Zeros of 6-j symbols: Atoms, nuclei, and bosons

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The absence of certain LS states in atoms leads to the vanishing of several 6-j symbols. One of these vanishing 6-j's explains the absence of a certain jj coupling state in a nucleus, while the other explains the vanishing of a certain state for a system of three bosons. This is part of a continuing study of "companion problems." It is noted that the vanishing 6-j's play an important role for establishing partial dynamical symmetries. Whenever possible we offer alternate explanations that do not involve 6-j symbols. Extensions to vanishing 9-j symbols are also shown. Regge symmetries help to make connections between different topics.

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I. INTRODUCTION

In this work we address the problem of missing states for certain configurations of fermions—both in LS coupling and jj coupling—and also missing states for bosons. Unity is brought to these *a priori* different topics by relations involving 6-j symbols, which have similar appearances in the above three categories. Many, but not all, of the relations involve the vanishings of 6-j symbols. Extensions to 9-j symbols are also shown, as well as applications to partial dynamical symmetries. Some results presented here are familiar; some are not. The main virtue of this work is to bring these diverse topics all into one place. The topics addressed are fermions in LS coupling, fermions in jj coupling, and spinless bosons.

II. *LS* COUPLING IN ATOMS AND NUCLEI

In a 1989 paper by Judd and Li [1] it was noted that for three electrons in the g^3 configuration (LS coupling) there is no quartet state ^4D. For those not familiar with the atomic notation the superscript refers to the spin degeneracy. In the context of nuclear physics we would say that the spin S is equal to $3/2$. They were able to show that the nonexistence of the quartet state was due to the vanishing of two 6-j symbols:

$$\begin{Bmatrix} 4 & 2 & 5 \\ 4 & 4 & 3 \end{Bmatrix} = 0, \qquad \begin{Bmatrix} 4 & 8 & 7 \\ 4 & 4 & 5 \end{Bmatrix} = 0. \tag{1}$$

This was generalized to the following relations for even L:

$$\begin{Bmatrix} L & 2 & (L+1) \\ L & L & (L-1) \end{Bmatrix} = 0, \tag{2}$$

$$\begin{Bmatrix} L & (3L-4) & (2L-1) \\ L & L & (2L-3) \end{Bmatrix} = 0. \tag{3}$$

We will consider not only three g electrons but three L electrons where L is even. We take as a given that there are no states of this configuration with $S = \frac{3}{2}$ and $L_T = 2$, the latter being the total angular momentum; i.e., there are no ^4D states of the L^3 configuration for any even L. We then find out what are the mathematical consequences. This will involve relations among the 6-j symbols. The results will of course also apply to three identical nucleons in LS coupling—three neutrons or three protons. In the nuclear case LS coupling is a better approximation for light nuclei, whereas jj coupling is better for heavier nuclei.

Note that $S = \frac{3}{2}$ is the maximum spin for three electrons. The $S = \frac{3}{2}$ spin-wave function must be symmetric since the $M_S = \frac{3}{2}$ state must have all three electrons with spin up. Hence the orbital part of the wave function must be antisymmetric.

First we couple two of the electrons to a godparent angular momentum L_G, which must be odd so that the two electrons have an antisymmetric wave function. The possible values of L_G are $L-1$ and $L+1$, and there is no loss in generality in choosing the former.

We then antisymmetrize the state $[(LL)^{L-1}L]^2$:

$$\Psi = \left[1 - 2(2L-1)\begin{Bmatrix} L & L & (L-1) \\ L & 2 & (L-1) \end{Bmatrix}\right][(LL)^{L-1}L]^2 -2\sqrt{(2L-1)(2L+3)}\begin{Bmatrix} L & L & (L-1) \\ L & 2 & (L+1) \end{Bmatrix}[(LL)^{L+1}L]^2. \tag{4}$$

Since Ψ is zero, the coefficients of the two basic states in Eq. (4) must vanish. This leads to Eq. (2) for the second term, while for the first term we obtain

$$1 - 2(2L-1)\begin{Bmatrix} L & L & (L-1) \\ L & 2 & (L-1) \end{Bmatrix} = 0. \tag{5}$$

These relations can be verified case by case from tables of 6-j symbols.

We next show that the nonexistence of states with $S = \frac{3}{2}$ and $L_T = (3L-4)$, also of the L^3 configuration with even L, leads to other relations involving 6-j symbols. We choose the value of the godparent as $L_G = (2L-3)$. Antisymmetrizing $\Psi = A[(LL)^{2L-3}L]^{3L-4} + B[(LL)^{2L-1}\,L]^{3L-4}$ gives us

$$A = 1 - 2(4L-5)\begin{Bmatrix} L & L & (2L-3) \\ L & (3L-4) & (2L-3) \end{Bmatrix}, \tag{6}$$

$$B = 2\sqrt{(4L-5)(4L-1)}\begin{Bmatrix} L & L & (2L-3) \\ L & (3L-4) & (2L-1) \end{Bmatrix}. \tag{7}$$

Since the state Ψ does not exist we must have $A = 0$ and $B = 0$.

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But we can here actually prove that the state with $L_T = (3L-4)$ does not exist. The maximum M state for three L electrons with $S = \frac{3}{2}$ is equal to $L + (L-1) + (L-2) = 3L-3$. Thus we have a state with $L_T = L_{\max} = (3L-3)$ and $M_T = (3L-3)$. There is only one way to form a state with $M = 3L-4$. The M values of the three electrons are L, $(L-1)$, and $(L-3)$. This state must be part of the $(3L-3)$ multiplet. Thus, we cannot have a state with $S = \frac{3}{2}$ and $L_T = (3L-4)$.

From what was mentioned above there also cannot be states with $L_T = 3L$, $3L-1$, and $3L-2$. Concerning the latter we can nevertheless try to antisymmetrize the state $\Psi = [(LL)^{2L-1}L]^{L_T}$, where L_T can be either $(3L-2)$ or $(3L-1)$. This leads to the following relation:

$$1 - 2(4L-1)\begin{Bmatrix} L & L & (2L-1) \\ L & L_T & (2L-1) \end{Bmatrix} = 0. \quad (8)$$

As an example of the considerations in this section we see that for three electrons in the g shell we cannot have quartet states with L_T equal to 2, 8, 10, 11, and 12.

III. BOSONS—THE ODD-L CASE

We can make use of the vanishing 6-j of Eq. (2) for odd L. Consider three spinless L bosons with L odd with total angular momentum $J = 2$. We construct a symmetric wave function $\{[L(1)L(2)]^{J_0}L(3)\}^2 + \{[L(3)L(2)]^{J_0}L(1)\}^2 + \{[L(1)L(3)]^{J_0}L(2)\}^2$. Here J_0 can be $L-1$ or $L+1$. Using Racah algebra this equals

$$\left\{1 + 2\Sigma[(2J_0+1)(2J_a+1)]^{1/2}\begin{Bmatrix} L & L & J_0 \\ L & 2 & J_a \end{Bmatrix}\right\} \times \{[L(1)L(2)]^{J_a}L(3)\}^2 \qquad (J_a\text{even}). \quad (9)$$

Taking $J_0 = L-1$, we find

$$\begin{Bmatrix} L & L & L-1 \\ L & 2 & L-1 \end{Bmatrix} = -\frac{1}{2(2L-1)}, \quad (10)$$

so the $J_a = L-1$ term vanishes, but what about the $J_a = L+1$ term? We find that indeed

$$\begin{Bmatrix} L & L & L-1 \\ L & 2 & L+1 \end{Bmatrix} = 0. \quad (11)$$

This is one of many cases of nontrivial vanishings of certain 6-j symbols. Hence there will be no $J = 2$ states of three bosons with odd L. Note that this is the same condition as Eq. (2). For $L = 2$ this condition explains why there is no ^4D state for three electrons (or identical fermions, e.g., neutrons) of the g^3 configuration. For $L = 3$ it explains why there are no $J = 2$ states of three spinless bosons of the L^3 configuration.

It is also true that for bosons in a single L shell there is no state with $J = J_{\max} - 1$. One can see this from the tables of Bayman and Lande [2] and the online book by Dommelen [3]. One can also show it analytically.

The value of $J_{\max}$ is nL where n is the number of bosons. This is also the value of $M_{\max.}$ One can construct a state with $M = M_{\max} - 1$ by changing the M value of the ith particle to $L-1$. Call such a many-particle state ψ_i. There are n such states but the only symmetric wave function is $\sum \psi_i$. This state must belong to the $J_{\max}$ multiplet and so there cannot be any state of n bosons in a single L shell with $J = J_{\max} - 1$—the same as in the fermion case.

One can nevertheless try to construct such a state by coupling two L bosons to $L_G = 2L$ and symmetrizing the state $[[(LL)^{2L}L]^{(3L-1)}]$. The nonexistence of this state leads to a condition that is valid not only for odd L but also for even L and half-integer L:

$$1 + 4(4L+1)(-1)^{2L}\begin{Bmatrix} L & L & 2L \\ L & (3L-1) & 2L \end{Bmatrix} = 0 \quad (12)$$

IV. FERMIONS IN jj COUPLING

The conditions in Eqs. (2) and (3) which were originally derived for even L not only hold for odd L but also for half-integer spin and are therefore useful in jj coupling situations. Indeed Talmi [4] obtained this result by constructing a coefficient of fractional parentage to a state of three neutrons in a single-j shell which he knew did not exist. In particular $J_{\max}$ for three identical fermions is equal to $M_{\max} = j + (j-1) + (j-2) = 3j-3$. There is only one state with $M = M_{\max} - 1$. One moves a nucleon with $M = (j-2)$ to the state with $M = (j-3)$. This state must belong to the $J_{\max}$ multiplet, so there cannot be a state with $J = J_{\max} - 1$. Trying to calculate coefficients of fractional parentage to the nonexistent state leads to the result

$$\begin{Bmatrix} j & (3j-4) & (2j-1) \\ j & j & (2j-3) \end{Bmatrix} = 0. \quad (13)$$

This is the same as Eq. (3) but for half integer j is greater than 3/2.

Sometimes the vanishing 6-j's are part of a bigger picture. Robinson and Zamick [5] used this relationship along with some "diagonal conditions" to demonstrate that for a system of two protons and one neutron in a single-j shell a partial dynamical symmetry (PDS) occurred when one sets all two-body matrix elements with $T = 0$ to zero in a shell-model calculation. It turns out that not only J but also J_p and J_n separately are good quantum numbers. Furthermore, states with the same J_p and J_n are degenerate. The diagonal conditions are

$$\begin{Bmatrix} j & j & (2j-1) \\ j & I & (2j-1) \end{Bmatrix} = \frac{(-1)^{2j}}{8j-2}, \quad (14)$$

where $I = (2j-1)$, $(3j-2)$, and $(3j-4)$.

Another known fact is that there are no $J = \frac{1}{2}$ states for three identical fermions in a single-j shell. This problem has been addressed by Talmi [4,6] and Zhao and Arima [7]. We include this case here for completeness. To show the consequences of this for 6-j relations we first couple two fermions to an even angular momentum J_0. If $j + 1/2$ is even, J_0 must be $j + 1/2$; if odd, J_0 must be $j - 1/2$. We then add the third fermion and couple the combination to $J = \frac{1}{2}$. We then antisymmetrize. The fact that $J = \frac{1}{2}$ states for three fermions do not exist leads to the following relations: if

$j+1/2$ is even we get

$$1+2(2j+2)\begin{Bmatrix} j & j & (j+1/2) \\ j & 1/2 & (j+1/2)\end{Bmatrix}=0. \quad (15)$$

If $j+1/2$ is odd we get

$$1+4j\begin{Bmatrix} j & j & (j-1/2) \\ j & 1/2 & (j-1/2)\end{Bmatrix}=0. \quad (16)$$

V. VANISHING 9-j'S

From the fact that some states do not exist for four fermions in a j shell Robinson and Zamick [5,8] were able to show that certain 9-j symbols vanished (see also the related work by Zhao and Arima [9]). This was an extension of the above arguments from Talmi about three fermions [4]. One of their results is

$$\begin{Bmatrix} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-3) & (4j-4)\end{Bmatrix}=0. \quad (17)$$

They also used this relation for a different physical problem—a system of two neutrons and two protons. If one sets all the two-body interaction matrix elements with isospin $T=0$ to zero then a PDS emerges. There are certain angular momenta for this system that cannot occur for a system of four identical fermions.

The PDS applies to these angular momenta. It turns out that not only total J but also J_p and J_n separately are good quantum numbers, and this is carried by the vanishing of the above 9-j symbol. Furthermore, states with the same J_p and J_n are degenerate.

The diagonal conditions are

$$\begin{Bmatrix} j & j & (2j-3) \\ j & j & (2j-1) \\ (2j-3) & (2j-1) & I\end{Bmatrix}=\frac{1}{4(4j-5)(4j-1)}, \quad (18)$$

for $I=(4j-4)$, $(4j-5)$, and $(4j-7)$, and

$$\begin{Bmatrix} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-1) & I\end{Bmatrix}=\frac{1}{2(4j-1)^2}, \quad (19)$$

for $I=(4j-4)$ and $(4j-2)$.

Let us now apply these results in more detail. For three identical particles in a j shell the maximum J is $j+(j-1)+(j-2)=(3j-3)$. For one proton and two neutrons the maximum value is $(2j-1)+j=(3j-1)$. Hence states with $J=3j-2$ and $3j-1$ are part of the PDS. These have high spins and so the single-j model might work better. Also belonging to the PDS are states with $J=\frac{1}{2}$ and $J_{\max}=3j-4$; the last one belongs because there are no states with $J=J_{\max}-1$ for identical fermions (this is also true for identical bosons).

For four nucleons (or holes) the maximum J is $j+(j-1)+(j-2)+(j-3)=4j-6$. However, for two protons and two neutrons the maximum J is $(2j-1)+(2j-1)=4j-2$. Hence states with $J=(4j-5)$, $(4j-4)$, $(4j-3)$, and $(4j-2)$ belong to the PDS. These are again high-spin states so the single-j shell might work fairly well for these. There might be other states with PDS, e.g., as noted above, $J=3$ and 7 in the $f_{7/2}$ shell.

Consider next three nucleons in the $g_{9/2}$ shell. If they are identical, $J_{\max}=\frac{21}{2}$. For a system of two protons and one neutron the value of $J_{\max}=\frac{25}{2}$. We get a degenerate set $J_p=9$, $J_n=\frac{9}{2}$ with total angular momenta $J=\frac{19}{2}$, $\frac{23}{2}$, and $\frac{25}{2}$, all with isospin $T=\frac{1}{2}$.

Consider four nucleons in the $g_{9/2}$ shell. If they are all identical, $J_{max}=12$. For two protons and two neutrons, $J_{\max}=16$. Here are selected sets of degenerate states for four nucleons in the $g_{9/2}$ shell:

J_p	J_n		
8	8	$J=14, 16$	$T=0$
8	6	$J=11, 13, 14$	$T=0$

There are more. Above, (8,6) is an abbreviation for (8,6)+$(-1)^{J+T}$(6,8). For the (8,6) configuration there is also a degeneracy of $J=8$ and 9. The above considerations do not explain this.

The discovery of the $J=16^+$ isomeric state in ^{96}Cd has been recently reported [10]. It lies below the $J=14^+$ state and so the long lifetime is due to a spin gap. It can only decay by gamma emission to a 10^+ state via an E6 transition, and this is highly inhibited. If we set all $T=0$ 2 body matrix elements in the $g9/2$ shell to zero, keeping only $T=1$, then the J $=16^+$ and J $=14^+$ would be degenerate, as just noted. Evidently the $T=0$ part of the two-body interaction moves the $J=14^+$ state above $J=16^+$.

VI. VANISHING 6-j IN THE f SHELL—RACAH, JUDD AND ELLIOTT, AND REGGE

Racah noted that for electrons in the f shell the calculation of coefficients of fractional parentage could be greatly simplified by noting that the exceptional group G2 is a subgroup of SO(7) [11].

The proof involved noting the following 6-j relation: $\{\begin{smallmatrix}5&5&3\\3&3&3\end{smallmatrix}\}=0$. Regge [12] found several symmetry relations for 6-j symbols, one of which is

$$\begin{Bmatrix} a & b & e \\ d & c & f\end{Bmatrix}=\begin{Bmatrix} a & 1/2(b+c+e-f) & 1/2(b-c+e+f) \\ d & 1/2(b+c-e+f) & 1/2(-b+c+e+f)\end{Bmatrix}. \quad (20)$$

Early on, Judd and Elliott [13] used this to show that

$$\begin{Bmatrix} 5 & 5 & 3 \\ 3 & 3 & 3\end{Bmatrix}=\begin{Bmatrix} 5 & 4 & 4 \\ 3 & 4 & 2\end{Bmatrix}. \quad (21)$$

See also the work of Judd and Li [1]. Furthermore we emphasized at the beginning of this work that for quartet states of three electrons in the g shell the space wave function has to be antisymmetric. This leads to the vanishing of the 6-j on the right-hand side above. This is easier to understand than the f-shell result of Racah [11]. Thus we have an amusing connection between electrons in the f shell and those in the

g shell, and some of the mystery of the vanishing Racah has been removed. The above result has also been used by Vanden Berghe *et al*. [14]. They show many other examples of vanishing 6-j's.

It should be noted that the following Regge symmetry relation (previously used by Robinson and Zamick [5]) can shed some light on the relation between Eqs. (2) and (3):

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} (b+c+f-e)/2 & (a+c+d-f)/2 & (a+b+e-d)/2 \\ (c+e+f-b)/2 & (a+d+f-e)/2 & (d+c+b-a)/2 \end{Bmatrix}. \tag{22}$$

This leads to the relation

$$\begin{Bmatrix} j & j & (2j-3) \\ (3j-4) & j & (2j-1) \end{Bmatrix} = \begin{Bmatrix} (2j-2) & (2j-3) & 2 \\ (2j-2) & (2j-1) & (2j-2) \end{Bmatrix}. \tag{23}$$

This is true for even j, odd j, and half-integer j. We have already shown that the 6-j on the left which appears in Eq. (3) vanishes for even j and half integer j by constructing all m states for $M_{\max} = J_{\max}$ and $J_{\max} - 1$. This did not involve 6-j symbols explicitly. We can now use this relation to show selected vanishings in Eq. (2), e.g.,

$$\begin{Bmatrix} 4 & 4 & 5 \\ 8 & 4 & 7 \end{Bmatrix} = \begin{Bmatrix} 6 & 2 & 5 \\ 6 & 6 & 7 \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} 7/2 & 7/2 & 4 \\ 13/2 & 7/2 & 6 \end{Bmatrix} = \begin{Bmatrix} 5 & 4 & 2 \\ 5 & 6 & 5 \end{Bmatrix} = 0.$$

We thus establish connections between 6-j's whose vanishing can be obtained from m-state arguments to selected states which have "two" in them, as per Eq. (2).

We note that explicit expressions for 6-j symbols with a "two" in them have been worked out by Biedenharn *et al.* [15]. Using their notation, we find from their results that $\{\begin{smallmatrix} l_1 & J_1 & 2 \\ J_2 & l_2 & L \end{smallmatrix}\}$ for $l_2 = J_1 + 1$ and $l_1 = J_1 + 1$ is proportional to X where $X = [(J_1+1)(J_1-J_2) - L(L+1) + J_2(J_2+2)]$. We have $L = 2j-2, J_1 = 2j-3, l_1 = 2j-2, J_2 = 2j-2$, and $l_2 = 2j-1$. With these values we see that X vanishes.

VII. CLOSING

In summary we have in this work mainly addressed the problem of missing states for fermions in LS coupling, especially electrons, fermions in jj coupling, especially for particles of one kind, e.g., neutrons only or protons only, and of bosons. We note that very similar expressions apply in the different cases. For example, the nonexistence of quartet $S = 3/2$ states with total angular orbital momentum $L_T = 2$ for an L^3 configuration with even L is closely associated with the nonexistence of spinless boson states also with $L_T = 2$ but for an odd L, L^3 configuration. We also have shown that in all three cases states with $J = J_{\max} - 1$ did not exist. We were able to obtain these results not only in terms of vanishing 6-j symbols but also by counting the number of m states. On the other side we have shown that the value of having 6-j symmetries is greatly enhanced by the Regge symmetry relations. They help to establish connections with what were *a priori* diverse subjects. By putting all these results in one place we hope we have conveyed the beautiful unity that pervades the problem of missing states.

This work can be regarded as an extension of previous work on companion problems [16]. In the previous contributions we showed how similar expressions have consequences on different physical problems and different branches of physics, e.g., how isospin can be used to get the same results as quasispin [16,17]. We find such associations fascinating. In this work we show that an expression involving even L which was used to explain the absence of certain states in LS coupling can be generalized to odd L in order to explain the absence of certain bosonic states and can also be generalized to half-integer angular momenta to explain the absence of certain states in jj coupling, the latter being most relevant to nuclear physics.

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[1] J. B. Judd and S. Li, J. Phys. B **22**, 2057 (1989).
[2] B. F. Bayman and A. Lande, Nucl. Phys. **77**, 1 (1966).
[3] L. Van Dommelen, *Dommelen Quantum Mechanics for Engineers*, 5.40 alpha, Online Book, [http://www.eng.fsu.edu/~dommelen/quantum/].
[4] I. Talmi, *Simple Models of Complex Nuclei* (Harwood Academic, Switzerland, 1993).
[5] S. J. Q. Robinson and L. Zamick, Phys. Rev. C **63**, 064316 (2001).
[6] I. Talmi, Phys. Rev. C **72**, 037302 (2005).
[7] Y. M. Zhao and A. Arima, Phys. Rev. C **68**, 044310 (2003).
[8] S. J. Q. Robinson and L. Zamick, Phys. Rev. C **64**, 057302 (2001).

[9] Y. M. Zhao and A. Arima, Phys. Rev. C **72**, 054307 (2005).
[10] B. S. Nara Singh *et al*., Phys. Rev. Lett. **107**, 172502 (2011).
[11] G. Racah, Phys. Rev. **76**, 1352 (1949).
[12] T. Regge, Il Nuovo Cimento **11**, 116 (1959).
[13] B. R. Judd and J. P. Elliott, *Topics in Atomic and Nuclear Theory* (Caxton, Christchurch, 1970).
[14] G. Vanden Berghe, H. D. De Meyer, and J. Van der Jeugt, J. Math. Phys. **25**, 2585 (1984).
[15] L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Rev. Mod. Phys. **24**, 212 (1952).
[16] L. Zamick and A. Escuderos, Phys. Rev. C **71**, 014315 (2005).
[17] G. Rosensteel and D. J. Rowe, Phys. Rev. C **67**, 014303 (2003).