Neutrino emissivity of $^3P_2$-$^3F_2$ superfluid cores in neutron stars

L. B. Leinson

Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation RAS, 142190 Troitsk, Moscow Region, Russia
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The influence of the admixture of the $^3F_2$ state onto collective spin oscillations and neutrino emission processes in the triplet superfluid neutron liquid is studied in the BCS approximation. The eigenmode of spin oscillations with $\omega \simeq \sqrt{58/35}\Delta$ is predicted to exist in the triplet superfluid neutron condensate besides the already known mode $\omega \simeq \Delta/\sqrt{5}$. Excitation of the high-frequency spin oscillations in the condensate occurs through the tensor interactions between quasiparticles. Neutrino energy losses through neutral weak currents are found to consist of three separate contributions caused by a recombination of broken Cooper pairs and by weak decays of the collective modes of spin oscillations. Neutrino decays of the low-frequency spin waves can play an important role in the cooling scenario of neutron stars. Weak decays of the high-frequency oscillations that occur only if the tensor forces are taken into account in the pairing interactions does not modify substantially the total energy losses. Simple expressions are suggested for the total neutrino emissivity.

I. INTRODUCTION

It is considered well established that the inner core of neutron stars contains a condensate of superfluid neutrons below the critical condensation temperature T_c. A superfluid energy gap Δ arising in the quasiparticle spectrum suppresses most of the neutrino emission mechanisms in the volume of the star [1], especially when the temperature falls substantially below the critical value. In this case the number of broken Cooper pairs rapidly decreases and leads to a strong quenching of the neutrino emission caused by pair breaking and formation (PBF) processes, which is considered as the most efficient cooling mechanism of superfluid neutron cores [2–5]. According to this scenario, at temperatures $T \simeq 0.1T_c$, the neutron star enters the epoch of a surface cooling.

The neutron superfluidity in the inner core of neutron stars is believed to arise owing to pairing of fermions into a triplet state. It is natural to expect the existence of low-frequency collective modes associated with spin fluctuations of such a condensate. Previously, spin modes have been thoroughly studied in the p-wave superfluid liquid ^{3}He with a central interaction between quasiparticles [6–9]. These results cannot be applied without revision to the triplet superfluid condensate of neutrons, where the pairing occurs mostly owing to a short-range negative spin-orbit force of the interaction in the channel of $j = 2$.

In a series of papers [10–12] we have investigated the collective spin oscillations in the $^3P_2$ superfluid neutron liquid which can be formed because of the strong attractive spin-orbit interaction between neutrons at high densities. Spin waves with the excitation energy $\omega = \Delta/\sqrt{5}$ were predicted to exist in such a superfluid condensate and it has been shown that the spin-wave decay (SWD) through neutral weak currents leads to a substantial neutrino emission at the lowest temperatures $T \ll T_c$, when all other mechanisms of the neutrino energy losses are killed by the superfluidity.

In this paper the problem is considered for the case of $^3P_2$-$^3F_2$ pairing. We consider the spin eigenmodes of the superfluid condensate in the case of pairing owing to spin-orbit

and tensor forces. The neutrino emission owing to PBF and SWD processes is calculated. The calculations are made within the BCS approximation by assuming a projection of the total angular momentum of the bound pairs $m_j = 0$ as the preferable one at supernuclear densities.

The paper is organized as follows. Section II contains some preliminary notes and outlines some of the important properties of the Green's functions and the one-loop integrals used below. We recollect the gap equations for the case of spin-orbit and tensor pairing forces. In Sec. III we derive, in the BCS approximation, the equations for anomalous three-point vertices responsible for the interaction of the neutron superfluid liquid with an external axial-vector field. In Sec. IV we discuss the renormalizations which transform the standard gap equations to a very simple form valid near the Fermi surface. In Sec. V we apply the angle average approximation to make the equations solvable analytically. In Sec. VI we analyze the poles of anomalous vertices to derive the dispersion of spin-density oscillations in the condensate. In Sec. VII we calculate the linear response of the superfluid neutron liquid onto an external axial-vector field. In Sec. VIII we derive neutrino losses caused by the recombination of broken Cooper pairs and by the decay of spin waves. In Sec. IX, we evaluate neutrino energy losses in the $^3P_2$-$^3F_2$ superfluid neutron liquid. Section X contains a short summary of our findings and the conclusion.

Throughout this paper, we use the standard model of weak interactions, the system of units $\hbar = c = 1$ and the Boltzmann constant $k_B = 1$.

II. GENERAL APPROACH AND NOTATION

A. Green's functions and loop integrals

The order parameter, $\hat{D} \equiv D_{\alpha\beta}$, arising owing to triplet pairing of quasiparticles, represents a 2×2 symmetric matrix in spin space, ($\alpha, \beta = \uparrow, \downarrow$). The spin-orbit interaction among quasiparticles is known to dominate in the nucleon matter of

a high density. Therefore, it is conventional to represent the triplet order parameter of the system $\hat{D} = \sum_{jlm_j} \Delta_{jlm_j} \Phi_{\alpha\beta}^{(jlm_j)}$ as a superposition of standard spin-angle functions of the total angular momentum (j, m_j),

$$
\left[\Phi_{jlm_j}(\mathbf{n})\right]_{\alpha\beta} \equiv \sum_{m_s+m_l=m_j} \left(\frac{1}{2}\frac{1}{2}\alpha\beta|sm_s\right)(slm_sm_l|jm_j)Y_{l,m_l}(\mathbf{n}).
$$

(1)

The angular dependence of the order parameter is represented by the unit vector $\mathbf{n} = \mathbf{p}/p$, which defines the polar angles (θ, φ) on the Fermi surface.

For our calculations it is more convenient to use vector notation that involves a set of mutually orthogonal complex vectors $\mathbf{b}_{jlm_j}(\mathbf{n})$ defined as

$$
\mathbf{b}_{jlm_j}(\mathbf{n}) = -\frac{1}{2}\mathrm{Tr}\left(\hat{g}\hat{\boldsymbol{\sigma}}\hat{\Phi}_{jlm_j}\right), \quad \mathbf{b}_{jl,-m_j} = (-)^{m_j}\mathbf{b}_{jlm_j}^*,
$$

(2)

where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ are Pauli spin matrices and $\hat{g} = i\hat{\sigma}_2$. The vectors $\mathbf{b}_{jlm_j}$ obey the normalization condition

$$
\int \frac{d\mathbf{n}}{4\pi} \mathbf{b}_{j'l'm'_j}^* \mathbf{b}_{jlm_j} = \delta_{jj'}\delta_{ll'}\delta_{m_jm'_j}.
$$

(3)

Using the vector notation the order parameter is $\hat{D}(\mathbf{n}) = \Delta \bar{\mathbf{b}}\hat{\boldsymbol{\sigma}}\hat{g}$, where the vector $\bar{\mathbf{b}}$ in spin space is defined by the relation

$$
\Delta(p)\bar{\mathbf{b}}(\mathbf{n}) = \sum_{jlm_j} \Delta_{jlm_j}(p)\mathbf{b}_{jlm_j}(\mathbf{n}).
$$

(4)

Because the ground-state order parameter is to be a unitary triplet [13,14], $\bar{\mathbf{b}}(\mathbf{n})$ is a real vector, which we normalize by the condition

$$
\int \frac{d\mathbf{n}}{4\pi}\bar{b}^2(\mathbf{n}) = 1.
$$

(5)

Making use of the adopted graphical notation for the ordinary and anomalous propagators, $\hat{G} = \longrightarrow$, $\hat{G}^-(p) = \longleftarrow$, $\hat{F}^{(1)} = \rightarrow\!\!\leftarrow$, and $\hat{F}^{(2)} = \leftarrow\!\!\rightarrow$, it is convenient to employ the Matsubara calculation technique for the system in thermal equilibrium. Then the analytic form of the propagators is as follows [15,16]:

$$
\hat{G}(p_\eta, \mathbf{p}) = G(p_\eta, \mathbf{p})\delta_{\alpha\beta}, \quad \hat{G}^-(p_\eta, \mathbf{p}) = G^-(p_\eta, \mathbf{p})\delta_{\alpha\beta},
$$

(6)

$$
\hat{F}^{(1)}(p_\eta, \mathbf{p}) = F(p_\eta, \mathbf{p})\bar{\mathbf{b}}\hat{\boldsymbol{\sigma}}\hat{g}, \quad \hat{F}^{(2)}(p_\eta, \mathbf{p}) = F(p_\eta, \mathbf{p})\hat{g}\hat{\boldsymbol{\sigma}}\bar{\mathbf{b}},
$$

where the scalar Green's functions are of the form $G^-(p_\eta, \mathbf{p}) = G(-p_\eta, -\mathbf{p})$ and

$$
G(p_\eta, \mathbf{p}) = \frac{-ip_\eta - \varepsilon_{\mathbf{p}}}{p_\eta^2 + E_{\mathbf{p}}^2}, \quad F(p_\eta, \mathbf{p}) = \frac{-\Delta}{p_\eta^2 + E_{\mathbf{p}}^2}.
$$

(7)

Here $p_\eta \equiv i\pi(2\eta + 1)T$ with $\eta = 0, \pm 1, \pm 2, \dots$ is the Matsubara's fermion frequency, and the quasiparticle energy is given by

$$
E_{\mathbf{p}}^2 = \varepsilon_p^2 + \Delta^2\bar{b}^2(\mathbf{n}),
$$

(8)

where ε_p is the single-particle spectrum of the normal Fermi liquid. Near the Fermi surface one has

$$
\varepsilon_p = \frac{p^2}{2M^*} - \frac{p_F^2}{2M^*} \simeq \upsilon_F(p - p_F).
$$

(9)

The effective mass of a neutron quasiparticle is defined as $M^* = p_F/\upsilon_F$, where $\upsilon_F \ll 1$ is the Fermi velocity of the nonrelativistic neutrons. In the absence of external fields, the gap amplitude $\Delta(T)$ is real.

The following notation is used below. We denote as $L_{X,X}(\omega, \mathbf{q}; \mathbf{p})$ the analytical continuation of the Matsubara sums:

$$
L_{XX'}\left(\omega_\kappa, \mathbf{p} + \frac{\mathbf{q}}{2}; \mathbf{p} - \frac{\mathbf{q}}{2}\right)
$$

$$
= T\sum_\eta X\left(p_\eta + \omega_\kappa, \mathbf{p} + \frac{\mathbf{q}}{2}\right)X'\left(p_\eta, \mathbf{p} - \frac{\mathbf{q}}{2}\right),
$$

(10)

where $X, X' \in G, F, G^-$, and $\omega_\kappa = 2i\pi\kappa T$ with $\kappa = 0, \pm 1, \pm 2\dots$. We divide the integration over the momentum space into integration over the solid angle and over the energy according to

$$
\int \frac{d^3p}{(2\pi)^3}\cdots = \varrho\int \frac{d\mathbf{n}}{4\pi}\frac{1}{2}\int_{-\infty}^{\infty}d\varepsilon_{\mathbf{p}}\cdots,
$$

(11)

where $\varrho = p_F M^*/\pi^2$ is the density of states near the Fermi surface in the normal state, and operate with integrals over the quasiparticle energy:

$$
\mathcal{I}_{XX'}(\omega, \mathbf{n}, \mathbf{q}; T) \equiv \frac{1}{2}\int_{-\infty}^{\infty}d\varepsilon_p L_{XX'}\left(\omega, \mathbf{p} + \frac{\mathbf{q}}{2}, \mathbf{p} - \frac{\mathbf{q}}{2}\right).
$$

(12)

These are functions of ω, $\mathbf{q}$, and the direction of a quasiparticle momentum $\mathbf{n}$.

The loop integrals (12) possess the following properties, which can be verified by a straightforward calculation:

$$
\mathcal{I}_{G^-G} = \mathcal{I}_{GG^-}, \quad \mathcal{I}_{GF} = -\mathcal{I}_{FG},
$$

$$
\mathcal{I}_{G^-F} = -\mathcal{I}_{FG^-},
$$

(13)

$$
\mathcal{I}_{G^-F} + \mathcal{I}_{FG} = \frac{\omega}{\Delta}\mathcal{I}_{FF},
$$

(14)

$$
\mathcal{I}_{G^-F} - \mathcal{I}_{FG} = -\frac{\mathbf{qv}}{\Delta}\mathcal{I}_{FF}.
$$

(15)

For arbitrary ω, $\mathbf{q}$, T one can obtain also

$$
\mathcal{I}_{GG^-} + \bar{b}^2\mathcal{I}_{FF} = A + \frac{\omega^2 - (\mathbf{qv})^2}{2\Delta^2}\mathcal{I}_{FF},
$$

(16)

where $\mathbf{v}$ is a vector with the magnitude of the Fermi velocity υ_F and the direction of $\mathbf{n}$, and

$$
A(\mathbf{n}) \equiv [\mathcal{I}_{GG^-}(\mathbf{n}) + \bar{b}^2(\mathbf{n})\mathcal{I}_{FF}(\mathbf{n})]_{\omega=0, \mathbf{q}=0}.
$$

(17)

B. Gap equation

The block of the interaction diagrams irreducible in the channel of two quasiparticles, $\Gamma_{\alpha\beta,\gamma\delta}$, is usually generated by the expansion over spin-angle functions (1). The spin-orbit interaction among quasiparticles is known to dominate at high densities. In this case the spin $\mathbf{s}$ and orbital momentum $\mathbf{l}$ of the

pair cease to be conserved separately. Thus, the complete list of channels participating in the triplet-spin P-wave pairing includes the pair states with $j = 0, 1, 2$, and $|m_j| \leqslant j$. The pairing occurs in the state with $j = 2$ because the attractive interaction in this channel strongly dominates. The tensor components of the neutron-neutron interaction are known also to exert some influence on pair formation in dense neutron matter, favoring the condensation of pairs in the $^3P + {}^3F_2$ state, but the contributions from $^3P_2 \rightarrow {}^3P_0$ or $^3P_2 \rightarrow {}^3P_1$ transitions are deemed to be unimportant. Hence, we take the approximation to neglect the $j = 0, 1$ coupling throughout this paper. From now on we omit the suffix j everywhere by assuming that the pairing occurs into the state with $j = 2$. Then, in the vector notation, the pairing interaction is of the form

$$
\begin{aligned}
\varrho\Gamma_{\alpha\beta,\gamma\delta}(\mathbf{p}, \mathbf{p}') \\
= \sum_{l'lm_j}(-1)^{\frac{l-l'}{2}}\mathcal{V}_{ll'}(p, p')\big[\mathbf{b}_{lm_j}(\mathbf{n})\hat{\sigma}\hat{g}\big]_{\alpha\beta}\big[\hat{g}\hat{\sigma}\,\mathbf{b}^*_{l'm_j}(\mathbf{n}')\big]_{\gamma\delta},
\end{aligned}
$$

(18)

where the pairing matrix elements $\mathcal{V}_{ll'}(p, p')$ with $l, l' = j \pm 1 = 1, 3$ are the corresponding interaction amplitudes.

The ground-state problem is normally treated in terms of the set of equations for the coupled partial-wave amplitudes Δ_{lm_j} [17–22]. Making use of the identity

$$
\frac{1}{2E_\mathbf{p}}\tanh\frac{E_\mathbf{p}}{2T} \equiv T\sum_\eta \frac{1}{p_\eta^2 + E_\mathbf{p}^2},
$$

(19)

one can obtain the standard set of equations for the triplet partial amplitudes Δ_{lm_j} in the form

$$
\begin{aligned}
\Delta_{lm_j}(p) = -\sum_{l'=1,3}\frac{1}{2\varrho}\int dp'\, p'^2(-1)^{\frac{l-l'}{2}}\mathcal{V}_{ll'}(p, p') \\
\times \Delta(p')\left\langle \mathbf{b}^*_{l'm_j}(\mathbf{n}')\bar{\mathbf{b}}(\mathbf{n}')T\sum_\eta \frac{1}{p_\eta^2 + E_{\mathbf{p}'}^2}\right\rangle.
\end{aligned}
$$

(20)

Here and in what follows we use the angle brackets to denote angle averages,

$$
\langle\cdots\rangle \equiv \frac{1}{4\pi}\int d\mathbf{n}\ldots.
$$

Notice that

$$
\frac{1}{p_\eta^2 + E_\mathbf{p}^2} \equiv G(p_\eta, \mathbf{p})G^-(p_\eta, \mathbf{p}) + \bar{b}^2 F(p_\eta, \mathbf{p})F(p_\eta, \mathbf{p}),
$$

(21)

and the gap equation (20) can be identically written as

$$
\begin{aligned}
\Delta_{lm_j}(p) = -\sum_{l'}\frac{1}{2\varrho}\int dp'\, p'^2(-1)^{\frac{l-l'}{2}}\mathcal{V}_{ll'}(p, p')\Delta(p') \\
\times \left\langle \mathbf{b}^*_{l'm_j}(\mathbf{n}')\bar{\mathbf{b}}(\mathbf{n}')[L_{GG^-} + \bar{b}^2 L_{FF}]_{\omega=0,\mathbf{q}=0}\right\rangle.
\end{aligned}
$$

(22)

III. VERTEX EQUATIONS

We are interested in the linear medium response onto an external axial-vector field. The field interaction with a
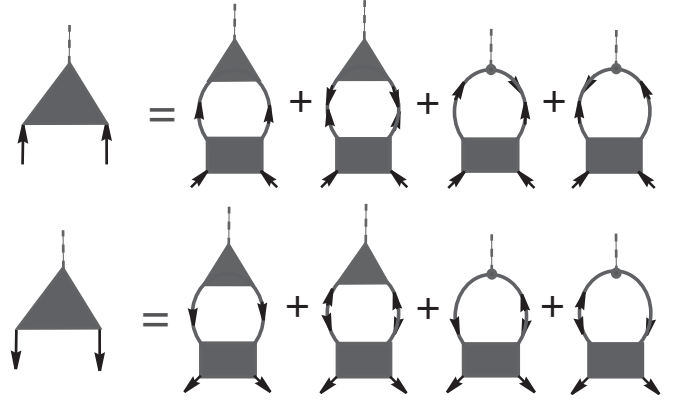

FIG. 1. Dyson's equations for the anomalous vertices. The ordinary vertices are shown by small filled circles. The shaded rectangle represents the pairing interaction.

superfluid should be described with the aid of two ordinary and two anomalous three-point effective vertices. In the BCS approximation, the ordinary axial-vector vertices of a nonrelativistic particle and a hole are to be taken as $\hat{\sigma}$ and $\hat{\sigma}^T$, respectively. The anomalous effective vertices, $\hat{\mathbf{T}}^{(1)}(\mathbf{n};\omega, \mathbf{q})$ and $\hat{\mathbf{T}}^{(2)}(\mathbf{n};\omega, \mathbf{q})$ are given by the infinite sums of the diagrams taking account of the pairing interaction in the ladder approximation [23]. These 2×2 vector matrices are to satisfy the Dyson's equations symbolically depicted by graphs in Fig. 1.

The analytic form of the diagrams in Fig. 1 is derived in Ref. [10]. We are interested in the neutrino energy losses through neutral weak currents. In the case of a nonrelativistic medium the relevant input for this calculation are the effective weak vertices at zero momentum transfer. This substantially simplifies the problem. After some algebraic manipulations the BCS equations for anomalous vertices at $\mathbf{q} = 0$ can be found in the following form (we omit for brevity the dependence of functions on ω):

$$
\begin{aligned}
\hat{\mathbf{T}}^{(1)}(\mathbf{n}) = \sum_{lm_j}\hat{\sigma}\,\mathbf{b}_{lm_j}(\mathbf{n})\hat{g}\sum_{l'}\mathcal{V}_{ll'}\frac{1}{2}\big\langle \mathcal{I}_{GG^-}\mathrm{Tr}\big[\hat{g}\big(\hat{\sigma}\mathbf{b}^*_{l'm_j}\big)\hat{\mathbf{T}}^{(1)}\big] \\
- \mathcal{I}_{FF}\mathrm{Tr}\big[(\hat{\sigma}\mathbf{b}^*_{l'm_j})(\hat{\sigma}\bar{\mathbf{b}})\hat{g}\hat{\mathbf{T}}^{(2)}(\hat{\sigma}\bar{\mathbf{b}})\big] \\
- \frac{\omega}{\Delta}\mathcal{I}_{FF}2i\big(\mathbf{b}^*_{l'm_j}\times\bar{\mathbf{b}}\big)\big\rangle,
\end{aligned}
$$

(23)

$$
\begin{aligned}
\hat{\mathbf{T}}^{(2)}(\mathbf{n}) = \sum_{lm_j}\hat{g}\hat{\sigma}\,\mathbf{b}^*_{lm_j}(\mathbf{n})\sum_{l'}\mathcal{V}_{ll'}\frac{1}{2}\big\langle \mathcal{I}_{G^-G}\mathrm{Tr}\big[(\hat{\sigma}\mathbf{b}_{l'm_j})\hat{g}\hat{\mathbf{T}}^{(2)}\big] \\
- \mathcal{I}_{FF}\mathrm{Tr}\big[(\hat{\sigma}\mathbf{b}_{l'm_j})(\hat{\sigma}\bar{\mathbf{b}})\hat{\mathbf{T}}^{(1)}\hat{g}(\hat{\sigma}\bar{\mathbf{b}})\big] \\
- \frac{\omega}{\Delta}\mathcal{I}_{FF}2i\big(\mathbf{b}_{l'm_j}\times\bar{\mathbf{b}}\big)\big\rangle.
\end{aligned}
$$

(24)

Inspection of the equations reveals that the anomalous axial-vector vertices can be found in the following form:

$$
\hat{\mathbf{T}}^{(1)}(\mathbf{n}, \omega) = \sum_{lm_j}\mathbf{B}^{(1)}_{lm_j}(\omega)\big(\hat{\sigma}\,\mathbf{b}_{lm_j}\big)\hat{g},
$$

(25)

$$
\hat{\mathbf{T}}^{(2)}(\mathbf{n}, \omega) = \sum_{lm_j}\mathbf{B}^{(2)}_{lm_j}(\omega)\hat{g}\big(\hat{\sigma}\,\mathbf{b}^*_{lm_j}\big).
$$

(26)

Insertion of these expressions into Eqs. (23) and (24) makes it possible to obtain the following equations:

$$
\mathbf{B}_{lm_j}^+ = -\sum_{l'} \mathcal{V}_{ll'} \Bigg[\sum_{l''m_j'} \big\langle \big(\mathcal{I}_{GG^-} + \bar{b}^2 \mathcal{I}_{FF}\big)\big(\mathbf{b}_{l'm_j}^* \mathbf{b}_{l''m_j'}\big)\big\rangle \mathbf{B}_{l''m_j'}^+
$$
$$
- 2\sum_{l''m_j'} \big\langle \big(\bar{\mathbf{b}}\mathbf{b}_{l'm_j}^*\big)\big(\bar{\mathbf{b}}\mathbf{b}_{l''m_j'}\big)\mathcal{I}_{FF}\big\rangle \mathbf{B}_{l''m_j'}^+
$$
$$
+ \frac{\omega}{\Delta}\big\langle\mathcal{I}_{FF} i\big(\mathbf{b}_{l'm_j}^* \times \bar{\mathbf{b}}\big)\big\rangle \Bigg],
$$

$$
(27)
$$

$$
\mathbf{B}_{lm_j}^- = -\sum_{l'} \mathcal{V}_{ll'} \sum_{l''m_j'} \big[\big\langle \big(\mathcal{I}_{GG^-} - \bar{b}^2 \mathcal{I}_{FF}\big)\big(\mathbf{b}_{l'm_j}^* \mathbf{b}_{l''m_j'}\big)\big\rangle
$$
$$
+ 2\big\langle \big(\bar{\mathbf{b}}\mathbf{b}_{l'm_j}^*\big)\big(\bar{\mathbf{b}}\mathbf{b}_{l''m_j'}\big)\mathcal{I}_{FF}\big\rangle \big]\mathbf{B}_{l''m_j'}^-, \qquad (28)
$$

where the new unknown vector functions are defined as

$$
\mathbf{B}_{lm_j}^{(\pm)} = \tfrac{1}{2}\big[\mathbf{B}_{lm_j}^{(1)} \pm (-)^{m_j}\mathbf{B}_{l,-m_j}^{(2)}\big]. \qquad (29)
$$

The uniform Eqs. (28) have nontrivial solutions if the determinant of the system equals zero. This condition could be considered as the dispersion equation for the eigenmodes of oscillations. However, the solutions obtained in this way would be spurious, because the physical solution must develop a pole in the vertex function when the frequency approaches the eigenvalue. In the case of uniform equations the solution remains finite at the resonant frequency. Therefore, only trivial solutions $\mathbf{B}_{lm_j}^- = 0$ are physically meaningful. We then obtain

$$
\mathbf{B}_{lm_j}^+ = \mathbf{B}_{lm_j}^{(1)} = (-)^{m_j}\mathbf{B}_{l,-m_j}^{(2)}. \qquad (30)
$$

From now on we omit "plus" in the notation by assuming $\mathbf{B}_{lm_j} \equiv \mathbf{B}_{lm_j}^+$.

By making use of Eq. (16) and denoting $\mathcal{I}_{FF}(\mathbf{n},\omega, \mathbf{q}=0) = \mathcal{I}_0(\mathbf{n},\omega;T)$, where

$$
\mathcal{I}_0(\mathbf{n},\omega) = \int_0^\infty \frac{d\varepsilon_p}{E_\mathbf{p}} \frac{\Delta^2}{4E_\mathbf{p}^2 - (\omega + i0)^2}\tanh\frac{E_\mathbf{p}}{2T}, \quad (31)
$$

one can obtain Eq. (27) in the form

$$
\mathbf{B}_{lm_j} = -\sum_{l'} \mathcal{V}_{ll'} \Bigg\{ \sum_{l''m_j'} \mathbf{B}_{l''m_j'}\big\langle A(\mathbf{n})\big(\mathbf{b}_{l'm_j}^* \mathbf{b}_{l''m_j'}\big)\big\rangle
$$
$$
+ \sum_{l''m_j'} \mathbf{B}_{l''m_j'} \Big\langle \mathcal{I}_0(\mathbf{n})\Big[\frac{\omega^2}{2\Delta^2}\big(\mathbf{b}_{l'm_j}^* \mathbf{b}_{l''m_j'}\big)
$$
$$
- 2\big(\bar{\mathbf{b}}\mathbf{b}_{l'm_j}^*\big)\big(\bar{\mathbf{b}}\mathbf{b}_{l''m_j'}\big)\Big]\Big\rangle \Bigg\} + i\frac{\omega}{\Delta}\big\langle\mathcal{I}_0(\mathbf{n})\big(\mathbf{b}_{l'm_j}^* \times \bar{\mathbf{b}}\big)\big\rangle \Bigg\}.
$$

$$
(32)
$$

This equation is to be solved together with the gap equation (22).

IV. RENORMALIZATIONS

Both the gap equation (22) and the vertex equation (32) involve integrations over the regions far from the Fermi surface

while we are interested in the processes occurring in a vicinity of the Fermi sphere. To get rid of the integration over the far regions we renormalize the interaction as suggested in Refs. [24,25]: We define

$$
\mathsf{V}_{ll'}(p, p') = \mathcal{V}_{ll'}(p, p') - \sum_{l''} \int \frac{dp'' p''^2}{\pi^2}\mathcal{V}_{ll''}(p, p'')
$$
$$
\times L_{GG^-}^{(N)}(p'')\mathsf{V}_{l''l'}(p'', p')
$$
$$
= \mathcal{V}_{ll'}(p, p') - \sum_{l''} \int \frac{dp'' p''^2}{\pi^2}\mathsf{V}_{ll''}(p, p'')
$$
$$
\times L_{GG^-}^{(N)}(p'')\mathcal{V}_{l''l'}(p'', p'),
$$

where the loop $L_{GG^-}^{(N)}(p'')$ is evaluated in the normal (nonsuperfluid) state. Using the identity

$$
\frac{\Delta(p)}{\Delta_{lm_j}(p)} \int \frac{d\mathbf{n}}{4\pi}\big(\mathbf{b}_{lm_j}^* \bar{\mathbf{b}}\big) \equiv 1
$$

one can recast the above as

$$
\mathsf{V}_{ll'}(p, p') = \mathcal{V}_{ll'}(p, p') - \sum_{l''} \int \frac{d^3\mathbf{p}''}{4\pi^3}\mathsf{V}_{ll''}(p, p'')
$$
$$
\times \frac{\Delta(p'')}{\Delta_{l''m_j}(p'')}\big(\mathbf{b}_{l''m_j}^* \bar{\mathbf{b}}\big)_{\mathbf{n}''}L_{GG^-}^{(N)}(p'')\mathcal{V}_{l''l'}(p'', p').
$$

Then it can be shown [11] that we may everywhere substitute $\mathsf{V}_{ll'}$ for $\mathcal{V}_{ll'}$ provided that at the same time, we understand by the L_{GG^-} element, the subtracted quantity $L_{GG^-} - L_{GG^-}^{(N)}(p'')$ [$L_{GG^-}^{(N)}$ is to be evaluated for $\omega = 0$, $\mathbf{q} = 0$ in all cases]. The gap equation (22) becomes of the form

$$
\Delta_{lm_j} = -\sum_{l'} i^{l-l'}\mathsf{V}_{ll'}\Delta\big\langle\mathbf{b}_{l'm_j}^*(\mathbf{n})\bar{\mathbf{b}}(\mathbf{n})A(\mathbf{n})\big\rangle, \qquad (33)
$$

which is valid in the narrow vicinity of the Fermi surface, where the smooth functions $\Delta_{lm_j}(p)$, $\mathsf{V}_{ll'}(p, p')$, and $\Delta(p)$ may be replaced with constants $\Delta(p) \simeq \Delta(p_F) \equiv \Delta$, etc. The function (17) is now to be understood as

$$
A(\mathbf{n}) \to \big[\mathcal{I}_{GG^-} - \mathcal{I}_{GG^-}^{(N)} + \bar{b}^2\mathcal{I}_{FF}\big]_{\omega=0,\mathbf{q}=0}. \qquad (34)
$$

It can be found explicitly by performing the Matsubara summation:

$$
A(\mathbf{n})
$$
$$
= \frac{1}{2}\int_0^\infty d\varepsilon\left(\frac{1}{\sqrt{\varepsilon^2 + \Delta^2\bar{b}^2}}\tanh\frac{\sqrt{\varepsilon^2 + \Delta^2\bar{b}^2}}{2T} - \frac{1}{\varepsilon}\tanh\frac{\varepsilon}{2T}\right).
$$

$$
(35)
$$

The renormalization of Eq. (32) also reduces to the replacements $\mathcal{V}_{ll'} \to \mathsf{V}_{ll'}$. The function $A(\mathbf{n})$ should be replaced by the expression (35).

V. ANGLE AVERAGE APPROXIMATION

The angle dependence of the functions $A(\mathbf{n})$ and $\mathcal{I}_{FF}(\mathbf{n})$ arises owing to anisotropy of the square of the energy gap $\Delta^2(p')\bar{b}^2(\mathbf{n})$ entering the energy of a quasiparticle. The consideration can be substantially simplified using the angle

average approximation, that is, replacing the anisotropic energy gap with its angle average, $\Delta^2 \bar{b}^2 \to \langle \Delta^2 \bar{b}^2 \rangle = \Delta^2$. In Refs. [26–28], it has been shown that the angle average approximation is an excellent approximation to the true solution, as long as one is only interested in the average value of the gap near the Fermi surface and not the angular dependence of the gap functions. After this replacement the angle integration becomes trivial. Making use of the orthogonality relations (3) after the renormalizations from Eq. (32) we get a set of linear equations for each value of m_j

$$\mathbf{B}_{lm_j} = -\sum_{l'} \mathsf{V}_{ll'} \left\{ \mathbf{B}_{l'm_j} \left(A_{\mathrm{av}} + \frac{\omega^2}{2\Delta^2} \mathcal{I}_{\mathrm{av}} \right) - 2 \sum_{l''m'_j} \mathbf{B}_{l''m'_j} \right.$$
$$\left. \times \langle (\bar{\mathbf{b}}\mathbf{b}^*_{l'm_j})(\bar{\mathbf{b}}\mathbf{b}_{l''m'_j}) \rangle \mathcal{I}_{\mathrm{av}} + i \frac{\omega}{\Delta} \mathcal{I}_{\mathrm{av}} \langle \mathbf{b}^*_{l'm_j} \times \bar{\mathbf{b}} \rangle \right\}, \quad (36)$$

where

$$A_{\mathrm{av}} = \frac{1}{2} \int_0^\infty d\varepsilon \left(\frac{1}{E} \tanh \frac{E}{2T} - \frac{1}{\varepsilon} \tanh \frac{\varepsilon}{2T} \right), \quad (37)$$

and

$$\mathcal{I}_{\mathrm{av}} = \int_0^\infty \frac{d\varepsilon}{E} \frac{\Delta^2}{4E^2 - \omega^2} \tanh \frac{E}{2T}, \quad (38)$$

with $E = \sqrt{\varepsilon^2 + \Delta^2}$.

The gap equation (33) becomes of the form

$$\Delta_{lm_j} = -\sum_{l'} i^{l-l'} \mathsf{V}_{ll'} \Delta_{l'm_j} A_{\mathrm{av}}. \quad (39)$$

It is convenient to write the Eqs. (36) and (39) as two matrix equations. The corresponding vertex equation is

$$\begin{pmatrix} \mathbf{B}_{1m_j} \\ \mathbf{B}_{3,m_j} \end{pmatrix} = -\begin{pmatrix} \mathsf{V}_{11} & -\mathsf{V}_{13} \\ -\mathsf{V}_{13} & \mathsf{V}_{33} \end{pmatrix} \left\{ \begin{pmatrix} \left(A_{\mathrm{av}} + \frac{\omega^2}{2\Delta^2} \mathcal{I}_{\mathrm{av}} \right) \mathbf{B}_{1m_j} \\ \left(A_{\mathrm{av}} + \frac{\omega^2}{2\Delta^2} \mathcal{I}_{\mathrm{av}} \right) \mathbf{B}_{3m_j} \end{pmatrix} \right.$$
$$- \begin{pmatrix} 2\mathcal{I}_{\mathrm{av}} \sum_{lm'_j} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{1m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \mathbf{B}_{lm'_j} \\ 2\mathcal{I}_{\mathrm{av}} \sum_{lm'_j} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{3m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \mathbf{B}_{lm'_j} \end{pmatrix}$$
$$\left. + \frac{\omega}{\Delta} i \begin{pmatrix} \mathcal{I}_{\mathrm{av}} \langle \mathbf{b}^*_{1m_j} \times \bar{\mathbf{b}} \rangle \\ \mathcal{I}_{\mathrm{av}} \langle \mathbf{b}^*_{3m_j} \times \bar{\mathbf{b}} \rangle \end{pmatrix} \right\}, \quad (40)$$

and the gap equation becomes of the form

$$\begin{pmatrix} \Delta_{1m_j} \\ \Delta_{3m_j} \end{pmatrix} = -\begin{pmatrix} \mathsf{V}_{11} & -\mathsf{V}_{13} \\ -\mathsf{V}_{13} & \mathsf{V}_{33} \end{pmatrix} \begin{pmatrix} \Delta_{1m_j} A_{\mathrm{av}} \\ \Delta_{3m_j} A_{\mathrm{av}} \end{pmatrix}. \quad (41)$$

In obtaining the equations the fact is used that the interaction matrix is symmetric on the Fermi surface, $\mathsf{V}_{31} = \mathsf{V}_{13}$.

The interaction matrix which enters Eqs. (40) and (41) can be diagonalized by unitary transformations $V' = UVU^\dagger$, with U being the unitary matrix

$$U = (U^{-1})^\dagger = \frac{1}{(V_+ + V_-)^{\frac{1}{2}}} \begin{pmatrix} \sqrt{V_+} & \sqrt{V_-} \\ -\sqrt{V_-} & \sqrt{V_+} \end{pmatrix}, \quad (42)$$

where $V_\pm = \sqrt{(\mathsf{V}_{33} - \mathsf{V}_{11})^2 + 4\mathsf{V}_{13}^2} \pm (\mathsf{V}_{33} - \mathsf{V}_{11})$.

One has $UVU^\dagger = \mathrm{diag}(W_-, W_+)$, with

$$W_\pm = \frac{1}{2} \left(\mathsf{V}_{33} + \mathsf{V}_{11} \pm \sqrt{(\mathsf{V}_{33} - \mathsf{V}_{11})^2 + 4\mathsf{V}_{13}^2} \right). \quad (43)$$

Applying the unitary transformation U to the gap equations (41) yields two coupled equations:

$$\sqrt{V_+}\Delta_{1m_j} + \sqrt{V_-}\Delta_{3m_j} = -W_-(\sqrt{V_+}\Delta_{1m_j} + \sqrt{V_-}\Delta_{3m_j})A_{\mathrm{av}}, \quad (44)$$

$$\sqrt{V_-}\Delta_{1m_j} - \sqrt{V_+}\Delta_{3m_j} = -W_+(\sqrt{V_-}\Delta_{1m_j} - \sqrt{V_+}\Delta_{3m_j})A_{\mathrm{av}}. \quad (45)$$

In Eq. (40), the interaction matrix can be also diagonalized by the unitary transformation (42). Further simplification is possible owing to the fact that, by virtue of Eqs. (44) and (45), the coupling constants $W_\pm$ can be removed out of the equations. This results in the set of equations

$$\frac{\omega^2}{4\Delta^2} \left(\sqrt{V_+}\mathbf{B}_{1m_j} + \sqrt{V_-}\mathbf{B}_{3m_j} \right) - \sum_{lm'_j} \left[\sqrt{V_+} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{1m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \right.$$
$$+ \left. \sqrt{V_-} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{3m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \right] \mathbf{B}_{lm'_j}$$
$$= -\frac{\omega}{2\Delta} i \left(\sqrt{V_+} \langle \mathbf{b}^*_{1m_j} \times \bar{\mathbf{b}} \rangle + \sqrt{V_-} \langle \mathbf{b}^*_{3m_j} \times \bar{\mathbf{b}} \rangle \right), \quad (46)$$

$$\frac{\omega^2}{4\Delta^2} \left(\sqrt{V_-}\mathbf{B}_{1m_j} - \sqrt{V_+}\mathbf{B}_{3m_j} \right) - \sum_{lm'_j} \left[\sqrt{V_-} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{1m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \right.$$
$$- \left. \sqrt{V_+} \langle (\bar{\mathbf{b}}\mathbf{b}^*_{3m_j})(\bar{\mathbf{b}}\mathbf{b}_{lm'_j}) \rangle \right] \mathbf{B}_{lm'_j}$$
$$= -\frac{\omega}{2\Delta} i \left(\sqrt{V_-} \langle \mathbf{b}^*_{1m_j} \times \bar{\mathbf{b}} \rangle - \sqrt{V_+} \langle \mathbf{b}^*_{3m_j} \times \bar{\mathbf{b}} \rangle \right). \quad (47)$$

Equations (46) and (47) represent a closed set of equations for determining the vertex amplitudes $\mathbf{B}_{lm_j}(\omega)$.

VI. EIGENMODES OF SPIN OSCILLATIONS

For further progress we have to define the ground state of the condensate which is specified by the vector $\bar{\mathbf{b}}(\mathbf{n})$. We focus on the condensation with $m_j = 0$, which is conventionally considered as the preferable one in the bulk matter of neutron stars [2,3,20]. In this case only the partial gap amplitudes with $l = 1, 3$ and $m_j = 0$ contribute. To simplify the notation we denote them as $\Delta_{10} \equiv \Delta_1$ and $\Delta_{30} \equiv \Delta_3$. Taking into account spin-orbit and tensor interactions the ground state of such a triplet condensate is given by the vector $\bar{\mathbf{b}}(\mathbf{n})$ of the form

$$\bar{\mathbf{b}}(\mathbf{n}) = \frac{\Delta_1}{\Delta} \mathbf{b}_{10}(\mathbf{n}) + \frac{\Delta_3}{\Delta} \mathbf{b}_{30}(\mathbf{n}), \quad (48)$$

where $\Delta^2 = \Delta_1^2 + \Delta_3^2$. For this particular form of the ground state the following relations can be verified by a straightforward calculation:

$$\langle (\bar{\mathbf{b}}\mathbf{b}^*_{lm_j})(\bar{\mathbf{b}}\mathbf{b}_{l'm'_j}) \rangle = \delta_{m_j m'_j} (\bar{\mathbf{b}}\mathbf{b}^*_{lm_j})(\bar{\mathbf{b}}\mathbf{b}_{l'm_j}). \quad (49)$$

We denote

$$\beta^{l,l'}_{m_j} \equiv \langle (\bar{\mathbf{b}}\mathbf{b}^*_{lm_j})(\bar{\mathbf{b}}\mathbf{b}_{l'm_j}) \rangle, \quad (50)$$

and

$$\Omega = \frac{\omega}{2\Delta}.$$

From Eqs. (46) and (47) one can obtain five sets of linear equations corresponding to different values of $m_j = 0, \pm 1, \pm 2$.

For $m_j = 0$ we obtain a set of two equations:

$$\left[\sqrt{V_+}\left(\Omega^2 - \beta_0^{1,1}\right) - \sqrt{V_-}\beta_0^{3,1}\right]\mathbf{B}_{10}$$
$$+ \left[\sqrt{V_-}\left(\Omega^2 - \beta_0^{3,3}\right) - \sqrt{V_+}\beta_0^{1,3}\right]\mathbf{B}_{30} = 0, \quad (51)$$

$$\left[\sqrt{V_-}\left(\Omega^2 - \beta_0^{1,1}\right) + \sqrt{V_+}\beta_0^{3,1}\right]\mathbf{B}_{10}$$
$$- \left[\sqrt{V_+}\left(\Omega^2 - \beta_0^{3,3}\right) + \sqrt{V_-}\beta_0^{1,3}\right]\mathbf{B}_{30} = 0. \quad (52)$$

The sets of equations for $m_j = \pm 2$ are of the form

$$\left[\sqrt{V_+}\left(\Omega^2 - \beta_{\pm 2}^{1,1}\right) - \sqrt{V_-}\beta_{\pm 2}^{3,1}\right]\mathbf{B}_{1,\pm 2}$$
$$+ \left[\sqrt{V_-}\left(\Omega^2 - \beta_{\pm 2}^{3,3}\right) - \sqrt{V_+}\beta_{\pm 2}^{1,3}\right]\mathbf{B}_{3,\pm 2} = 0, \quad (53)$$

$$\left[\sqrt{V_-}\left(\Omega^2 - \beta_{\pm 2}^{1,1}\right) + \sqrt{V_+}\beta_{\pm 2}^{3,1}\right]\mathbf{B}_{1,\pm 2}$$
$$- \left[\sqrt{V_+}\left(\Omega^2 - \beta_{\pm 2}^{3,3}\right) + \sqrt{V_-}\beta_{\pm 2}^{1,3}\right]\mathbf{B}_{3,\pm 2} = 0. \quad (54)$$

For each of the values of $m_j = \pm 1$ we get a set of two equations:

$$\left[\sqrt{V_+}\left(\Omega^2 - \beta_{\pm 1}^{1,1}\right) - \sqrt{V_-}\beta_{\pm 1}^{3,1}\right]\mathbf{B}_{1,\pm 1}$$
$$+ \left[\sqrt{V_-}\left(\Omega^2 - \beta_{\pm 1}^{3,3}\right) - \sqrt{V_+}\beta_{\pm 1}^{1,3}\right]\mathbf{B}_{3,\pm 1}$$
$$= -\Omega i\left(\sqrt{V_+}\langle\mathbf{b}_{1,\pm 1}^* \times \bar{\mathbf{b}}\rangle + \sqrt{V_-}\langle\mathbf{b}_{3,\pm 1}^* \times \bar{\mathbf{b}}\rangle\right), \quad (55)$$

$$\left[\sqrt{V_-}\left(\Omega^2 - \sqrt{V_-}\beta_{\pm 1}^{1,1}\right) + \sqrt{V_+}\beta_{\pm 1}^{3,1}\right]\mathbf{B}_{1,\pm 1}$$
$$- \left[\sqrt{V_+}\left(\Omega^2 - \beta_{\pm 1}^{3,3}\right) + \sqrt{V_-}\beta_{\pm 1}^{1,3}\right]\mathbf{B}_{3,\pm 1}$$
$$= -\Omega i\sqrt{V_-}\langle\mathbf{b}_{1,\pm 1}^* \times \bar{\mathbf{b}}\rangle - \sqrt{V_+}\langle\mathbf{b}_{3,\pm 1}^* \times \bar{\mathbf{b}}\rangle). \quad (56)$$

By the same reason as in the case of Eqs. (28) the uniform equations for $m_j = 0$ and for $m_j = \pm 2$ have only trivial physical solutions $\mathbf{B}_{10} = \mathbf{B}_{30} = 0$ and $\mathbf{B}_{1,\pm 2} = \mathbf{B}_{3,\pm 2} = 0$. The solutions to Eqs. (55) and (56) are

$$\mathbf{B}_{1,\pm 1} = \frac{-i\Omega}{\chi_{\pm 1}(\Omega)}\left[\left(\Omega^2 - \beta_{\pm 1}^{3,3}\right)\langle\mathbf{b}_{1,\pm 1}^* \times \bar{\mathbf{b}}\rangle + \beta_{\pm 1}^{1,3}\langle\mathbf{b}_{3,\pm 1}^* \times \bar{\mathbf{b}}\rangle\right],$$
$$(57)$$

$$\mathbf{B}_{3,\pm 1} = \frac{-i\Omega}{\chi_{\pm 1}(\Omega)}\left[\left(\Omega^2 - \beta_{\pm 1}^{1,1}\right)\langle\mathbf{b}_{3,\pm 1}^* \times \bar{\mathbf{b}}\rangle + \beta_{\pm 1}^{3,1}\langle\mathbf{b}_{1,\pm 1}^* \times \bar{\mathbf{b}}\rangle\right],$$
$$(58)$$

where

$$\langle\mathbf{b}_{1,\pm 1}^* \times \bar{\mathbf{b}}\rangle = \frac{1}{2}\sqrt{\frac{3}{2}}\frac{\Delta_1}{\Delta}(i, \pm 1, 0), \quad (59)$$

$$\langle\mathbf{b}_{3,\pm 1}^* \times \bar{\mathbf{b}}\rangle = -\frac{1}{6}\sqrt{6}\frac{\Delta_3}{\Delta}(i, \pm 1, 0), \quad (60)$$

and

$$\chi_{\pm 1}(\Omega) = \left(\Omega^2 - \beta_{\pm 1}^{1,1}\right)\left(\Omega^2 - \beta_{\pm 1}^{3,3}\right) - \beta_{\pm 1}^{1,3}\beta_{\pm 1}^{3,1}. \quad (61)$$

Notice that the interaction parameters $V_\pm$ drop out of the final result in Eqs. (57) and (58).

Poles of the effective vertices at $\chi_{\pm 1}(\omega) = 0$ signal the existence of collective modes. Frequencies of the eigenoscillations are independent of the sign of m_j. For each of the values of $m_j = \pm 1$ we obtain two eigenmodes:

$$\Omega_{1,2}^2 = \frac{1}{2}\left(\beta_{m_j}^{1,1} + \beta_{m_j}^{3,3} \pm \sqrt{\left(\beta_{m_j}^{3,3} - \beta_{m_j}^{1,1}\right)^2 + 4\beta_{m_j}^{1,3}\beta_{m_j}^{3,1}}\right).$$
$$(62)$$

Straightforward calculations give

$$\beta_{\pm 1}^{1,1} = \frac{1}{20}\frac{\Delta_1^2}{\Delta^2}\left(1 - \frac{2}{7}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{58\Delta_3^2}{7\Delta_1^2}\right), \quad (63)$$

$$\beta_{\pm 1}^{3,3} = \frac{29}{70}\frac{\Delta_1^2}{\Delta^2}\left(1 - \frac{32}{87}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{28}{87}\frac{\Delta_3^2}{\Delta_1^2}\right), \quad (64)$$

$$\beta_{\pm 1}^{1,3} = \beta_{\pm 1}^{3,1} = -\frac{1}{140}\sqrt{6}\frac{\Delta_1^2}{\Delta^2}\left(1 - 11\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{32}{3}\frac{\Delta_3^2}{\Delta_1^2}\right).$$
$$(65)$$

It is interesting to compare the resonant frequencies (62) with the spin oscillation frequency $\omega_s = \Delta/\sqrt{5}$ obtained in Refs. [10,11] in a simple model restricted to excitations of the condensate with $l = 1$. By taking $\Delta_3 = 0$ and $\Delta_1 = \Delta$ in the matrix elements we find $4\beta_{\pm 1}^{1,3}\beta_{\pm 1}^{3,1} \ll \left(\beta_{\pm 1}^{3,3} - \beta_{\pm 1}^{1,1}\right)^2$. By neglecting the small term $4\beta_{\pm 1}^{1,3}\beta_{\pm 1}^{3,1}$ under the root in Eq. (62) we obtain

$$\omega_1(\Delta_3 = 0) = 2\Delta\beta_{\pm 1}^{1,1}(\Delta_3 = 0) = \frac{1}{\sqrt{5}}\Delta \simeq 0.45\Delta \quad (66)$$

and

$$\omega_2(\Delta_3 = 0) = 2\Delta\beta_{\pm 1}^{3,3}(\Delta_3 = 0) = \sqrt{\frac{58}{35}}\Delta \simeq 1.29\Delta. \quad (67)$$

We see that the extending of the decomposition scheme of the excited states with the total angular momentum $j = 2$ up to $l = 1, 3$ leads to a very small frequency shift of the known mode, $\omega_s = \Delta/\sqrt{5}$, but opens the new additional mode of collective spin oscillations. Inclusion of the tensor interaction implies $\Delta_3 \neq 0$ and $\Delta^2 = \Delta_1^2 + \Delta_3^2$. In this case from Eq. (62) we obtain two twofold ($m_j = \pm 1$) frequency:

$$\omega_1^2 = \Delta_1^2\left(\frac{13}{14} - \frac{1}{3}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{23}{21}\frac{\Delta_3^2}{\Delta_1^2} - \sqrt{\frac{15}{28} - \frac{25}{49}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{485}{147}\frac{\Delta_3^2}{\Delta_1^2} - \frac{370}{441}\sqrt{6}\frac{\Delta_3^3}{\Delta_1^3} + \frac{55}{63}\frac{\Delta_3^4}{\Delta_1^4}}\right), \quad (68)$$

$$\omega_2^2 = \Delta_1^2\left(\frac{13}{14} - \frac{1}{3}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{23}{21}\frac{\Delta_3^2}{\Delta_1^2} + \sqrt{\frac{15}{28} - \frac{25}{49}\sqrt{6}\frac{\Delta_3}{\Delta_1} + \frac{485}{147}\frac{\Delta_3^2}{\Delta_1^2} - \frac{370}{441}\sqrt{6}\frac{\Delta_3^3}{\Delta_1^3} + \frac{55}{63}\frac{\Delta_3^4}{\Delta_1^4}}\right). \quad (69)$$

According to calculations of different authors, at the Fermi surface one has $\Delta_3 \simeq 0.17\Delta_1$ (see, e.g., Ref. [18]). In this case our theoretical analysis predicts two degenerate modes with $\omega = \omega_1 \simeq 0.42\Delta$ and two degenerate modes with $\omega = \omega_2 = 1.19\Delta$.

VII. ANOMALOUS VERTICES AND POLARIZATION FUNCTIONS

Making use of Eqs. (25), (26), (29), and (30) we find

$$\hat{\mathbf{T}}^{(1)}(\mathbf{n}, \omega) = \sum_{m_j = \pm 1} \left[\mathbf{B}_{1m_j}\left(\hat{\boldsymbol{\sigma}}\mathbf{b}_{1m_j}\right)\hat{g} + \mathbf{B}_{3m_j}\left(\hat{\boldsymbol{\sigma}}\mathbf{b}_{3m_j}\right)\hat{g} \right], \quad (70)$$

$$\hat{\mathbf{T}}^{(2)}(\mathbf{n}, \omega) = \sum_{m_j = \pm 1} \left[\mathbf{B}_{1m_j}\hat{g}\left(\hat{\boldsymbol{\sigma}}\mathbf{b}_{1m_j}\right) + \mathbf{B}_{3m_j}\hat{g}\left(\hat{\boldsymbol{\sigma}}\mathbf{b}_{3m_j}\right) \right], \quad (71)$$

where the functions $\mathbf{B}_{1m_j}(\omega)$ and $\mathbf{B}_{3m_j}(\omega)$ are given in Eqs. (57) and (58).

The general expression of the axial polarization tensor in the BCS approximation has been already discussed before. It can be obtained in the form [10–12]

$$\Pi_{\mathrm{A}}^{ij}(\omega) = -4\varrho \mathcal{I}_{\mathrm{av}}(\delta_{ij} - \langle\bar{b}_i\bar{b}_j\rangle) - \varrho\Omega\mathcal{I}_{\mathrm{av}}\langle\mathrm{Tr}\left[\hat{\sigma}_i \hat{T}_j^{(1)}\hat{g}\left(\hat{\boldsymbol{\sigma}}\bar{\mathbf{b}}\right)\right]$$
$$- \mathrm{Tr}\left[\hat{\sigma}_i\left(\hat{\boldsymbol{\sigma}}\bar{\mathbf{b}}\right)\hat{g}\hat{T}_j^{(2)}\right]\rangle, \quad (72)$$

where $i, j = 1, 2, 3$ and the anomalous axial-vector vertices $\hat{\mathbf{T}}^{(1,2)}$ are given by Eqs. (70) and (71). We omit for brevity the dependence on $\mathbf{n}$ and ω. Calculation of the traces results in the expression

$$\Pi_{\mathrm{A}}^{ij}(\omega) = -4\varrho \mathcal{I}_{\mathrm{av}}(\delta^{ij} - \langle\bar{b}^i\bar{b}^j\rangle) + 4\varrho\Omega\mathcal{I}_{\mathrm{av}} \sum_{m_j = \pm 1} \left(i\langle\mathbf{b}_{1m_j}\times\bar{\mathbf{b}}\rangle^i\right.$$
$$\left.\times B_{1m_j}^j + i\langle\mathbf{b}_{3m_j}\times\bar{\mathbf{b}}\rangle^i B_{3m_j}^j\right). \quad (73)$$

Inserting the vectors $\mathbf{B}_{l,m_j}$, given by Eqs. (57) and (58), we write the function (61) as

$$\chi_{m_j}(\omega) = \left(\Omega^2 - \Omega_1^2\right)\left(\Omega^2 - \Omega_2^2\right), \quad (74)$$

where $m_j = \pm 1$, and

$$\Omega_1^2 = \frac{\omega_1^2}{4\Delta^2}, \quad \Omega_2^2 = \frac{\omega_2^2}{4\Delta^2}. \quad (75)$$

In this way we obtain

$$\Pi_{\mathrm{A}}^{ij}(\omega) = -4\varrho\mathcal{I}_{\mathrm{av}}(\delta^{ij} - \langle\bar{b}^i\bar{b}^j\rangle)$$
$$+ 4\varrho\mathcal{I}_{\mathrm{av}} \frac{\Omega^2}{\left(\Omega^2 - \Omega_1^2 + i0\right)\left(\Omega^2 - \Omega_2^2 + i0\right)}$$
$$\times \sum_{m_j = \pm 1} \left[\left(\Omega^2 - \beta_{m_j}^{3,3}\right)\langle\mathbf{b}_{1m_j}\times\bar{\mathbf{b}}\rangle^i\langle\mathbf{b}_{1m_j}^*\times\bar{\mathbf{b}}\rangle^j\right.$$
$$+ \beta_{m_j}^{1,3}\langle\mathbf{b}_{1m_j}\times\bar{\mathbf{b}}\rangle^i\langle\mathbf{b}_{3m_j}^*\times\bar{\mathbf{b}}\rangle^j$$
$$+ \left(\Omega^2 - \beta_{m_j}^{1,1}\right)\langle\mathbf{b}_{3m_j}\times\bar{\mathbf{b}}\rangle^i\langle\mathbf{b}_{3m_j}^*\times\bar{\mathbf{b}}\rangle^j$$
$$\left.+ \beta_{m_j}^{3,1}\langle\mathbf{b}_{3m_j}\times\bar{\mathbf{b}}\rangle^i\langle\mathbf{b}_{1m_j}^*\times\bar{\mathbf{b}}\rangle^j\right]. \quad (76)$$

The poles location on the complex plane of Ω is chosen so that to obtain the retarded polarization function.

Summation over $m_j = \pm 1$ can be done using the fact that the parameters $\beta_{m_j}^{l,l'}$ entering this equation are independent of the sign of m_j, and $\beta_{\pm 1}^{1,3} = \beta_{\pm 1}^{3,1}$. Then a simple calculation gives

$$\sum_{m_j = \pm 1} \langle\mathbf{b}_{lm_j}\times\bar{\mathbf{b}}\rangle^i\langle\mathbf{b}_{l'm_j}^*\times\bar{\mathbf{b}}\rangle^j = \lambda_{ll'}(\delta_{ij} - \delta_{i3}\delta_{j3}), \quad (77)$$

with

$$\lambda_{11} = \frac{3}{4}\frac{\Delta_1^2}{\Delta^2}, \quad \lambda_{33} = \frac{1}{3}\frac{\Delta_3^2}{\Delta^2}, \quad \lambda_{13} = \lambda_{31} = -\frac{1}{2}\frac{\Delta_1\Delta_3}{\Delta^2}, \quad (78)$$

and

$$\langle\bar{b}^i\bar{b}^j\rangle = \frac{1}{6}\frac{\Delta_1^2}{\Delta^2}\left[\left(1 + \frac{12}{7}\frac{\Delta_3^2}{\Delta_1^2}\right)\delta^{ij} + \left(3 + \frac{6}{7}\frac{\Delta_3^2}{\Delta_1^2}\right)\delta^{i3}\delta^{3j}\right]. \quad (79)$$

We finally obtain the expression

$$\Pi_{\mathrm{A}}^{ij}(\omega) = -4\varrho\mathcal{I}_{\mathrm{av}}(\delta^{ij} - \langle\bar{b}^i\bar{b}^j\rangle) + 4\varrho\mathcal{I}_{\mathrm{av}}(\delta^{ij} - \delta^{i3}\delta^{3j})$$
$$\times \frac{4\Delta^2\omega^2}{\left(\omega^2 - \omega_1^2 + i0\right)\left(\omega^2 - \omega_2^2 + i0\right)}$$
$$\times \left[\lambda_{11}\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{3,3}\right) + \lambda_{33}\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{1,1}\right)\right.$$
$$\left.+ 2\lambda_{13}\beta_1^{1,3}\right]. \quad (80)$$

Below we use the retarded polarization tensor for a calculation of the neutrino emissivity of a nonrelativistic superfluid matter. In this calculation one can neglect the temporal and mixed components of the polarization tensor occurring as small relativistic corrections.

VIII. NEUTRINO ENERGY LOSSES

We examine the neutrino energy losses in the standard model of weak interactions. Then after integration over the phase volume of freely escaping neutrinos and antineutrinos the total energy which is emitted per unit volume and time can be obtained in the form (see details, e.g., in Ref. [29])

$$\epsilon = -\frac{G_F^2 C_A^2 \mathcal{N}_\nu}{192\pi^5} \int_0^\infty d\omega \int d^3q \frac{\omega\Theta(\omega - q)}{\exp\left(\frac{\omega}{T}\right) - 1}\mathrm{Im}\Pi_{\mathrm{A}}^{\mu\nu}(\omega, \mathbf{q})$$
$$\times \left(k_\mu k_\nu - k^2 g_{\mu\nu}\right), \quad (81)$$

where G_F is the Fermi coupling constant, $C_A = 1.26$ is the axial-vector weak coupling constant of neutrons, $\mathcal{N}_\nu = 3$ is the number of neutrino flavors, $\Theta(x)$ is the Heaviside step function, and $k^\mu = (\omega, \mathbf{q})$ is the total energy and momentum of the freely escaping neutrino pair ($\mu, \nu = 0, 1, 2, 3$).

In Eq. (81), we have neglected the neutrino emission in the vector channel, which is strongly suppressed owing to conservation of the vector current [10,30]. Therefore, the energy losses are connected to the imaginary part of the retarded polarization tensor in the axial channel, $\mathrm{Im}\Pi_{\mathrm{A}}^{\mu\nu} \simeq \delta^{\mu i}\delta^{\nu j}\mathrm{Im}\Pi_{\mathrm{A}}^{ij}$. The latter is caused by the PBF processes and by the SWDs. These processes operate in different kinematical

domains so that the imaginary part of the polarization tensor consists of two clearly distinguishable contributions, $\text{Im}\Pi_A^{ij} = \text{Im}\Pi_{\text{PBF}}^{ij} + \text{Im}\Pi_{\text{SWD}}^{ij}$, which we now consider.

A. PBF channel

The imaginary part of $\mathcal{I}_{\text{av}}$, which arises from the poles of the integrand in Eq. (38) at $|\omega| = 2E$ is given by

$$\text{Im}\mathcal{I}_{\text{av}}(\omega > 0) = \frac{\pi}{2}\frac{\Delta^2\Theta(\omega^2 - 4\Delta^2)}{\omega\sqrt{\omega^2 - 4\Delta^2}}\tanh\frac{\omega}{4T}. \quad (82)$$

With the aid of this expression we find

$$\text{Im}\Pi_{ij}^{\text{PBF}}(\omega) = -2\pi\varrho\frac{\Delta^2\Theta(\omega^2 - 4\Delta^2)}{\omega\sqrt{\omega^2 - 4\Delta^2}}\tanh\frac{\omega}{4T}\left\{\delta_{ij} - \langle\bar{b}_i\bar{b}_j\rangle\right.$$
$$-(\delta_{ij} - \delta_{i3}\delta_{3j})\frac{3}{4}\frac{\Delta_1^2}{\Delta^2}\frac{4\Delta^2\omega^2}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$
$$\times\left[\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{3,3}\right) + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{1,1}\right)\right.$$
$$\left.\left.-\frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3}\right]\right\}. \quad (83a)$$

Inserting the imaginary part of the polarization tensor into Eq. (81), we calculate the contraction of $\text{Im}\Pi_{\text{PBF}}^{\mu\nu}$ with the symmetric tensor $k_\mu k_\nu - k^2 g_{\mu\nu}$. This gives

$$\epsilon_{\text{PBF}} = \frac{1}{96\pi^6}G_F^2 C_A^2 \mathcal{N}_\nu p_F M^* \Delta^2 \int_{2\Delta}^\infty d\omega\frac{1}{\sqrt{\omega^2 - 4\Delta^2}}$$
$$\times\frac{1}{\exp\left(\frac{\omega}{T}\right) - 1}\tanh\frac{\omega}{4T}\int_{q<\omega}d^3q\left\{2\omega^2 - q^2\right.$$
$$-\frac{1}{6}\frac{\Delta_1^2}{\Delta^2}\left[\left(1 + \frac{12}{7}\frac{\Delta_3^2}{\Delta_1^2}\right)q_\perp^2 + \left(4 + \frac{18}{7}\frac{\Delta_3^2}{\Delta_1^2}\right)q_z^2\right]$$
$$-\frac{3}{4}[2(\omega^2 - q_z^2) - q_\perp^2]\frac{\Delta_1^2}{\Delta^2}\frac{4\Delta^2\omega^2}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$
$$\times\left[\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{3,3}\right) + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}\left(\frac{\omega^2}{4\Delta^2} - \beta_1^{1,1}\right)\right.$$
$$\left.\left.-\frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3}\right]\right\}. \quad (84)$$

Integration over d^3q can be done in cylindrical frame, where $q_1 = q_\perp\cos\Phi$, $q_2 = q_\perp\sin\Phi$, and $q_3 = q_z$. This results in the neutrino energy losses of the form

$$\epsilon_{\text{PBF}} = \frac{2}{3\pi^5}G_F^2 C_A^2 \mathcal{N}_\nu p_F M^* T^7 y^2 \int_0^\infty\frac{z^4 dx}{(1 + \exp z)^2}$$
$$\times\left\{\frac{4}{5} - \frac{3}{5}\frac{\Delta_1^2}{\Delta^2}\frac{x^2 + y^2}{[x^2 + y^2(1 - \Omega_1^2)][x^2 + y^2(1 - \Omega_2^2)]}\right.$$
$$\times\left[x^2 + y^2(1 - \beta_1^{3,3}) + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}[x^2 + y^2(1 - \beta_1^{1,1})]\right.$$
$$\left.\left.-\frac{4}{3}\frac{\Delta_3}{\Delta_1}y^2\beta_1^{1,3}\right]\right\}, \quad (85)$$

where $z = \sqrt{x^2 + y^2}$, $y = \Delta(T)/T$, and $\Omega_{1,2}$ are defined in Eq. (75). In obtaining Eq. (85) the change is used $\omega = 2T\sqrt{x^2 + \Delta^2/T^2}$.

For a practical usage from Eq. (85), we find

$$\epsilon_{\text{PBF}} = 5.85 \times 10^{20}\left(\frac{M^*}{M}\right)\left(\frac{p_F}{Mc}\right)T_9^7\mathcal{N}_\nu C_A^2\mathcal{F}_{\text{PBF}}(y)\frac{\text{erg}}{\text{cm}^3\,\text{s}}, \quad (86)$$

where M is the bare nucleon mass; $T_9 = T/10^9$ K, and

$$\mathcal{F}_{\text{PBF}}(y) = y^2\int_0^\infty dx\frac{z^4}{(1 + \exp z)^2}$$
$$\times\left\{4 - 3\frac{\Delta_1^2}{\Delta^2}\frac{x^2 + y^2}{[x^2 + y^2(1 - \Omega_1^2)][x^2 + y^2(1 - \Omega_2^2)]}\right.$$
$$\times\left[x^2 + y^2(1 - \beta_1^{3,3}) + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}[x^2 + y^2(1 - \beta_1^{1,1})]\right.$$
$$\left.\left.-\frac{4}{3}\frac{\Delta_3}{\Delta_1}y^2\beta_1^{1,3}\right]\right\}. \quad (87)$$

In the limit $\Delta_3 = 0$, the neutrino energy losses, as given by Eq. (85) reproduce the result obtained in Ref. [12] for the one-component phase $m_j = 0$. It is necessary to notice that Eq. (87) obtained in the angle-average approximation is much simpler for numerical evaluations than the "exact" expression which contains additionally the angle integration [11]. To avoid possible misunderstanding we stress that the gap amplitude $\Delta(T)$ in Eq. (87) is $\sqrt{2}$ times larger than the gap amplitude Δ_{YKL} used in Ref. [31], where the same anisotropic gap $\Delta_{\mathbf{n}} = \Delta\bar{b}(\mathbf{n})$ is written in the form $\Delta_{\mathbf{n}} = \Delta_{YKL}\sqrt{1 + 3\cos^2\theta} \equiv \Delta_{YKL}\sqrt{2}\,\bar{b}(\mathbf{n})$. In other words, $\langle\Delta_{\mathbf{n}}^2\rangle = \Delta^2 = 2\Delta_{YKL}^2$.

B. SWD channel

In the frequency domain $0 < \omega < 2\Delta$, the imaginary part of the weak polarization tensor (80) arises from the poles of the denominator at $\omega = \omega_1$ and $\omega = \omega_2$ and consists of two terms:

$$\text{Im}\Pi_{ij}^{\text{SWD}}(\omega > 0)$$
$$= -2\pi\frac{3}{4}\varrho(\delta^{ij} - \delta^{i3}\delta^{3j})\mathcal{I}_{\text{av}}(\omega_1)\frac{\Delta_1^2}{\Delta^2}\frac{\omega_1\delta(\omega - \omega_1)}{(\Omega_1^2 - \Omega_2^2)}$$
$$\times\left[\Omega_1^2 - \beta_1^{3,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_1^2 - \beta_1^{1,1}) - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3}\right]$$
$$- 2\pi\frac{3}{4}\varrho(\delta^{ij} - \delta^{i3}\delta^{3j})\mathcal{I}_{\text{av}}(\omega_2)\frac{\Delta_1^2}{\Delta^2}\frac{\omega_2\delta(\omega - \omega_2)}{(\Omega_2^2 - \Omega_1^2)}$$
$$\times\left[\Omega_2^2 - \beta_1^{3,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_2^2 - \beta_1^{1,1}) - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3}\right]. \quad (88)$$

According to Eqs. (67) at $\Delta_3 = 0$ one has $\Omega_2 = \beta_{\pm 1}^{3,3}$ and $\text{Im}\Pi_{ij}^{\text{SWD}}(\omega \to \omega_2) = 0$. In other words, the high-frequency spin oscillations cannot be excited if the tensor interactions between the pairing particles are not taken into account.

Inserting Eq. (88) into Eq. (81) and performing trivial calculations, we find two contributions to the neutrino energy

losses. The first contribution is caused by the decay into neutrino pairs of the lowest mode of spin oscillations at $\omega = \omega_1$:

$$
\epsilon_{\text{SWD}}^{(1)} = \frac{1}{320\pi^5} G_F^2 C_A^2 \mathcal{N}_\nu p_F M^* \frac{\Delta_1^2}{\Delta^2} \frac{1}{\Omega_1^2 - \Omega_2^2}
$$
$$
\times \left[\Omega_1^2 - \beta_1^{3,3} - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_1^2 - \beta_1^{1,1}) \right]
$$
$$
\times \frac{\omega_1^7}{\exp\left(\frac{\omega_1}{T}\right) - 1} \int_0^\infty \frac{d\varepsilon}{E} \frac{\Delta^2}{E^2 - \omega_1^2/4} \tanh\frac{E}{2T}. \quad (89)
$$

According to this equation, in the case of $\Delta_3 \to 0$, $\Delta_1 \to \Delta$, and $\omega_1 \to \Delta/\sqrt{5}$, the energy losses are twice less than that found in Ref. [12]:

$$
\epsilon_{\text{SWD}}^{(1)}(\Delta_3 = 0) = \frac{1}{320\pi^5} G_F^2 C_A^2 \mathcal{N}_\nu p_F M^* \frac{\omega_1^7}{\exp\left(\frac{\omega_1}{T}\right) - 1}
$$
$$
\times \int_0^\infty \frac{d\varepsilon}{E} \frac{\Delta^2}{E^2 - \omega_1^2/4} \tanh\frac{E}{2T}. \quad (90)
$$

We use the opportunity to point out the error in Eqs. (79) and (81) of Ref. [12], where the factor of 1/2 is lost.

The second contribution originates from weak decays of the second (higher) mode at $\omega = \omega_2$:

$$
\epsilon_{\text{SWD}}^{(2)} = \frac{1}{320\pi^5} G_F^2 C_A^2 \mathcal{N}_\nu p_F M^* \frac{\Delta_1^2}{\Delta^2} \frac{1}{\Omega_2^2 - \Omega_1^2}
$$
$$
\times \left[\Omega_2^2 - \beta_1^{3,3} - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_2^2 - \beta_1^{1,1}) \right]
$$
$$
\times \frac{\omega_2^7}{\exp\left(\frac{\omega_2}{T}\right) - 1} \int_0^\infty \frac{d\varepsilon}{E} \frac{\Delta^2}{E^2 - \omega_2^2/4} \tanh\frac{E}{2T}. \quad (91)
$$

Because excitation of the high-frequency spin oscillations occurs through the tensor interactions, the contribution of the second mode vanishes if the tensor forces are switched off (i.e., when $\Delta_3 = 0$).

The expressions (89) and (91) can be written in the traditional form,

$$
\epsilon_{\text{SWD}}^{(1)} = 1.76 \times 10^{21} \left(\frac{M^*}{M}\right) \left(\frac{p_F}{Mc}\right) \mathcal{N}_\nu C_A^2 T_9^7 y^7 \frac{\Delta_1^2}{\Delta^2}
$$
$$
\times \left[\Omega_1^2 - \beta_1^{3,3} - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_1^2 - \beta_1^{1,1}) \right]
$$
$$
\times \frac{1}{\Omega_1^2 - \Omega_2^2} \frac{\Omega_1^7}{\exp(2y\Omega_1) - 1} \mathcal{F}_{\text{SWD}}(\Omega_1, y) \frac{\text{erg}}{\text{cm}^3\,\text{s}}, \quad (92)
$$

$$
\epsilon_{\text{SWD}}^{(2)} = 1.76 \times 10^{21} \left(\frac{M^*}{M}\right) \left(\frac{p_F}{Mc}\right) \mathcal{N}_\nu C_A^2 T_9^7 y^7 \frac{\Delta_1^2}{\Delta^2}
$$
$$
+ \left[\Omega_2^2 - \beta_1^{3,3} - \frac{4}{3}\frac{\Delta_3}{\Delta_1}\beta_1^{1,3} + \frac{4}{9}\frac{\Delta_3^2}{\Delta_1^2}(\Omega_2^2 - \beta_1^{1,1}) \right]
$$
$$
\times \frac{1}{\Omega_2^2 - \Omega_1^2} \frac{\Omega_2^7}{\exp(2y\Omega_2) - 1} \mathcal{F}_{\text{SWD}}(\Omega_2, y) \frac{\text{erg}}{\text{cm}^3\,\text{s}}, \quad (93)
$$

where $\beta_1^{l,l'}$, given by Eqs. (63)–(65), are functions of the gap components Δ_1 and Δ_3; $y \equiv \Delta(T)/T$, and

$$
\mathcal{F}_{\text{SWD}}(\Omega, y) = y^7 \int_0^\infty \frac{du}{\sqrt{u^2+1}} \frac{1}{u^2 + 1 - \Omega^2}
$$
$$
\times \tanh\frac{y}{2}\sqrt{u^2+1}. \quad (94)
$$

IX. EFFICIENCY OF THE NEUTRINO EMISSION

In general, the temperature dependence of the gap amplitudes Δ_1 and Δ_3 is to be found with the aid of the gap equations. For simple estimates we take the approximation that the ratio Δ_3/Δ_1 remains constant when the temperature varies, and the temperature dependence of the gap is given by the function $y = \Delta(T)/T$. This function is well investigated for a $^3P_2$ pairing. Because the tensor contribution can be considered as a perturbation [21], in a zero approximation we can use, for example, the simple fit to $\Delta_{YKL}(T)/T = \mathsf{v}_B(\tau)$, as suggested in Ref. [31], where $\tau \equiv T/T_c$. Taking into account that, in Ref. [31], the gap amplitude $\Delta_{YKL}(T)$ is defined by the relation $\Delta_\mathbf{n}^2 = \Delta_{YKL}^2(1 + 3\cos^2\theta)$, while our definition is $\Delta_\mathbf{n}^2(\Delta_3 = 0) = \frac{1}{2}\Delta^2(1 + 3\cos^2\theta)$, we obtain $y(\tau) = \sqrt{2}\mathsf{v}_B(\tau)$.

In Fig. 2 we show the neutrino emissivity ϵ caused by the PBF processes and by the decay of the lowest mode (SWD$_1$) and the higher mode (SWD$_2$) of spin oscillations. The temperature dependence of the emissivity is evaluated at $p_F = 2.1$ fm^{-1} assuming $\Delta_3/\Delta_1 = 0.17$. We set the effective nucleon masses $M^* = 0.7M$; the critical temperature for
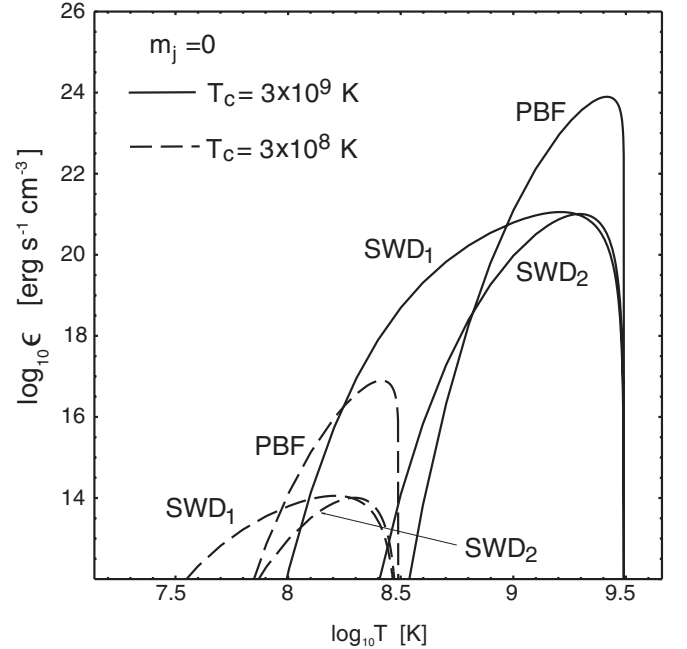
FIG. 2. Temperature dependence of the neutrino emissivity owing to recombination of Cooper pairs (PBF) and owing to decay of spin waves (SWD$_1$, the emissivity of the lower mode; SWD$_2$, the emissivity of the upper mode) at $\Delta_3/\Delta_1 = 0.17$ and $p_F = 2.1$ fm^{-1}.

neutron pairing is chosen to be $T_c = 3 \times 10^9$ K and $T_c = 3 \times 10^8$ K.

One can see that the decay of the low-frequency spin waves into neutrino pairs (SWD$_1$) is very effective at low temperatures, when other known mechanisms of neutrino energy losses in the bulk neutron matter are strongly suppressed by the superfluidity. As discussed in Ref. [12] the neutrino emission caused by the decay of the low-frequency spin waves can dominate the γ radiation within a wide range of low temperatures, which was considered before as the photon-cooling era. A simple estimate has shown that the decays of spin waves can modify the cooling trajectory of neutron stars (see Fig. 5 in Ref. [12]).

Weak decays of the high-frequency mode of spin oscillations occur only if the tensor forces are taken into account in the pairing interaction, that is, if $\Delta_3 \neq 0$. Although the maximal neutrino emission caused by the SWD$_2$ processes is as large as in the SWD$_1$, the neutrino energy losses from the decay of the upper mode decrease more rapidly along with lowering of the temperature. As a result the SWD$_2$ contribution into the total energy losses is negligible in comparison with the sum of the PBF and SWD$_1$ contributions. We found that the latter can be excellently described by the expressions obtained in Ref. [12] for the case of $^3P_2$ pairing with $m_j = 0$ [However, see note after Eq. (90).] We quote these simple expressions for
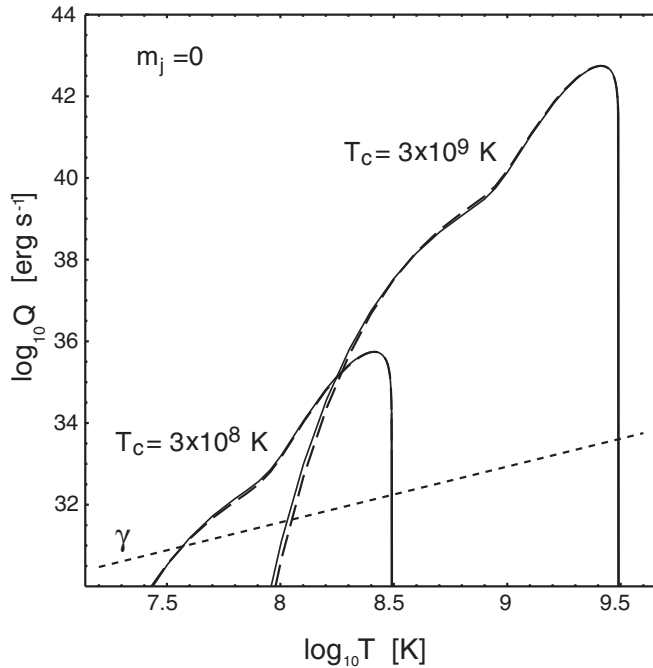

FIG. 3. Temperature dependence of the total bulk neutrino luminosity from a homogeneous superfluid core owing to recombination of Cooper pairs (PBF) and owing to decay of spin waves (SWD) for $p_F = 2.1$ fm^{-1}. The effective mass is taken to be $M^* = 0.7M$. Solid lines are calculated according to the exact expressions (86), (92), (93). Dashed lines are calculated by simplified formulas (95) and (97). Volume of the triplet condensate is estimated as 7×10^{18} cm^3. The short-dashed line is the energy losses per unit of time owing to the surface γ radiation, as calculated in Ref. [2].

references:

$$\epsilon_{PBF} = 5.85 \times 10^{20} \left(\frac{M^*}{M}\right)\left(\frac{p_F}{Mc}\right)T_9^7 \mathcal{N}_\nu C_A^2 F_{PBF}(y)\frac{\text{erg}}{\text{cm}^3\,\text{s}},$$
(95)

with

$$F_{PBF}(y) = y^2 \int_0^\infty dx \frac{z^4}{(1 + \exp z)^2},$$
(96)

and

$$\epsilon_{SWD} = 1.37 \times 10^{19} \left(\frac{M^*}{M}\right)\left(\frac{p_F}{Mc}\right)T_9^7 \mathcal{N}_\nu C_A^2$$
$$\times \frac{(y/\sqrt{5})^7 \mathcal{I}_0(y)}{\exp(y/\sqrt{5}) - 1}\frac{\text{erg}}{\text{cm}^3\,\text{s}},$$
(97)

where

$$\mathcal{I}_0(y) = \int_0^\infty \frac{du}{(u^2 + 1)^{3/2}} \tanh \frac{y}{2}\sqrt{u^2 + 1}.$$
(98)

To get an idea of the accuracy of the simplified expressions, in Fig. 3 we demonstrate the total neutrino energy losses caused by PBF and SWD neutrino emission from the superfluid core of the volume 7×10^{18} cm^3. The total neutrino luminosity, as calculated by the exact Eqs. (86), (92), and (93), is shown in comparison with the sum of the PBF and SWD neutrino losses calculated with the aid of simple expressions given by Eqs. (95) and (97). We display also the the luminosity of the surface photon radiation. The latter is taken as in Fig. 20 of Ref. [2].

X. SUMMARY AND CONCLUSION

According to modern theories the triplet-spin superfluid condensate in the inner core of neutron stars arises owing to pairing of neutrons caused by attractive spin-orbit and tensor forces and consists of the orbital contributions corresponding to $l = 1, 3$. Assuming the projection of the total angular momentum $m_j = 0$, the superfluid energy gap Δ arising in the $^3P_2$-$^3F_2$ state is $\Delta^2 = \Delta_1^2 + \Delta_3^2$, where the contribution $\Delta_3 \neq 0$ is caused by the tensor interactions. We have studied the influence of the admixture of the $^3F_2$ state onto the collective spin oscillations and neutrino emission processes in the triplet superfluid neutron liquid.

To evaluate the rate of neutrino energy losses out of the $^3P_2$-$^3F_2$ superfluid neutron liquid we have calculated the anomalous three-point vertices responsible for the interaction of the superfluid liquid with an external axial-vector field. The BCS-like calculation has done in the angle average approximation. The polelike behavior of the vertices points out the existence of two twofold eigenmodes of oscillations. The oscillation frequencies in terms of the gap components are given by Eqs. (68) and (69). According to the obtained expressions the known low-frequency mode $\omega_1 \simeq \Delta/\sqrt{5}$ undergoes only a small frequency shift owing to the tensor interactions. The frequency of the new, upper mode $\omega_2(\Delta_3 = 0) \simeq \sqrt{58/35}\Delta$ decreases slightly along with increasing of the tensor contribution into the energy gap. We found that the excitation of the high-frequency spin oscillations is strongly quenched if the tensor interactions between the

pairing particles are not taken into account, that is, if $\Delta_3 = 0$. According to calculations of different authors, at the Fermi surface one has $\Delta_3 \simeq 0.17\Delta_1$ (see, e.g., Ref. [18]). In this case our theoretical analysis predicts two twofold modes: $\omega = \omega_1 \simeq 0.42\Delta$ and $\omega = \omega_2 = 1.19\Delta$.

We have derived the linear response of the superfluid liquid onto an external axial-vector field. At the timelike momentum transfer the imaginary part of this polarization function consists of three contributions originating from a recombination of broken Cooper pairs and from weak decays of the collective modes of spin oscillations. Accordingly, the neutrino energy losses through neutral weak currents consist of three contributions caused by the above decay processes. The neutrino energy losses owing to PBF, SWD$_1$, and SWD$_2$ processes are presented analytically by Eqs. (86), (92), and (93).

Neutrino decays of the low-energy spin waves (SWD$_1$) can play an important role in the cooling scenario of neutron stars.

Previously we have demonstrated (see Fig. 5 in Ref. [12]) that the decays of spin waves with $\omega = \Delta/\sqrt{5}$ can become the dominant cooling mechanism in a wide range of low temperatures and modify the cooling trajectory of neutron stars.

Weak decays of the high-frequency mode (SWD$_2$) occur only if the tensor forces are taken into account in the pairing interaction, that is, if $\Delta_3 \neq 0$. The maximal neutrino emission caused by the SWD$_2$ processes is of the same order as in the SWD$_1$; however, the neutrino energy losses from the decay of the upper mode decrease more rapidly along with lowering of the temperature. As a result the SWD$_2$ contribution into the total energy losses is negligible in comparison with the sum of the PBF and SWD$_1$ contributions. This fact makes it possible to neglect the SWD$_2$ contribution and describe the neutrino energy losses from the $^3P_2$-$^3F_2$ superfluid liquid by simple expressions given by Eqs. (95) and (97).

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