] reads, according to their Eq. (21),

$$\langle\sigma_\pm^{\rm sf}\rangle = \pi\left(\frac{\alpha\mu_p}{M_p}\right)^2\left[(2\pi\eta_C)^2\left(\frac{11}{6}-\ln 2\right) + \ln(l_{\rm max}/\eta_C)^2 \mp (2\pi\eta_C)^2\right]. \tag{48}$$

The logarithmic divergence in the angular momentum $l$ is regularized by choosing a finite $l_{\rm max}$ determined by the classical relation $l_{\rm max}=bp$, which corresponds to the choice of a minimum scattering angle in the present work. The spin-flip cross sections are displayed in Fig. 4. As expected, they show, with decreasing $T_h$, a growing strong influence of Coulomb effects via hyperfine and hadronic spin-orbit interactions. The latter is only important in the spin-independent part of the spin-flip cross section, while its influence in the spin-dependent part is negligible. The results of Milstein *et al.* [4], shown in the upper panels in Fig. 4 for opposite charges, are comparable to our results but display a slight overestimation, which is probably caused by the different approximations in [4]. The dependence of the integrated spin-flip cross section for opposite charges on the regularization parameter $b$ is exhibited in Fig. 5 for $b=10^9$, $10^{10}$, and $10^{11}$ fm. It appears to be quite weak.

The relevant quantity for a polarization buildup in a storage ring is the ratio of the spin-independent part to the spin-dependent part,

$$R^{\rm sf} = \frac{\langle\sigma_+^{\rm sf}\rangle-\langle\sigma_-^{\rm sf}\rangle}{\langle\sigma_+^{\rm sf}\rangle+\langle\sigma_-^{\rm sf}\rangle} = 2\frac{\langle S_2^-\rangle+\langle L_2^h\rangle}{2\langle L_0^h\rangle-\langle S_2\rangle}, \tag{49}$$

which is shown in Fig. 6. One readily notes the reduction in this ratio by the hadronic spin-orbit interaction, in particular, it is quite strong at higher energies but only about 12% at the lowest energy. This fact clearly shows the importance of the hadronic spin-orbit interaction besides the hyperfine contribution. Again, the agreement with Ref. [4] is satisfactory.

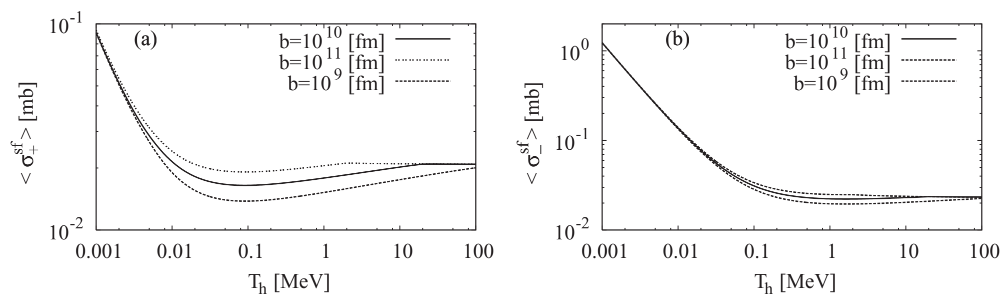


FIG. 5. Dependence of the integrated spin-flip cross sections [$\langle\sigma_+^{\rm sf}\rangle$ (a) and $\langle\sigma_-^{\rm sf}\rangle$ (b)] for opposite charges on the regularization parameter $b$.

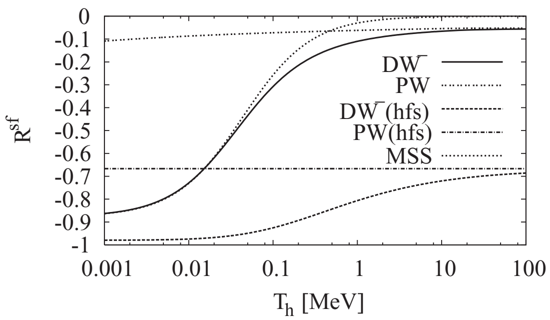


FIG. 6. Ratio of the spin-dependent part of the spin-flip cross section to its spin-independent part as a function of the proton laboratory kinetic energy $T_h$ for the plane-wave approximation (PW) and with Coulomb distortion for opposite charges: present calculation ($DW^-$) and the result from Ref. [4] (MSS). For the curves labeled "hfs," only the hyperfine amplitude is included.

## IV. CONCLUSIONS

Formal expressions for the polarization observables in electromagnetic hadron-lepton scattering have been presented within a nonrelativistic framework including the central Coulomb force as well as the lepton and hadron spin-orbit and hyperfine interactions. While the Coulomb force is included exactly, the latter have been treated in a DWA. Special emphasis has been placed on the triple-polarization cross section, with polarizations of the initial hadron and lepton and of the final hadron along the incoming hadron momentum. The structure functions which determine the differential triple-polarization cross section have been evaluated in the plane-wave approximation and DWA for hadron laboratory kinetic energies between 1 keV and 100 MeV.

For evaluation of the spin-dependent spin-orbit and hyperfine interactions with Coulomb distortion, two methods have been employed: (i) an integral representation of the nonrelativistic Coulomb scattering wave function and (ii) a partial-wave expansion. These two independent methods have served as a mutual check for the numerical accuracy of the results.

As expected, the distortion effects are very important at low energies in small polarization observables, which are driven by spin-orbit and hyperfine interactions, leading to sizable enhancements for opposite charges and suppressions for like charges according to the Coulomb attraction or repulsion. This is shown in detail for the structure functions of the triple-polarization cross section and for the special case of the spin-flip differential cross sections.

The leptonic spin-orbit interactions plays a minor role in the non-spin-flip cross section in its spin-dependent part, which, however, as a whole is smaller by many orders of magnitude than the spin-independent part, dominated by the Coulomb term $S_C$. The influence of the spin-orbit and hyperfine interactions on the unpolarized cross section is almost negligible in the whole range of energies studied here.

With respect to the integrated spin-flip cross sections, our previous work has been extended by the inclusion of the hadronic spin-orbit interaction, which shows a non-negligible effect on the spin-independent part, notably changing the ratio of the integrated strength of its spin-dependent to that of the spin-independent part.

## ACKNOWLEDGMENTS

I would like to thank Thomas Walcher for his continued interest in this work and for many useful discussions. This work was supported by SFB 443 of the Deutsche Forschungsgemeinschaft.

## APPENDIX A: THE GENERAL SCATTERING CROSS SECTION

Evaluation of the trace in Eq. (2) yields the general expression

$$\mathcal{O}\big(\vec{P}_h^f, \vec{P}_l^f, \vec{P}_h^i \vec{P}_l^i; \theta, \phi\big) = \sum_{\alpha,\beta\in\{C,LS_l,LS_h,SS\}} S_{\alpha,\beta}(\theta,\phi), \tag{A1}$$

where the various contributions are defined by

$$S_{\alpha,\beta}(\theta,\phi) = \frac{M_l^2 M_h^2}{\pi^2 W^2\big(1+|\vec{P}_l^f|\big)\big(1+|\vec{P}_h^f|\big)} \times \mathrm{Trace}\big[T_\alpha^\dagger \rho_h^f\big(\vec{P}_h^f\big)\rho_l^f\big(\vec{P}_l^f\big)T_\beta \rho_h^i\big(\vec{P}_h^i\big)\rho_l^i\big(\vec{P}_l^i\big)\big], \tag{A2}$$

with $T_\alpha$ defined in Eq. (12). One should note the relation $S_{\alpha,\beta} = S^*_{\beta,\alpha}$, from which it follows that $S_\alpha := S_{\alpha,\alpha}$ is real. Separating the diagonal contributions ($S_\alpha$) from the interference terms ($S_{\alpha,\beta}$ for $\alpha \neq \beta$), one obtains for the "quadruple-polarization" cross section

$$\frac{d\sigma_{\vec{P}_h^f,\vec{P}_l^f,\vec{P}_h^i,\vec{P}_l^i}(\theta,\phi)}{d\Omega} = \sum_{\alpha\in\{C,LS_l,LS_h,SS\}} S_\alpha(\theta,\phi) + \sum_{\alpha<\beta\in\{C,LS_l,LS_h,SS\}} 2\mathrm{Re}S_{\alpha,\beta}(\theta,\phi). \tag{A3}$$

Explicitly, one finds, in terms of the different contributions to the $T$ matrix in Eq. (11), for the various diagonal terms

$$S_C(\theta,\phi) = V_0\,|a_C|^2\,\Pi_h^+\Pi_l^+, \tag{A4}$$

$$S_{LS_{l/h}}(\theta,\phi) = V_0\,\Pi_{h/l}^+\big(\Pi_{l/h}^-\,\vec{b}_{l/h}^*\cdot\vec{b}_{l/h} + 2\mathrm{Re}\big[\big(\vec{b}_{l/h}\cdot\vec{P}_{l/h}^f\big)^*\big(\vec{b}_{l/h}\cdot\vec{P}_{l/h}^i\big)\big]\big), \tag{A5}$$

$$\begin{aligned} S_{SS}(\theta,\phi) = V_0\,\Big(&\Pi_h^-\Pi_l^- D_0 - \vec{P}_h^-\cdot\overleftrightarrow{G}\cdot\vec{P}_l^- \\ &+ \big[\Pi_h^-\vec{P}_l^f\cdot\overleftrightarrow{D}\cdot\vec{P}_l^i + (l\leftrightarrow h)\big] \\ &- \mathrm{Im}\Big[\Pi_h^-\big(\vec{P}_l^-\cdot\vec{H}\big) \\ &+ 2\sum_{jst} P_{l,j}^- P_{h,s}^f P_{h,t}^i\,E_{jst} + (l\leftrightarrow h)\Big] \\ &+ 2\mathrm{Re}\big[\big(\vec{P}_l^f\cdot\overleftrightarrow{d}^*\cdot\vec{P}_h^f\big)\big(\vec{P}_l^i\cdot\overleftrightarrow{d}\cdot\vec{P}_h^i\big) \\ &+ \big(\vec{P}_l^i\cdot\overleftrightarrow{d}^*\cdot\vec{P}_h^f\big)\big(\vec{P}_h^i\cdot\overleftrightarrow{d}\cdot\vec{P}_l^f\big)\big]\Big), \end{aligned} \tag{A6}$$

where $V_0 = 4\alpha^2 M_l^2 M_h^2/[W^2(1+|\vec{P}_l^f|)(1+|\vec{P}_h^f|)]$, and the following quantities depend on the polarization parameters:

$$\Pi^\pm_{h/l} = 1 \pm \vec{P}^f_{h/l}\cdot\vec{P}^i_{h/l}, \quad \vec{P}^\pm_{h/l} = \vec{P}^i_{h/l} \pm \vec{P}^f_{h/l},$$
$$\vec{Q}_{h/l} = \vec{P}^+_{h/l} - i\,\vec{P}^f_{h/l}\times\vec{P}^i_{h/l}. \tag{A7}$$

Furthermore, for convenience I have introduced the following quantities, which depend on the hyperfine interaction tensor $d_{ij}$:

$$D_{ij} = 2\mathrm{Re}\left(\sum_k d^*_{ik}d_{kj}\right), \quad D_0 = \sum_{ij} d^*_{ij}d_{ji} = \frac{1}{2}\mathrm{Trace}(\overset{\leftrightarrow}{D}), \tag{A8}$$

$$E_{jst} = \sum_{kl}\varepsilon_{jkl}\,d^*_{ks}d_{lt}, \quad G_{ij} = \sum_{lmst}\varepsilon_{ils}\,\varepsilon_{jmt}\,d^*_{lm}d_{st}\,,$$
$$H_i = \sum_{klm}\varepsilon_{ikl}\,d^*_{km}d_{ml}, \tag{A9}$$

where $\varepsilon_{ikl}$ denotes the totally antisymmetric Levi-Civita tensor in three dimensions. These functions depend on the scattering angles $(\theta,\phi)$. Correspondingly, one finds for the interference terms

$$S_{C,LS_{l/h}}(\theta,\phi) = V_0\,a^*_C\Pi^+_{h/l}\,\vec{b}_{l/h}\cdot\vec{Q}_{l/h}, \tag{A10}$$

$$S_{C,SS}(\theta,\phi) = V_0\,a^*_C\,\vec{Q}_h\cdot\overset{\leftrightarrow}{d}\cdot\vec{Q}_l, \tag{A11}$$

$$S_{LS_l,LS_h}(\theta,\phi) = V_0\,(\vec{b}_l\cdot\vec{Q}_l)^*(\vec{b}_h\cdot\vec{Q}_h), \tag{A12}$$

$$S_{LS_{l/h},SS}(\theta,\phi) = V_0\,\vec{Q}_{h/l}\cdot\overset{\leftrightarrow}{d}\cdot\big(\Pi^-_{l/h}\vec{b}^*_{l/h} - i\,\vec{b}^*_{l/h}\times\vec{P}^-_{l/h} + \big(\vec{b}^*_{l/h}\cdot\vec{P}^i_{l/h}\big)\,\vec{P}^f_{l/h} + \big(\vec{b}^*_{l/h}\cdot\vec{P}^f_{l/h}\big)\,\vec{P}^i_{l/h}\big). \tag{A13}$$

Using the separation of the tensor $\overset{\leftrightarrow}{d}$ into a scalar and a traceless symmetric tensor according to Eq. (13), one finds

$$D_0 = 3|d_0|^2 + \sum_{i,k} d^{[2]*}_{ik}d^{[2]}_{ki},$$
$$D_{ij} = 2|d_0|^2\delta_{ij} + 4\mathrm{Re}\big(d^*_0 d^{[2]}_{ij}\big) + 2\mathrm{Re}\sum_k d^{[2]*}_{ik}d^{[2]}_{kj}, \tag{A14}$$

$$E_{jst} = \varepsilon_{jst}\,|d_0|^2 + \sum_{kl}\varepsilon_{jkl}\,d^{[2]*}_{ks}d^{[2]}_{lt} + \left(\sum_l \varepsilon_{jsl}d^*_0 d^{[2]}_{lt} - (s\leftrightarrow t)^*\right), \tag{A15}$$

$$G_{ij} = 2|d_0|^2\delta_{ij} - 2\mathrm{Re}\big(d^*_0 d^{[2]}_{ij}\big) + \sum_{lmst}\varepsilon_{ils}\varepsilon_{jmt}d^{[2]*}_{lm}d^{[2]}_{st},$$
$$H_i = \sum_{klm}\varepsilon_{ikl}\,d^{[2]*}_{km}d^{[2]}_{ml}. \tag{A16}$$

It is now easy to see that the vector $\vec{H}$ is purely imaginary and that the tensor $G_{ij}$ is real and symmetric. Furthermore, one notes the symmetry property $E^*_{jst} = -E_{jts}$. It suffices to evaluate the spin-orbit vector $\vec{b}$ and the hyperfine tensor $\overset{\leftrightarrow}{d}$ for $\phi = 0$, because then the values for an arbitrary $\phi$ can be generated by a rotation around the $z$ axis exploiting their rotation properties. For further details see Ref. [8].

### 1. The triple-polarization cross section

I now specialize to the case where only the final lepton polarization is not analyzed, i.e., $\vec{P}_l^f = 0$, but all other particles are completely polarized ($|\vec{P}_h^{i/f}| = 1,\ |\vec{P}_l^i| = 1$). This case is of particular interest for polarization transfer in a storage ring [3]. The corresponding "triple-polarization" cross section has the form

$$\frac{d\sigma^{\mathrm{triple}}_{\vec{P}^f_h,\vec{P}^i_h\vec{P}^i_l}(\theta,\phi)}{d\Omega} = \mathcal{O}\big(\vec{P}^f_h,\vec{0},\vec{P}^i_h\vec{P}^i_l;\theta,\phi\big) = \sum_{\alpha\in\{C,LS_l,LS_h,SS\}} S^{\mathrm{triple}}_\alpha(\theta,\phi) + \sum_{\alpha<\beta\in\{C,LS_l,LS_h,SS\}} 2\mathrm{Re}S^{\mathrm{triple}}_{\alpha,\beta}(\theta,\phi). \tag{A17}$$

In this case, the lepton polarization quantities in (A7) become $\Pi^\pm_l = 1$, $\vec{Q}_l = \vec{P}^i_l$, $\vec{P}^\pm_l = \vec{P}^i_l$, and one finds for the diagonal terms

$$S^{\mathrm{triple}}_C(\theta) = V\,\Pi^+_h|a_C(\theta)|^2, \tag{A18}$$

$$S^{\mathrm{triple}}_{LS_l}(\theta,\phi) = V\,\Pi^+_h\,\vec{b}^*_l\cdot\vec{b}_l, \tag{A19}$$

$$S^{\mathrm{triple}}_{LS_h}(\theta,\phi) = V\,\big(\Pi^-_h\,\vec{b}^*_h\cdot\vec{b}_h + 2\mathrm{Re}\big[(\vec{b}_h\cdot\vec{P}^f_h)^*\big(\vec{b}_h\cdot\vec{P}^i_h\big)\big]\big), \tag{A20}$$

$$S^{\mathrm{triple}}_{SS}(\theta,\phi) = V\big(\Pi^-_h D_0 - \vec{P}^-_h\cdot\overset{\leftrightarrow}{G}\cdot\vec{P}^i_l + \vec{P}^f_h\cdot\overset{\leftrightarrow}{D}\cdot\vec{P}^i_h - \mathrm{Im}\big[\vec{P}^-_h\cdot\vec{H} + \Pi^-_h\big(\vec{P}^i_l\cdot\vec{H}\big) + 2\sum_{jst}P^i_{l,j}P^f_{h,s}P^i_{h,t}\,E_{jst}\big]\big), \tag{A21}$$

and for the interference terms

$$S^{\mathrm{triple}}_{C,LS_l}(\theta,\phi) = V\,a^*_C\Pi^+_h\,\vec{b}_l\cdot\vec{P}^i_l, \tag{A22}$$

$$S^{\mathrm{triple}}_{C,LS_h}(\theta,\phi) = V\,a^*_C\vec{b}_h\cdot\vec{Q}_h, \tag{A23}$$

$$S^{\mathrm{triple}}_{C,SS}(\theta,\phi) = V\,a^*_C\,\vec{Q}_h\cdot\overset{\leftrightarrow}{d}\cdot\vec{P}^i_l, \tag{A24}$$

$$S^{\mathrm{triple}}_{LS_l,LS_h}(\theta,\phi) = V\,\big(\vec{b}_l\cdot\vec{P}^i_l\big)^*\,(\vec{b}_h\cdot\vec{Q}_h), \tag{A25}$$

$$S^{\mathrm{triple}}_{LS_l,SS}(\theta,\phi) = V\,\vec{Q}_h\cdot\overset{\leftrightarrow}{d}\cdot\big(\vec{b}^*_l - i\vec{b}^*_l\times\vec{P}^l_i\big), \tag{A26}$$

$$S^{\mathrm{triple}}_{LS_h,SS}(\theta,\phi) = V\vec{P}^i_l\cdot\overset{\leftrightarrow}{d}\cdot\big(\Pi^-_h\vec{b}^*_h - i\,\vec{b}^*_h\times\vec{P}^-_h + \big(\vec{b}^*_h\cdot\vec{P}^i_h\big)\vec{P}^f_h + \big(\vec{b}^*_h\cdot\vec{P}^f_h\big)\vec{P}^i_h\big), \tag{A27}$$

where $V = \frac{2\alpha^2 M_l^2 M_h^2}{W^2}$.

## APPENDIX B: EVALUATION OF HYPERFINE AND SPIN-ORBIT INTERACTION IN DWBA

For evaluation of the spin-orbit interaction amplitude $\vec{b}_{l/h}$ and the hyperfine tensor amplitude $d^{[2]}$ with Coulomb distortion in a DWBA, two methods have been applied: (i) an integral representation for the Coulomb scattering wave function and (ii) a partial-wave expansion of the Coulomb wave function. For convenience, in this Appendix I set $\eta = \eta_C$.

### 1. Integral representation

In Ref. [1] a detailed description of this method for the evaluation of $d^{[2]}$ has been given. Therefore, I only summarize the result. The method is based on an integral representation of the confluent hypergeometric function as proposed in Ref. [7]. With the help of this representation, the hyperfine tensor $d^{[2]}_{ij}$ and the spin orbit vector $\vec{b}_{l/h}$ can be expressed as two-dimensional integrals, as described below.

#### *a. Hyperfine interaction*

For the hyperfine tensor one finds

$$d^{[2]}_{ij}(\eta) = c^{SS}\, N(\eta)\Bigg[\widetilde{d}^{[2]}_{ij}(1,\eta) + i\eta\int_0^1 \frac{dt}{1-t}\, e^{-i\eta\ln(1-t)} \times \left(\widetilde{d}^{[2]}_{ij}(1,\eta) - e^{i\eta\ln t}\widetilde{d}^{[2]}_{ij}(t,\eta)\right)\Bigg], \tag{B1}$$

where $N(\eta) = e^{-\pi\eta}\sinh(\pi\eta)/\pi\eta$ is a normalization factor and

$$\widetilde{d}^{[2]}_{ij}(t,\eta) = A_{ij}(t,1) + i\eta\int_0^1 \frac{dt'}{1-t'}\, e^{-i\eta\ln(1-t')} \times (A_{ij}(t,1) - e^{i\eta\ln t'}A_{ij}(t,t')). \tag{B2}$$

Here I have introduced the tensor

$$A_{ij}(t,t') = (3\hat{a}_i(t,t')\,\hat{a}_j(t,t') - \delta_{ij})I_{SS}(c(t,t')), \quad \text{with}$$
$$\hat{\vec{a}}(t,t') = \frac{\vec{p}\,t - \vec{p}\,'t'}{p[t^2+t'^2-2tt'\cos\theta]^{1/2}}, \tag{B3}$$

and $I_{SS}(c)$ denotes the integral

$$I_{SS}(c) = \int_0^\infty \frac{dx}{x}\, e^{icx}\, j_2(x) = \frac{1}{3} - \frac{1}{2}c^2 - \frac{1}{4}c(1-c^2) \times \left(\ln\left|\frac{c+1}{c-1}\right| - i\pi\,\Theta(1-c)\right), \tag{B4}$$

with $\Theta(x)$ as the Heaviside step function and $c(t,t') = (2-t-t')/[t^2+t'^2-2tt'\cos\theta]^{1/2}$. One should note that $d^{[2]}_{ij}(\eta)$ and $\widetilde{d}^{[2]}_{ij}(t,\eta)$ are also functions in $\theta$ and $\phi$, the scattering angles in the c.m. frame. However, as mentioned above, it suffices to choose $\phi = 0$. The remaining integrations over $t$ and $t'$ in Eqs. (B1) and (B2) are done numerically. Details are presented in Refs. [1] and [8]. In particular, the numerical problems arising for large negative $\eta$ are discussed in Ref. [8].

#### *b. Spin-orbit interaction*

Following the analogous steps for the spin-orbit interaction, one finds

$$\vec{b}_{l/h}(\theta,\phi) = i\, b_0^{l/h}(\eta,\theta)\frac{\vec{p}\,'\times\vec{p}}{|\vec{p}\,'\times\vec{p}|}, \tag{B5}$$

where

$$b_0(\eta,\theta) = \sin\theta\, c^{LS}\, N(\eta)\Bigg[\widetilde{b}_0(\theta,1,\eta) + i\eta\int_0^1 \frac{dt}{1-t}\, e^{-i\eta\ln(1-t)} \times (\widetilde{b}_0(\theta,1,\eta) - e^{i\eta\ln t}t\,\widetilde{b}_0(\theta,t,\eta))\Bigg], \tag{B6}$$

$$\widetilde{b}_0(\theta,t,\eta) = H_{LS}(c(t,1)) + i\eta\int_0^1 \frac{dt'}{1-t'}\, e^{-i\eta\ln(1-t')} \times \big(H_{LS}(c(t,1)) - e^{i\eta\ln t'}H_{LS}(c(t,t'))\big), \tag{B7}$$

with $H_{LS}(c(t,t')) = t'I_{LS}(c(t,t'))/[t^2+t'^2-2tt'\cos\theta]$ and the radial integral

$$I_{LS}(c) = \int_0^\infty \frac{dx}{x}\, e^{icx}\, j_1(x) = 1 - \frac{c}{2}\ln\left|\frac{c+1}{c-1}\right| + i\frac{\pi c}{2}\,\Theta(1-c). \tag{B8}$$

The numerical evaluation is analogous to the one for the hyperfine interaction. For $\eta = 0$ one finds $b_0(0,\theta) = c^{LS}\sin\theta/(4\sin^2\theta/2)$, in agreement with Eq. (15).

### 2. Partial-wave expansion

The expansion of the Coulomb wave function into partial waves used in Ref. [4] reads [13]

$$\psi^{(+)}_{\vec{p}}(\vec{r}\,) = \frac{4\pi}{pr}\sum_{l,m} i^l e^{i\bar{\sigma}_l}\, F_l(\eta,pr)Y^*_{lm}(\hat{r})\,Y_{lm}(\hat{p}), \tag{B9}$$

where the radial function $F_l$ is given in terms of the confluent hypergeometric function ${}_1F_1(a,b,z)$ according to

$$F_l(\eta,\rho) = \frac{2^l}{(2l+1)!}e^{-\frac{\pi}{2}\eta}|\Gamma(l+1+i\eta)|\,e^{i\rho}\,\rho^{l+1} \times F_1(l+1+i\eta,2l+2,-2i\rho)\,. \tag{B10}$$

In the above expression, I have separated the $l=0$ phase $\sigma_0 = \sigma_C$ for convenience. The remaining partial-wave phase is given by $\bar{\sigma}_l = \sigma_l - \sigma_0$, where for $l > 0$,

$$e^{i\bar{\sigma}_l} = \frac{l+i\eta}{|l+i\eta|}\cdots\frac{1+i\eta}{|1+i\eta|}. \tag{B11}$$

Evaluation of the various contributions to the scattering matrix as listed in Eqs. (19) and (21) leads to the following expressions, which are still operators in spin space,

$$\vec{b}^{\,\mathrm{DW}}_{l/h}\cdot\vec{\sigma}_{l/h} = \sum_{l=1}^{\infty} G^l_{LS_{l/h}}\vec{\Omega}_{ll}(\hat{p}',\hat{p})\cdot\vec{\sigma}_{l/h}, \tag{B12}$$

$$\sum_{ij}\sigma_{l,i}d^{[2]}_{ij}\sigma_{h,j} = \sum_{l'=0}^{\infty}\sum_{l=0}^{\infty}G^{l'l}_{SS,2}\big[\Sigma^{[2]}(\vec{\sigma}_l,\vec{\sigma}_h)\times\Omega^{[2]}_{l'l}\big]^{[0]}(\hat{p}',\hat{p}), \tag{B13}$$

where, for convenience, I have introduced, in the notation of Fano and Racah [12] for irreducible spherical tensors,

$$\Omega^{[K]}_{l'l}(\hat{p}',\hat{p}) = [Y^{[l']}(\hat{p}')\times Y^{[l]}(\hat{p})]]^{[K]},$$
$$\Sigma^{[2]}(\vec{\sigma}_l,\vec{\sigma}_h) = \big[\sigma^{[1]}_l\times\sigma^{[1]}_h\big]^{[2]}. \tag{B14}$$

The coefficients are given in terms of the radial matrix elements $R_{l'l}$

$$G^l_{LS_{l/h}}(\eta) = (-)^{l+1}\frac{4\pi}{\sqrt{3}}c^{LS}_{l/h}\hat{l}\sqrt{l(l+1)}e^{2i\bar{\sigma}_l}\,R_{ll}, \tag{B15}$$

$$G^{l'l}_{SS,2}(\eta) = i^{l-l'} 16\pi\sqrt{6}\, c^{SS}\hat{l}'\hat{l}\, e^{i(\bar{\sigma}_{l'}+\bar{\sigma}_l)} \begin{pmatrix} l' & l & 2 \\ 0 & 0 & 0 \end{pmatrix} R_{l'l}, \tag{B16}$$

with

$$\begin{aligned} R_{l'l} &= \frac{4}{p^2}\int_0^\infty \frac{dr}{r^3}\, F_{l'}(\eta, pr) F_l(\eta, pr) \\ &= 4\int_0^\infty \frac{d\rho}{\rho^3}\, F_{l'}(\eta,\rho)F_l(\eta,\rho). \end{aligned} \tag{B17}$$

Besides the matrix elements $R_{ll}$, only $R_{l'l} = R_{ll'}$ for $|l - l'| = 2$ are needed in view of the selection rule of the $3j$ symbol in Eq. (B16). This type of radial matrix element is well known in Coulomb excitation (see, e.g., Ref. [14]) and is also derived in Ref. [4]. For $l' = l$ and $l > 0$ one has

$$R_{ll} = \frac{2}{l(l+1)}\left(1 + \frac{f_l(\eta)}{2l+1}\right)$$

$$\text{with}\quad f_l(\eta) = e^{-\pi\eta}\frac{\pi\eta}{\sinh(\pi\eta)} - 1 - 2\eta^2\sum_{k=1}^{l}\frac{1}{k^2+\eta^2}. \tag{B18}$$

One should note that $f_l$ vanishes for $\eta = 0$. For $|l' - l| = 2$ one has

$$R_{l,l+2} = \frac{2}{3|l+1+i\eta||l+2+i\eta|}. \tag{B19}$$

#### *a. The hyperfine contribution*

The tensor amplitude $d^{[2]}_{ij}$ of the hyperfine interaction is obtained by separating the spin dependence in Eq. (B13),

$$d^{[2]}_{ij} = \sum_{l'=0}^\infty\sum_{l=0}^\infty G^{l'l}_{SS,2}\frac{\partial^2}{\partial\sigma_{l,i}\partial\sigma_{h,j}}\left[\Sigma^{[2]}(\vec{\sigma}_l,\vec{\sigma}_h)\times\Omega^{[2]}_{l'l}\right]^{[0]}(\hat{p}',\hat{p}). \tag{B20}$$

It suffices to consider $d^{[2],0}_{ij}$ for the special case, for which the scattering plane coincides with the $x$-$z$ plane, i.e., $\phi = 0$. A straightforward evaluation (for details see Ref. [8]) yields for the nonvanishing components

$$d^{[2],0}_{33} = c^{SS}\sum_{l=0}^\infty S^{33}_l\, P_l(\cos\theta), \tag{B21}$$

$$d^{[2],0}_{11/22} = \pm c^{SS}\sum_{l=2}^\infty S^{11}_l\, P^2_l(\cos\theta) - \frac{1}{2}d^{[2],0}_{33}, \tag{B22}$$

$$d^{[2],0}_{13} = c^{SS}\sum_{l=1}^\infty S^{13}_l\, P^1_l(\cos\theta), \tag{B23}$$

with

$$\begin{aligned} S^{33}_l &= \frac{1}{2}\,S^0_l, \quad S^{11}_l = \frac{1}{2}\sqrt{\frac{3(l-2)!}{2(l+2)!}}\,S^2_l, \\ S^{13}_l &= -\frac{1}{2}\sqrt{\frac{3(l-1)!}{2(l+1)!}}\,S^1_l, \end{aligned} \tag{B24}$$

and where for $m = 0, 1, 2,$

$$\begin{aligned} S^m_l &= (-i)^l\hat{l}^2 e^{i\bar{\sigma}_l}\sum_{k=|l-2|}^{l+2} i^k\hat{k}^2 e^{i\bar{\sigma}_k} \\ &\quad\times\begin{pmatrix} l & k & 2 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & k & 2 \\ -m & 0 & m\end{pmatrix} R_{lk}. \end{aligned} \tag{B25}$$

It is useful to separate the $\eta$-independent contributions, constituting the plane-wave approximation. One finds explicitly the detailed expressions

$$d^{[2],0}_{33}(\eta) = c^{SS}\left(\sin^2(\theta/2) - \frac{1}{3} + \sum_{l=0}^\infty \widetilde{S}^{33}_l(\eta)\, P_l(\cos\theta)\right), \tag{B26}$$

$$\begin{aligned} d^{[2],0}_{11/22}(\eta) &= \pm c^{SS}\left(\frac{1}{2}\cos^2(\theta/2) + \sum_{l=2}^\infty \widetilde{S}^{11}_l(\eta)\, P^2_l(\cos\theta)\right) \\ &\quad - \frac{1}{2}d^{[2],0}_{33}, \end{aligned} \tag{B27}$$

$$d^{[2],0}_{13}(\eta) = c^{SS}\left(-\frac{1}{2}\sin(\theta) + \sum_{l=1}^\infty \widetilde{S}^{13}_l(\eta)\, P^1_l(\cos\theta)\right), \tag{B28}$$

where the coefficients $\widetilde{S}^{ij}_l(\eta)$ vanish for $\eta = 0$. In detail one finds for $i = j = 3$ and $l = 0, 1$

$$\widetilde{S}^{33}_0(\eta) = \frac{i\eta}{3}\,\frac{3 - i\eta}{(1-i\eta)(2-i\eta)}, \tag{B29}$$

$$\begin{aligned} \widetilde{S}^{33}_1(\eta) &= -\frac{3}{5}\left[\frac{1+i\eta}{1-i\eta}\,\frac{f_1(\eta)}{3} - \frac{i\eta(5-i\eta)}{6(2-i\eta)(3-i\eta)}\right. \\ &\quad\left. - \frac{2i\eta}{1-i\eta}\left(1 - \frac{1}{(2-i\eta)(3-i\eta)}\right)\right], \end{aligned} \tag{B30}$$

and for $l > 1$

$$\widetilde{S}^{33}_l(\eta) = -e^{2i\bar{\sigma}_l}\left[\frac{i\eta}{2}(b_l(\eta) - b_{l+2}(\eta)^*) + \frac{f_l(\eta)}{(2l-1)(2l+3)}\right]. \tag{B31}$$

Here I have introduced, for convenience,

$$b_l(\eta) = \frac{2l-1+i\eta}{(2l-1)(l-1)l(l-1+i\eta)(l+i\eta)}. \tag{B32}$$

One should note that the coefficients $\widetilde{S}^{33}_l$ behave as $1/l^2$ for $l\to\infty$. For $i = j = 1$ one obtains (note that $l > 1$)

$$\begin{aligned} \widetilde{S}^{11}_l(\eta) &= \frac{(2l+1)(1-e^{2i\bar{\sigma}_l})}{2(l-1)l(l+1)(l+2)} + e^{2i\bar{\sigma}_l}\left(\frac{i\eta}{4}(b_l(\eta)\right. \\ &\quad\left. - b_{l+2}(\eta)^*) - \frac{3f_l(\eta)}{2l(l+1)(2l-1)(2l+3)}\right). \end{aligned} \tag{B33}$$

The coefficient $\widetilde{S}^{11}_l$ behaves as $l^{-3}$ for $l\to\infty$. Finally, for $\widetilde{S}^{13}_l$ one obtains for $l = 1$

$$\begin{aligned} \widetilde{S}^{13}_1(\eta) &= \frac{3}{20}\left[\frac{1+i\eta}{1-i\eta}\,f_1(\eta) + \frac{5}{3}i\eta b_3(\eta)^*\right. \\ &\quad\left. + \frac{2i\eta}{1-i\eta}\left(3 + \frac{2}{(2-i\eta)(3-i\eta)}\right)\right], \end{aligned} \tag{B34}$$

and for $l > 1$

$$\widetilde{S}_l^{13}(\eta) = \frac{1}{2} e^{2i\bar{\sigma}_l}\left(\frac{i\eta}{4}(b_l(\eta) + b_{l+2}(\eta)^*) + \frac{3 f_l(\eta)}{l(l+1)(2l-1)(2l+3)}\right). \tag{B35}$$

The coefficient $\widetilde{S}_l^{13}$ behaves like $l^{-3}$ for $l \to \infty$. The convergence of the partial-wave series is quite good as demonstrated in Ref. [8] for $\eta = 2$.

#### *b. The spin-orbit contribution*

Acording to Eq. (B12) the spin-orbit strength is given by

$$\vec{b}_{l/h} = \sum_{l=1}^{\infty} G^l_{LS_{l/h}} \vec{\Omega}_{ll}(\hat{p}', \hat{p}). \tag{B36}$$

For the chosen reference frame one obtains the spin-orbit vector $\vec{b}_{l/h}$ in the form

$$\vec{b}_{l/h} = i\, b_0^{l/h} \frac{\vec{p}\,' \times \vec{p}}{|\vec{p}\,' \times \vec{p}\,|} \quad \text{with}$$
$$b_0^{l/h}(\eta, \theta) = -\frac{1}{2} c^{LS}_{l/h} \sum_{l=1}^{\infty} \beta_l(\eta) P_l^1(\cos\theta), \tag{B37}$$

where

$$\beta_l(\eta) = \frac{1}{2}\hat{l}^2 e^{2i\bar{\sigma}_l} R_{ll} = \frac{e^{2i\bar{\sigma}_l}}{l(l+1)}\big(2l + 1 + f_l(\eta)\big). \tag{B38}$$

This form is not well suited for a numerical evaluation, because even for $\eta = 0$ the sum extends up to infinity. Therefore, it is more advantageous to separate the $\eta$-independent part, writing

$$\beta_l(\eta) = \frac{2l+1}{l(l+1)} + \beta_l^\eta \quad \text{with}$$
$$\beta_l^\eta = \frac{e^{2i\bar{\sigma}_l}}{l(l+1)}((2l+1)(e^{2i\bar{\sigma}_l} - 1) + f_l(\eta)). \tag{B39}$$

The coefficient $\beta_l^\eta$ vanishes for $\eta = 0$. For the $\eta$-independent part one can evaluate the sum by using

$$P_l^1(x) = \frac{l(l+1)}{\hat{l}^2\sqrt{1-x^2}}\big(P_{l+1}(x) - P_{l-1}(x)\big), \tag{B40}$$

and one finds for $b_0^{l/h}(\eta, \theta)$

$$b_0^{l/h}(\eta, \theta) = \frac{1}{2} c^{LS}_{l/h}\left(\cot(\theta/2) - \sum_{l=1}^{\infty}\beta_l^\eta P_l^1(\cos\theta)\right), \tag{B41}$$

yielding $b_0^{l/h}(0, \theta)$ in accordance with Eq. (15). One can evaluate this expression directly or also rearrange the remaining sum using Eq. (B40), yielding

$$b_0^{l/h}(\eta, \theta) = \frac{1}{2} c^{LS}_{l/h}\left[\cot(\theta/2) - \frac{1}{\sin\theta}\sum_{l=0}^{\infty} e_l(\eta)\, P_l(\cos\theta)\right]. \tag{B42}$$

The coefficients $e_l(\eta)$ can be found in Ref. [8]. For $l \to \infty$ the coefficients $\beta_l^\eta$ and $e_l$ behave as $1/l$, resulting in a considerably slower convergence than for the hyperfine amplitude. Examples are presented in Ref. [8].


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