

General treatment of vortical, toroidal, and compression modesJ. Kvasil,^{1,*} V. O. Nesterenko,^{2,†} W. Kleinig,^{2,3} P.-G. Reinhard,⁴ and P. Vesely^{1,5}¹*Institute of Particle and Nuclear Physics, Charles University, CZ-18000, Praha 8, Czech Republic*²*Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow region, 141980, Russia*³*Technische Universität Dresden, Institut für Analysis, D-01062, Dresden, Germany*⁴*Institut für Theoretische Physik II, Universität Erlangen, D-91058, Erlangen, Germany*⁵*Department of Physics, P.O. Box 35 (YFL) FI-40014, University of Jyväskylä, Finland*

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The multipole vortical, toroidal, and compression modes are analyzed. Following the vorticity concept of Ravenhall and Wambach, the vortical operator is derived and related in a simple way to the toroidal and compression operators. The strength functions and velocity fields of the modes are analyzed in ²⁰⁸Pb within the random-phase approximation using the Skyrme force SLy6. Both convection and magnetization nuclear currents are taken into account. It is shown that the isoscalar (isovector) vortical and toroidal modes are dominated by the convection (magnetization) nuclear current while the compression mode is fully convective. The relation between the above concept of the vorticity and the hydrodynamical vorticity is briefly discussed.

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I. INTRODUCTION

An irrotational character of nuclear flow is a basic assumption in collective nuclear dynamics [1–4], which manifests itself in numerous examples of low-energy excitations and giant resonances (GR). At the same time, nuclear motion can also carry a vorticity (i.e., a deviation from irrotational flow) [5–8]. In hydrodynamics (HD), the vorticity is defined as a curl of the velocity field [9]. Instead, the nuclear theory deals with nuclear currents rather than velocities and so here the vorticity is often defined through the $j_{\lambda\lambda+1}(r)$ component of the multipole decomposition of the transverse nuclear current [7]. The component $j_{\lambda\lambda+1}(r)$ is treated as unrestricted by the continuity equation (containing the current divergence $\vec{\nabla} \cdot \vec{j}$) and so is believed to be of a vortical character. In this case, the vorticity and charge transition density represent two independent parts of the charge-current distribution. This consideration reminds the previous result [10] where the current component $j_{\lambda\lambda-1}(r)$ is proposed to be constrained by the continuity equation and thus determined by the charge distribution while the component $j_{\lambda\lambda+1}(r)$ is treated as independent.

Both definitions of the vorticity, from HD and Ref. [7] are widely used in the literature: the former in the nuclear fluid-dynamical models (see, e.g., Ref. [11] and references therein) and the latter in the microscopic studies (see, e.g., Refs. [7,8]). These two definitions are assumed to be closely related [7], though they are different observables by construction. Actually, they represent different aspects of the nuclear vorticity. In this article, we will concentrate on the $j_{\lambda\lambda+1}(r)$ -based vorticity [7]. The comparison with the HD case will be also done.

The most remarkable manifestation of vorticity is the electric dipole toroidal mode (TM) [12–14] intensively explored

during the last decades (see, e.g., the review in Ref. [15] and references therein). This exotic mode is treated as a vortical collective motion of the toroidal type. The TM operator is the second-order correction to the leading $E1$ operator in the long-wave approximation. Another kind of the second-order $E1$ flow is represented by the anisotropic compression mode (CM), often called the isoscalar dipole GR [16–19]. The mode is viewed as a compression wave in a definite direction and so is related to the nuclear incompressibility [16,19]. The isoscalar ($T = 0$) TM and CM were observed in the (α, α') reaction as broad low-energy (TM dominated) and high-energy (CM dominated) electric dipole distributions [14,18,20–25]. The TM was also investigated in the region of the pygmy resonance in ²⁰⁸Pb in a nuclear fluorescence experiment [26]. Perhaps, the TM can be discriminated in the transverse (e, e') form factors [27].

The TM and CM were examined in various models, including the fluid-dynamical and sum-rule approaches [11, 13,19,28], the method of Wigner function moments [29,30], the random-phase-approximation (RPA) [31] and multiphonon [26] methods with phenomenological single-particle potentials. More refined RPA studies within the self-consistent mean-field approaches were also performed, relativistic ones [32,33] and those based on Skyrme forces [34,35], for a review see Ref. [15]. A direct relation between the current-dependent TM and density-dependent CM operators was established in Refs. [13,31] and both modes were shown to be mixed. Most of the studies reproduce the observed bimodal (low-energy TM and high-energy CM) distribution. However, theoretical models generally overestimate the CM peak energy by ~ 4 MeV and underestimates the TM one by 1–2 MeV. Besides, they yield a much broader TM distribution [15].

A special effort was devoted to the nuclear vorticity as such [5–8]. The possibility to measure the vorticity in (e, e') experiments was discussed in Refs. [8,27,36].

Despite these thorough studies, some principle points concerning the vorticity and related modes deserve further inspection: (i) There is an essential difference in modeling the

*kvasil@ipnp.troja.mff.cuni.cz

†nester@theor.jinr.ru

vortical mode (VM) and their TM and CM counterparts. The TM and CM are usually deduced as second-order terms in a low-momentum expansion of the electric multipole transition operators [12,13]. To the best of our knowledge, an analogous way to the VM operator has yet to be developed. This would help to establish a formal relation between VM and TM/CM. (ii) Most of the previous studies (with the exception of Refs. [7,8,31]) employ only the convection part j_c of the nuclear current and skip its magnetization (spin) part j_m , though the latter can also produce a vorticity. The role of j_m in VM and other modes has yet to be clarified. (iii) Mainly the $T = 0$ channel of the modes were discussed, although their $T = 1$ counterpart is also interesting and deserves a closer look. (iv) The relation between two definitions of the vorticity, from HD [9] and $j_{\lambda\lambda+1}(r)$ current component [7], has yet to be clarified.

The aim of the present study is to explore the open problems listed above. First of all, the vortical operator unconstrained by the charge conservation is derived following the ideas of the authors of Ref. [7]. The operator has a simple relation with its TM/CM counterparts and is also deduced as a second-order term in a low-momentum (long wavelength) expansion of the dominant electric operator. Further, the difference in the vorticity criteria from the HD [9] and Ref. [7] is inspected. In the numerical calculations, both $T = 0$ and $T = 1$ channels of VM, TM, and CM are analyzed by using the full nuclear current $j = j_c + j_m$. The dominant role of j_m and thus the spin vorticity in forming the isovector VM and TM is worked out. Both single-particle and collective mechanisms of the vorticity are discussed.

The numerical analysis is done within the *self-consistent* separable random-phase-approximation (SRPA) approach based on the factorized Skyrme residual interaction [37,38]. The systematic studies of electric [37–42] and magnetic [43–45] GR in spherical and deformed nuclei have established this approach as a reliable and effective theoretical tool.

The article is organized as follows. In Sec. II, the nuclear vorticity is discussed in the context of the previous studies. In Sec. III, the VM, TM, and CM operators are derived on the same theoretical grounds, following the prescription in Ref. [7]. A simple relation between the operators is established. The mode velocities are analyzed and different criteria of the vorticity, from Ref. [7] and HD, are compared. Section IV provides an outline of the calculation scheme within the Skyrme SRPA. In Sec. V, the numerical results for the $T = 0$ and $T = 1$ VM, TM, and CM are discussed. A summary is given in Sec. VI. Appendix A justifies the procedure of building the vortical operator. Appendix B describes the procedure for extraction of the spurious center-of-mass corrections. Appendix C provides the explicit expressions for the density and current operators. Appendix D sketches the basics of SRPA.

II. VORTICITY IN TERMS OF VELOCITIES AND CURRENTS

The HD nuclear models, including the famous liquid-drop model, assume an irrotational character of the collective

nuclear flow [1–3]

$$\vec{\nabla} \times \vec{v}(\vec{r}) = 0, \quad (1)$$

where $\vec{v}(\vec{r})$ is the local velocity field. The deviation from the irrotational flow is quantified by the HD vorticity

$$\vec{\omega}(\vec{r}) = \vec{\nabla} \times \vec{v}(\vec{r}). \quad (2)$$

Unlike the HD models, nuclear theory prefers to deal with currents $\vec{j}_{\text{nuc}}(\vec{r})$ rather than velocities $\vec{v}(\vec{r})$. However, Eqs. (1) and (2) cannot be replaced by the similar expressions for the nuclear current since, as shown below, the curls of velocity and current have a different structure. Moreover, $\vec{\nabla} \times \vec{j}_{\text{nuc}}(\vec{r})$ is the key part of the electrical multipole operator $\hat{M}(E\lambda\mu, k)$ and, so, treating $\vec{\nabla} \times \vec{j}_{\text{nuc}}(\vec{r})$ as a vortical quantity will wrongly mean a fully vortical nature of any $E\lambda\mu$ excitations, which contradicts the predominantly irrotational character of electric GR.

For using the HD definition of vorticity (2), the quantum theory should express it through the nuclear current. A common way is to define the velocity field $\vec{v}_\nu(\vec{r})$ for the excitation mode ν through the current transition density $\delta\vec{j}_\nu(\vec{r})$ and ground-state density $\rho_0(\vec{r})$ [6,46]

$$\delta\vec{j}_\nu(\vec{r}) = \rho_0(\vec{r})\vec{v}_\nu(\vec{r}), \quad (3)$$

which casts Eq. (2) into the form

$$\vec{\nabla} \times \vec{v}_\nu(\vec{r}) = \frac{\rho_0(\vec{r})\vec{\nabla} \times \delta\vec{j}_\nu(\vec{r}) - \vec{\nabla}\rho_0(\vec{r}) \times \delta\vec{j}_\nu(\vec{r})}{\rho_0^2(\vec{r})}. \quad (4)$$

This illuminates the difference between the curls of the velocity and current and thus shows that $\vec{\nabla} \times \delta\vec{j}_\nu(\vec{r})$ cannot be a measure of vorticity. The difference is comprised in the gradient field $\vec{\nabla}\rho_0(\vec{r})$, which is maximal at the nuclear surface and minimal in the interior.

In Ref. [7], a concept of nuclear vorticity, alternative to the HD one and fully based on the nuclear current, was proposed. It aims to find a component of the nuclear current that is not restricted by the continuity equation

$$\vec{\nabla} \cdot \delta\vec{j}_\nu(\vec{r}) = -ikc\delta\rho_\nu(\vec{r}), \quad (5)$$

with k being the transfer momentum. By construction, this current component should not contribute to the irrotational motion and vanish in the divergence $\vec{\nabla} \cdot \delta\vec{j}_\nu(\vec{r})$. So it may be naturally used for building the vortical quantities.

Since the present study follows similar lines, it is worth outlining the concept [7] in more detail. In spherical nuclei, it exploits the multipole expansion of the nucleon and current transition densities

$$\begin{aligned} \delta\rho_{fi}(\vec{r}) &= \langle j_f m_f | \hat{\rho}(\vec{r}) | j_i m_i \rangle \\ &= \sum_{\lambda\mu} a_{\lambda\mu}^{fi} \rho_\lambda(r) Y_{\lambda\mu}(\hat{r}), \end{aligned} \quad (6)$$

$$\begin{aligned} \delta\vec{j}_{fi}(\vec{r}) &= \langle j_f m_f | \hat{j}_{\text{nuc}}(\vec{r}) | j_i m_i \rangle \\ &= -i \sum_{\lambda\mu} \sum_{L=\lambda\pm 1} a_{\lambda\mu}^{fi} j_{\lambda L}(r) \vec{Y}_{\lambda L\mu}(\hat{r}), \end{aligned} \quad (7)$$

with

$$a_{\lambda\mu}^{fi} = (-1)^\mu \frac{(j_i m_i \lambda - \mu | j_f m_f)}{\sqrt{2j_f + 1}}, \quad (8)$$

where j_i, m_i (j_f, m_f) are spins and their projections for the initial i (final f) state, $Y_{\lambda\mu}(\hat{r})$ and $\vec{Y}_{\lambda L\mu}(\hat{r})$ are ordinary and vector spherical harmonics [47]. The $\rho_\lambda(r)$ and $j_{\lambda L}(r)$ are multipole components of the transition density and current. Using the above expansions and the quantity

$$\delta \vec{S}_{fi}(\vec{r}) = \sum_{\lambda\mu} a_{\lambda\mu}^{fi} \sqrt{\frac{\lambda+1}{\lambda}} k c \rho_\lambda(r) \vec{Y}_{\lambda\lambda\mu}(\hat{r}), \quad (9)$$

the unconstrained vortical transition density

$$\begin{aligned} \delta \vec{w}_{fi}(\vec{r}) &= \vec{\nabla} \times \delta \vec{j}_{fi}(\vec{r}) - \delta \vec{S}_{fi}(\vec{r}) \\ &= \sum_{\lambda\mu} a_{\lambda\mu}^{fi} w_{\lambda\lambda}(r) \vec{Y}_{\lambda\lambda\mu}(\hat{r}), \end{aligned} \quad (10)$$

is built [7], where the vortical multipoles

$$w_{\lambda\lambda}(r) = \sqrt{\frac{2\lambda+1}{\lambda}} \left(\frac{d}{dr} + \frac{\lambda+2}{r} \right) j_{\lambda\lambda+1}(r), \quad (11)$$

are determined by the radial current component $j_{\lambda\lambda+1}(r)$. Finally, the vorticity strength is given by

$$v_\lambda = \int_0^\infty dr r^{\lambda+4} w_{\lambda\lambda}(r). \quad (12)$$

The continuity equation (5) in terms of the r^λ moments relates the current component $j_{\lambda\lambda-1}(r)$ to the transition density but leaves the component $j_{\lambda\lambda+1}(r)$ untouched [7]. So, just $j_{\lambda\lambda-1}(r)$ provides the charge-conservation constraint and the quantity (9) is constructed so as to remove $j_{\lambda\lambda-1}(r)$ from $\vec{\nabla} \times \delta \vec{j}_{fi}(\vec{r})$, see more discussion in Appendix A. The unconstrained vortical value (11) includes only $j_{\lambda\lambda+1}(r)$. Moreover, following Ref. [7], the motion is treated as vortical if its current involves $j_{\lambda\lambda+1}(r)$.

The formalism [7] treats the vorticity without using an explicit vortical operator analogous to the TM and CM ones. However, we need such an operator for reasons of comparison of vortical modes with TM and CM. In the next section, we will develop the explicit vortical operator by using the prescription of Ref. [7] and relate this operator with the toroidal and compression ones. The subsequent discussion will show that this operator is not truly vortical in the HD sense and the presence of $j_{\lambda\lambda+1}(r)$ in the current is not enough for the HD vorticity.

III. VM, TM, AND CM OPERATORS

A. Derivation of VM, TM, and CM operators

The standard electrical multipole operator may be written in different forms [48]

$$\begin{aligned} \hat{M}(E\lambda\mu, k) &= -i \frac{(2\lambda+1)!!}{ck^{\lambda+1}(\lambda+1)} \\ &\cdot \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot [\vec{\nabla} \times (\vec{r} \times \vec{\nabla}) j_\lambda(kr) Y_{\lambda\mu}(\hat{r})] \end{aligned} \quad (13)$$

$$\begin{aligned} &= \frac{(2\lambda+1)!!}{ck^{\lambda+1}} \sqrt{\frac{\lambda}{\lambda+1}} \\ &\cdot \int d^3r [j_\lambda(kr) \vec{Y}_{\lambda\lambda\mu}(\hat{r})] \cdot [\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})], \end{aligned} \quad (14)$$

where $j_\lambda(kr)$ is the spherical Bessel function.

The form (14) shows that $\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})$ cannot be a measure of the vorticity since otherwise $\hat{M}(E\lambda\mu, k)$ would indicate only vortical electric excitations. At the same time, the form (14) suggests that the vortical operator may be built from $\hat{M}(E\lambda\mu, k)$ by replacing $\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})$ with the truly vortical quantity [7]

$$\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r}) - \frac{i}{\lambda} k c [\vec{\nabla} \hat{\rho}(\vec{r}) \times \vec{r}]. \quad (15)$$

The density-dependent term in Eq. (15) subtracts the charge-conservation constraint. Actually it plays a similar role as the right-hand side (r.h.s.) second term in the vortical transition density (10). Both prescriptions, Eqs. (10) and (15), have the same intention, but being applied to different quantities (transition densities and operators), lead to formally different recipes. While Eq. (10) excludes the $j_{\lambda\lambda-1}(r)$ terms, the recipe (15) gives an exact compensation of the lowest-order k terms in the final vortical operator, see the derivation below. Both Eqs. (10) and (15) remind the r.h.s. of the HD expression (4). They are compared and discussed in more detail in Appendix A.

By using Eq. (15), the vortical operator is defined as

$$\hat{M}_{\text{vor}}(E\lambda\mu, k) = \hat{M}(E\lambda\mu, k) - \hat{M}_S(E\lambda\mu, k), \quad (16)$$

that is, as a difference of the electric operator (14) and the subsidiary operator

$$\begin{aligned} \hat{M}_S(E\lambda\mu, k) &= i \frac{(2\lambda+1)!!}{k^\lambda \sqrt{\lambda(\lambda+1)}} \\ &\cdot \int d^3r [j_\lambda(kr) \vec{Y}_{\lambda\lambda\mu}(\hat{r})] \cdot [\vec{\nabla} \hat{\rho}(\vec{r}) \times \vec{r}]. \end{aligned} \quad (17)$$

The second equation may be also written in the forms

$$\begin{aligned} \hat{M}_S(E\lambda\mu, k) &= -\frac{(2\lambda+1)!!}{k^\lambda \sqrt{\lambda(\lambda+1)}} \int d^3r \hat{\rho}(\vec{r}) j_\lambda(kr) \hat{l} \cdot \vec{Y}_{\lambda\lambda\mu}(\hat{r}) \end{aligned} \quad (18)$$

$$= -\frac{(2\lambda+1)!!}{k^\lambda} \int d^3r \hat{\rho}(\vec{r}) j_\lambda(kr) Y_{\lambda\mu}(\hat{r}) \quad (19)$$

$$= -i \frac{(2\lambda+1)!!}{ck^{\lambda+1}} \int dr^3 j_\lambda(kr) Y_{\lambda\mu}(\hat{r}) [\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})]. \quad (20)$$

The form (20) is obtained by using the operator continuity equation

$$\vec{\nabla} \cdot \hat{j}_{\text{nuc}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] = -ikc \hat{\rho}. \quad (21)$$

In the long-wavelength approximation ($k \rightarrow 0$), we keep only the first and second terms in the expansion of the spherical Bessel function

$$j_\lambda(kr) = \frac{(kr)^\lambda}{(2\lambda+1)!!} \left[1 - \frac{(kr)^2}{2(2\lambda+3)} + \dots \right], \quad (22)$$

and thus get for the electric and subsidiary operators

$$\hat{M}(E\lambda\mu, k) \approx \hat{M}(E\lambda\mu) + k \hat{M}_{\text{tor}}(E\lambda\mu), \quad (23)$$

$$\hat{M}_S(E\lambda\mu, k) \approx \hat{M}(E\lambda\mu) - k \hat{M}_{\text{com}}(E\lambda\mu), \quad (24)$$

where

$$\begin{aligned} \hat{M}(E\lambda\mu) &= \frac{i}{kc} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot \vec{\nabla}[r^\lambda Y_{\lambda\mu}(\hat{r})] \\ &= -\frac{i}{kc} \int d^3r [\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})] r^\lambda Y_{\lambda\mu}(\hat{r}) \\ &= -\int d^3r \hat{\rho}(\vec{r}) r^\lambda Y_{\lambda\mu}(\hat{r}), \end{aligned} \quad (25)$$

is the familiar electrical operator in the long-wavelength limit (lowest-order term) and

$$\begin{aligned} \hat{M}_{\text{tor}}(E\lambda\mu) &= \frac{i}{2c(\lambda+1)(2\lambda+3)} \\ &\cdot \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot [\vec{\nabla} \times (\vec{r} \times \vec{\nabla}) r^{\lambda+2} Y_{\lambda\mu}(\hat{r})] \\ &= \frac{i}{c(\lambda+1)} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot \vec{r} r^\lambda Y_{\lambda\mu}(\hat{r}) \\ &\quad + k \frac{\lambda+3}{2(\lambda+1)(2\lambda+3)} \int d^3r \hat{\rho}(\vec{r}) r^{\lambda+2} Y_{\lambda\mu}(\hat{r}), \end{aligned} \quad (26)$$

$$\begin{aligned} \hat{M}_{\text{com}}(E\lambda\mu) &= \frac{i}{2c(2\lambda+3)} \\ &\cdot \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot \vec{\nabla}[r^{\lambda+2} Y_{\lambda\mu}(\hat{r})], \end{aligned} \quad (27)$$

are toroidal and compressional operators, respectively. Note that both toroidal expressions (26) and (27) involve the function $r^{\lambda+2} Y_{\lambda\mu}(\hat{r})$ thus manifesting the relation between TM and CM. In Eq. (27), the second term precisely gives the CM operator [31].

In Eq. (16), the lowest-order k terms from Eqs. (23) and (24) exactly compensate each other and so we get

$$\hat{M}_{\text{vor}}(E\lambda\mu, k) = k [\hat{M}_{\text{tor}}(E\lambda\mu) + \hat{M}_{\text{com}}(E\lambda\mu)]. \quad (29)$$

Using the definition $\hat{M}_{\text{vor}}(E\lambda\mu, k) = k \hat{M}_{\text{vor}}(E\lambda\mu)$, we finally come to the relation

$$\hat{M}_{\text{vor}}(E\lambda\mu) = \hat{M}_{\text{tor}}(E\lambda\mu) + \hat{M}_{\text{com}}(E\lambda\mu), \quad (30)$$

where

$$\begin{aligned} \hat{M}_{\text{vor}}(E\lambda\mu) &= -\frac{i}{c(2\lambda+3)} \sqrt{\frac{2\lambda+1}{\lambda+1}} \\ &\cdot \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) r^{\lambda+1} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}), \end{aligned} \quad (31)$$

$$\begin{aligned} \hat{M}_{\text{tor}}(E\lambda\mu) &= -\frac{i}{2c} \sqrt{\frac{\lambda}{2\lambda+1}} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot r^{\lambda+1} \\ &\times \left[\vec{Y}_{\lambda\lambda-1\mu}(\hat{r}) + \sqrt{\frac{\lambda}{\lambda+1}} \frac{2}{2\lambda+3} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}) \right] \\ &= -\frac{1}{2c} \sqrt{\frac{\lambda}{\lambda+1}} \frac{1}{2\lambda+3} \int d^3r r^{\lambda+2} \vec{Y}_{\lambda\lambda\mu}(\hat{r}) \\ &\cdot (\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})), \end{aligned} \quad (32)$$

$$\begin{aligned} \hat{M}_{\text{com}}(E\lambda\mu) &= \frac{i}{2c} \sqrt{\frac{\lambda}{2\lambda+1}} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot r^{\lambda+1} \\ &\times \left[\vec{Y}_{\lambda\lambda-1\mu}(\hat{r}) - \sqrt{\frac{\lambda+1}{\lambda}} \frac{2}{2\lambda+3} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}) \right] \\ &= -\frac{i}{2c} \frac{1}{2\lambda+3} \int d^3r r^{\lambda+2} Y_{\lambda\mu}(\hat{r}) (\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})). \end{aligned} \quad (33)$$

Here, the TM and VM operators are the same as in Eqs. (26) through (28), but are written in the forms convenient for the comparison with the VM operator. Besides these forms demonstrate the $\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})$ and $\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})$ origin of the TM and CM operators, respectively.

The expression for the VM operator (31) and the relation (30) between VM, TM, and CM operators represent the main formal results of the present article.

Following Eq. (30), the operators $\hat{M}_{\text{vor}}(E\lambda\mu)$, $\hat{M}_{\text{tor}}(E\lambda\mu)$, and $\hat{M}_{\text{com}}(E\lambda\mu)$ are of the same second order by k . They are given in Eqs. (31)–(33) in the current-dependent form. Using the continuity equation (21), the current-dependent CM operator (33) is straightforwardly transformed to the familiar density-dependent CM operator [13]

$$\hat{M}'_{\text{com}}(E\lambda\mu) = \frac{1}{2(2\lambda+3)} \int d^3r \hat{\rho}(\vec{r}) r^{\lambda+2} Y_{\lambda\mu}(\hat{r}), \quad (34)$$

as

$$\hat{M}_{\text{com}}(E\lambda\mu) = -k \hat{M}'_{\text{com}}(E\lambda\mu). \quad (35)$$

Note that the relation (30) requires the compensation of the terms $\sim \vec{Y}_{\lambda\lambda-1\mu}(\hat{r})$ in the TM and CM operators. Thus a simultaneous use of these two operators is obligatory. The VM operator includes only $\vec{Y}_{\lambda\lambda+1\mu}(\hat{r})$ and so its matrix elements are determined by the current transition density $j_{\lambda\lambda+1}(r)$, as requested in Ref. [7]. It is easy to check that $\hat{M}_{\text{vor}}(E\lambda\mu)$ gives the transition vorticity (11) and so reproduces the results of Ref. [7].

The above formalism was derived for the case when the system is excited by the external electric field (i.e., for the excitation energy $\omega = \hbar ck > 0$). The case of deexcitation is easily obtained by replacement $k \rightarrow -k$ in the continuity equations (5) and (21), k -dependent terms in Eqs. (15), (27), (35), and the equations of Appendix C. The sign of $\hat{M}_S(E\lambda\mu, k)$ and density-dependent $\hat{M}(E\lambda\mu)$ is changed as well.

B. Dipole VM, TM, and CM operators

The VM, TM, and CM are usually studied for the electric $I^\pi = 1^-$ states [15]. Then the operators (31)–(34) are reduced to

$$\hat{M}_{\text{vor}}(E1\mu) = -\frac{i}{5c} \sqrt{\frac{3}{2}} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) r^2 \vec{Y}_{12\mu}(\hat{r}), \quad (36)$$

$$\begin{aligned} \hat{M}_{\text{tor}}(E1\mu) &= -\frac{i}{2\sqrt{3}c} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \\ &\cdot \left[\frac{\sqrt{2}}{5} r^2 \vec{Y}_{12\mu}(\hat{r}) + (r^2 - \delta_{T,0} \langle r^2 \rangle_0) \vec{Y}_{10\mu}(\hat{r}) \right], \end{aligned} \quad (37)$$

$$\hat{M}_{\text{com}}(E1\mu) = -\frac{i}{2\sqrt{3}c} \int d^3r \hat{j}_{\text{nuc}}(\vec{r}) \cdot \left[\frac{2\sqrt{2}}{5} r^2 \vec{Y}_{12\mu}(\hat{r}) - (r^2 - \delta_{T,0} \langle r^2 \rangle_0) \vec{Y}_{10\mu}(\hat{r}) \right], \quad (38)$$

$$\hat{M}'_{\text{com}}(E1\mu) = \frac{1}{10} \int d^3r \hat{\rho}(\vec{r}) \cdot \left[r^3 - \delta_{T,0} \frac{5}{3} \langle r^2 \rangle_0 r \right] Y_{1\mu}(\hat{r}), \quad (39)$$

where $\langle r^2 \rangle_0 = \int d^3r \rho_0(\vec{r}) r^2 / A$ is the ground-state squared radius.

In Eqs. (37)–(39), the terms $\sim \vec{Y}_{10\mu}(\hat{r})$, $Y_{1\mu}(\hat{r})$ include the center-of-mass corrections (c.m.c.) for $T = 0$ excitations [13]. In the TM and current-dependent CM operators, the c.m.c. have the same magnitude. For the VM, the c.m.c. is zero, see the discussion in Appendix B.

The expression in the square brackets of the TM operator (37) can be written as [13]

$$\vec{\nabla} \times (\vec{r} \times \vec{\nabla}) \left(r^3 - \frac{5}{3} \langle r^2 \rangle_0 r \right) Y_{1\mu}(\hat{r}), \quad (40)$$

which justifies a close relation between TM and CM.

C. Discussion of VM, TM, and CM operators and vorticity criteria

As shown above, the CM operator may be presented in the current-dependent (33) and density-dependent (34) forms. To the best of our knowledge, the former has not yet been used in the literature. Since the spin current j_m is a curl of the magnetization, it does not contribute to the continuity equation and CM operator. Though the current-dependent form (33) of CM formally involves j_m , its contribution is annihilated by $\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})$ or the corresponding combinations of vector spherical harmonics. So actually both forms, Eqs. (33) and (34), of the CM operator do not depend on j_m .

Both CM operators, Eqs. (33) and (34), are obtained through $\vec{\nabla} \cdot \hat{j}_{\text{nuc}}(\vec{r})$, which suggests their vorticity-free character. This is confirmed by the form of the CM velocity, which, following the prescription [46], reads

$$\vec{v}_{\text{com}}(\vec{r}) \propto \vec{\nabla} r^{\lambda+2} Y_{\lambda\mu}(\hat{r}), \quad (41)$$

and so gives $\vec{\nabla} \times \vec{v}_{\text{com}}(\vec{r}) = 0$.

The current-dependent CM operator (33) includes the $\vec{Y}_{\lambda\lambda+1\mu}(\hat{r})$ contribution which might be considered as an indicator of a vortical part. Indeed $\vec{Y}_{\lambda\lambda+1\mu}(\hat{r})$ leads to the current component $j_{\lambda\lambda+1}(r)$, which, following Ref. [7], is responsible for the vorticity. However, for the CM this is misleading. The velocity (41) can be straightforwardly cast into the form

$$\vec{v}_{\text{com}}(\vec{r}) \propto \sqrt{\frac{\lambda}{2\lambda+1}} r^{\lambda+1} \cdot \left[\vec{Y}_{\lambda\lambda-1\mu}(\hat{r}) - \sqrt{\frac{\lambda+1}{\lambda}} \frac{2}{2\lambda+3} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}) \right], \quad (42)$$

whose curl is zero despite the $\vec{Y}_{\lambda\lambda+1\mu}(\hat{r})$ term. So, following the HD criterion (2), the appearance of a term $\vec{Y}_{\lambda\lambda+1\mu}(\hat{r})$ in the mode operator and current is not yet a definitive signature of the vorticity.

Altogether, we see the apparent differences between two possible criteria for the vorticity: (i) the HD condition (2) in terms of velocities, $\vec{\nabla} \times \vec{v}(\vec{r}) \neq 0$, and (ii) the condition in Ref. [7] in terms of transition current density components, $j_{\lambda\lambda+1}(r) \neq 0$. As shown above for the CM example, a mode that is fully vorticity free in the HD definition can have a substantial vorticity of the Ref. [7] sort. The difference between these criteria might be understood if we take into account that the vorticity density $w_{\lambda\lambda}(r) \propto j_{\lambda\lambda+1}(r)$ was derived in Ref. [7], first of all, as a quantity completely unconstrained by the charge conservation rather than a purely vortical value in the HD sense.

In fluid-dynamical and HD models [11,28], the velocity fields are chosen in form of the relevant external fields exciting the proper modes. Following this practice, the TM velocity reads

$$\vec{v}_{\text{tor}}(\vec{r}) \propto \vec{\nabla} \times (\vec{r} \times \vec{\nabla}) r^{\lambda+2} Y_{\lambda\mu}(\hat{r}) \quad (43)$$

$$= i \sqrt{\frac{\lambda}{2\lambda+1}} (\lambda+1)(2\lambda+3) r^{\lambda+1} \cdot \left[\vec{Y}_{\lambda\lambda-1\mu}(\hat{r}) + \sqrt{\frac{\lambda}{\lambda+1}} \frac{2}{2\lambda+3} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}) \right]. \quad (44)$$

It is easy to check that $\vec{\nabla} \times \vec{v}_{\text{tor}} \propto r^\lambda \vec{Y}_{\lambda\lambda\mu}(\hat{r})$ and so TM carries the HD vorticity.

The VM velocity constructed in the same manner is

$$\vec{v}_{\text{vor}}(\vec{r}) \propto r^{\lambda+1} \vec{Y}_{\lambda\lambda+1\mu}(\hat{r}). \quad (45)$$

It has the similar nonzero curl $\vec{\nabla} \times \vec{v}_{\text{vor}} \propto r^\lambda \vec{Y}_{\lambda\lambda\mu}(\hat{r})$. So the VM has the vorticity of both HD and Ref. [7] sorts.

IV. CALCULATION SCHEME

The calculations of the excitation modes were performed within the separable random-phase approximation (SRPA) model using the Skyrme energy functional [37,38]. SRPA was earlier successfully applied to description of electric [37–41] and magnetic [43–45] GR in spherical and deformed nuclei. The approach was also used for the exploration of $E1$ strength near particle emission thresholds [42]. SRPA is *fully self-consistent* in the sense that both the static mean-field and factorized residual interaction are derived from the same Skyrme energy functional [49–51]. The functional, $\mathcal{E}_{\text{Sk}}(\rho, \tau, \vec{J}, \vec{j}, \vec{s}, \vec{T})$, includes time-even (nucleon ρ , kinetic-energy τ , spin-orbit \vec{J}) and time-odd (current \vec{j} , spin \vec{s} , vector kinetic-energy \vec{T}) densities. It also involves pairing (surface and volume), Coulomb (direct and exchange), and c.m.c. terms [37,38,43]. The Galilean invariance of the functional is maintained in SRPA. The tensor spin-orbit contribution is involved through the squared spin-orbit densities \vec{J}^2 . All

the functional terms are kept in the mean field and residual interaction.

The SRPA expands the RPA residual interaction self-consistently into a sum of separable terms, which dramatically reduces the computational effort while keeping the accuracy of the full (nonseparable) RPA [37,38]. This makes SRPA extremely useful for systematic calculations and tasks with very large configuration space (e.g., for description of GR in heavy deformed nuclei).

For GR studies, the computational expense can be even more reduced by a direct evaluation of the strength function, thus avoiding the solution of RPA equations for a large multitude of individual states. The SRPA strength function works with the Lorentz weight and has very simple form [37,38]. In the present study, the strength function for electric dipole modes reads

$$S_\alpha(E1; \omega) = (2\pi)^2 \sum_{\mu=0,\pm 1} \sum_{\nu} |\langle \Psi_\nu | \hat{M}_\alpha(E1\mu) | \Psi_0 \rangle|^2 \zeta(\omega - \omega_\nu), \quad (46)$$

where

$$\zeta(\omega - \omega_\nu) = \frac{1}{2\pi} \frac{\Delta}{(\omega - \omega_\nu)^2 + \frac{\Delta^2}{4}}, \quad (47)$$

is the Lorentz weight with the smoothing width Δ and $\hat{M}_\alpha(E1\mu)$ is the electric dipole transition operator whose type is determined by the index $\alpha = \{\text{vor}, \text{tor}, \text{com}, \text{com}'\}$. Further, Ψ_0 is the ground state, ν runs over the RPA spectrum with eigenfrequencies ω_ν and eigen-states $|\Psi_\nu\rangle$. The Lorentz smoothing uses a width $\Delta = 1$ MeV to simulate broadening effects beyond RPA (escape widths and coupling to complex configurations). The explicit form of the strength function (46), which does not directly involve the RPA solutions, is given elsewhere [37,38].

In general, the VM, TM, and CM appear at many different multipolarities $\lambda\mu$. Here we explore them in the $I^\pi = 1^-$ excitations of the doubly magic spherical nucleus ^{208}Pb . In this nucleus the pairing is absent. The calculations use the Skyrme parametrization SLy6 [52], which provides a satisfactory description of the $E1(T=1)$ GR in heavy nuclei [41]. For this parametrization, the tensor spin-orbit contribution is omitted.

The calculations employ a cylindrical coordinate-space grid with the mesh size 0.7 fm. A large single-particle basis is used. The particle-hole 1^- pairs extend up to ~ 65 MeV and, for $E1(T=1)$ excitations, the energy-weighted sum rule with the isovector effective mass is exhausted by $\sim 95\%$ [41].

The calculations involve both convection j_c and magnetization (spin) j_m parts of the nuclear current, see Appendix C for more details. For j_m , the isoscalar $g_s^{n,p}(T=0) = (g_s^p + g_s^n)/2 = 0.88\zeta$ and isovector $g_s^{n,p}(T=1) = (g_s^n - g_s^p)/2 = -4.70\zeta$ gyromagnetic factors are used, where $g_s^p = 5.58\zeta$ and $g_s^n = -3.82\zeta$ are bare proton and neutron g factors and $\zeta = 0.7$ is a quenching parameter taking approximately into account the meson degrees of freedom [53]. It is easy to see that $|g_s^{n,p}(T=1)| \gg |g_s^{n,p}(T=0)|$ and so the main effect of the spin nuclear current is expected for $T=1$ modes.

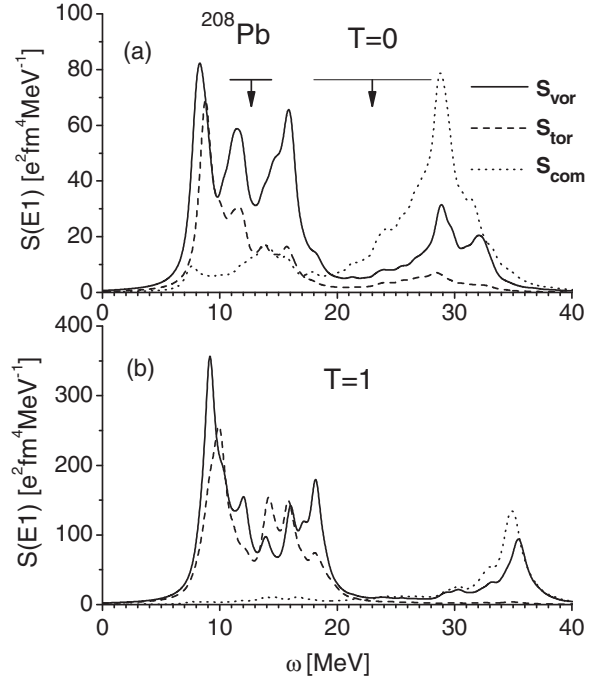


FIG. 1. Isoscalar ($T=0$) and isovector ($T=1$) vortical, toroidal, and compression dipole modes in ^{208}Pb , calculated with the SLy6 parametrization. The total nuclear current j_{nuc} is used. The CM is computed with the operator \hat{M}_{com} from Eq. (38). The lines with the arrows indicate widths and energy centroids of the low-energy and high-energy branches of isoscalar $E1$ excitations observed in the (α, α') reaction [24,25].

The proton and neutron ($q = n, p$) RPA velocity fields are determined in cylindrical coordinates (z, ρ) as

$$\vec{v}_v^q(z, \rho) = \frac{\delta \vec{j}_{vq}(z, \rho)}{\rho_0^q(z, \rho)}, \quad (48)$$

where

$$\delta \vec{j}_{vq}(z, \rho) = \sum_{ij \in q} \langle ij | j_{\text{nuc}}^{vq} | 0 \rangle (c_{ij}^{vq-} - c_{ij}^{vq+}), \quad (49)$$

is the current transition density for the RPA ν state with the normalized forward and backward particle-hole ($1ph$) amplitudes c_{ij}^{vq-} and c_{ij}^{vq+} . Further, $\rho_0^q(z, \rho)$ is the proton/neutron ground-state density.

The spurious c.m. admixtures are avoided by using the prescriptions from the Appendix B. The SRPA equations and some important points (e.g., a choice of the generating operators for the separable expansion) are sketched in Appendix D.

V. RESULTS AND DISCUSSION

The results of the calculations for the nucleus ^{208}Pb are presented in Figs. 1–9.

In Fig. 1, the VM, TM, and CM strengths in $T=0$ and 1 channels are compared. The strengths are computed for the transition operators (36)–(38). For the CM, the current-dependent operator \hat{M}_{com} is used. Unlike its familiar

density-dependent counterpart \hat{M}'_{com} , it has the same dimension as the VM and TM operators and so is more suitable for the comparison of the modes. All the modes are calculated with total nuclear current j_{nuc} . Figure 1 shows that all the modes have basically two broad branches, a low-energy branch (LEB) at 5–20 MeV and a high-energy branch (HEB) at 25–40 MeV. The VM is well presented in both branches while TM and CM are mainly localized in LEB and HEB, respectively. Such a double-branch structure was found for the TM and CM in most of the previous theoretical studies [15]. It is most probably related to $E1$ transitions with $\Delta N = 1$ and 3 where N is the principle shell number.

The double-branch structure of the $E1(T = 0)$ strength was confirmed by various experiments [14,18,20–25], mainly in (α, α') scattering at small angles. The recent results of this reaction [24,25] are depicted in Fig. 1(a). The reaction is considered as a common tool for measurements of the dipole CM($T = 0$) [53].

As seen in Fig. 1(a), our results do not reproduce the experimental energies and widths of the LEB and HEB. Neither of the mode centroids coincides with the experimental peak energies. Note that this is a common shortcoming of almost all theoretical studies performed within various theoretical approaches [15]. Namely, the theory (i) underestimates by 1–2 MeV the TM-dominated LEB and overestimates by about 4 MeV the CM-dominated HEB, (ii) yields a much broader TM distribution and a too narrow CM one. The reason for the discrepancies is still unclear. Perhaps this is partly caused by neglecting the coupling with complex configurations.

Figure 1 shows that the VM and TM strengths are of the same order of magnitude in the LEB left flank for $T = 0$ and in the whole LEB for $T = 1$ (the difference between $T = 0$ and 1 cases is explained below in the discussion of Figs. 2 and 3). Perhaps, in these regions the TM is mainly vortical. We also see that VM dominates at the right LEB flank and is significant in HEB where the TM contribution is weak. The CM strictly dominates in HEB and has a noticeable tail in LEB at $T = 0$. The later case is because of the coupling between TM and CM [13,31]. Obviously, the difference between VM, on the one hand, and TM and CM, on the other hand, is mainly explained by the terms $\sim \vec{Y}_{10\mu}(\hat{r})$ that are absent in \hat{M}'_{vor} , but active in \hat{M}'_{tor} and \hat{M}'_{com} . Because of these terms, Fig. 1 cannot be used for a direct check of the relation (30). For the same reason, the similarity of VM and CM strengths in the HEB cannot be considered as a signature of the CM vorticity because VM and CM represent essentially different kinds of the motion, vortical versus irrotational. Instead, this means that both kinds of motion are presented by $E1\Delta N = 3$ transitions.

The obtained results suggest that the VM($T = 0$) may be hopefully disentangled from other modes in (α, α') at the excitation energy ~ 16 MeV, where the VM($T = 0$) strictly dominates. For (e, e') , the pygmy region 7–10 MeV seems to be most promising to observe VM and TM. In this region a dominant contribution $j_{12}(r)$ of the nuclear current is expected.

In Fig. 2, the isoscalar VM and TM strengths, calculated with the complete $j_{\text{nuc}} = j_c + j_m$, convection j_c , and magnetization j_m nuclear currents, are compared. It is seen that the j_m contribution is weak and so the $T = 0$ VM and TM are mainly

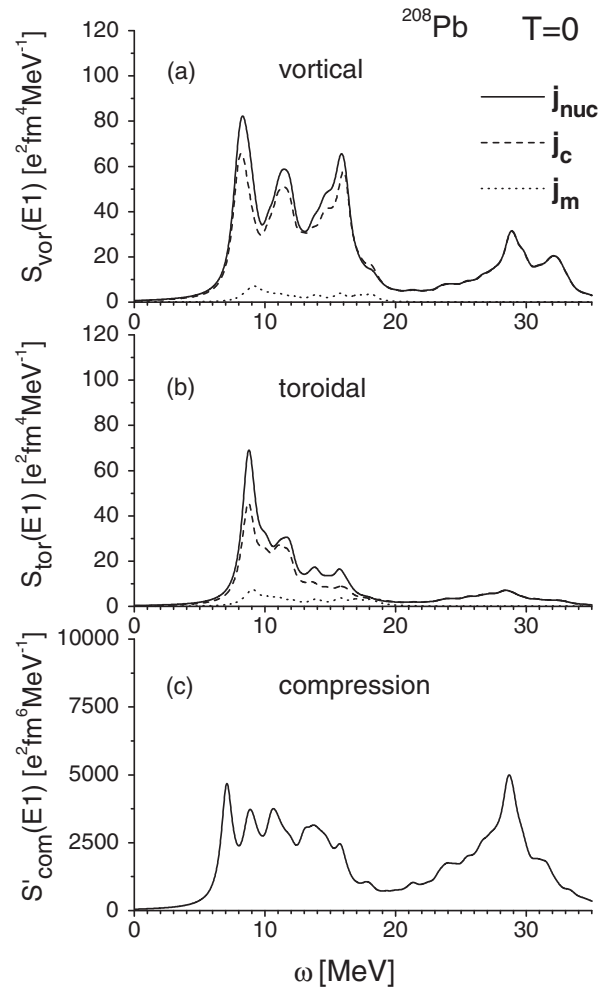
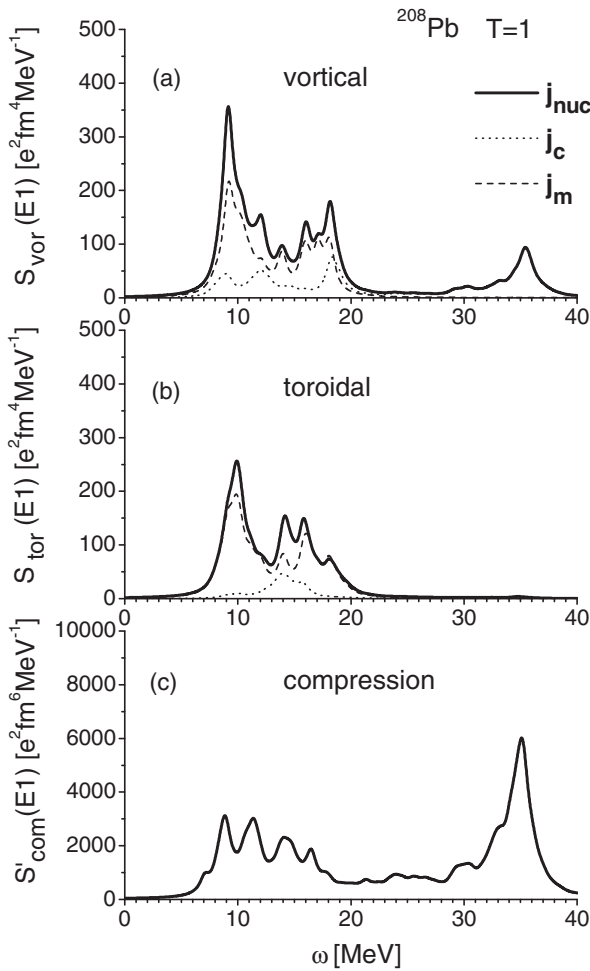


FIG. 2. Calculated isoscalar ($T = 0$) vortical, toroidal, and compression dipole modes in ^{208}Pb . For the VM and TM, the strengths with the total j_{nuc} , convection j_c , and magnetization j_m current contributions to the transition operators are shown.

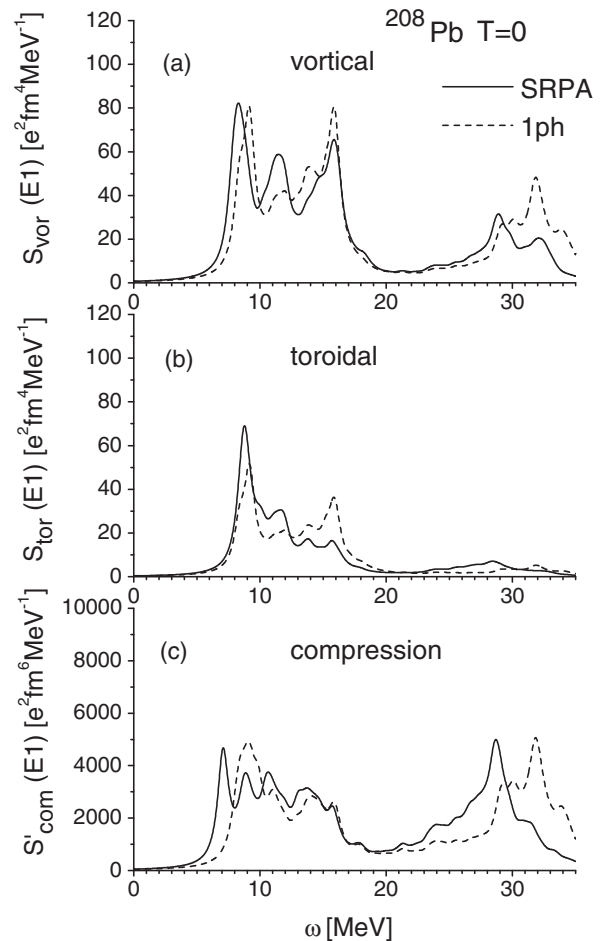
of the convection nature. This is especially the case for the HEB where the j_m contribution, being mainly of low-energy spin-flip character, vanishes at all. The weakness of the j_m weight in $T = 0$ channel is naturally explained by the low values of the gyromagnetic factors $g_s^q(T = 0)$, as mentioned in Sec. IV. Figure 2 also exhibits the CM. Unlike Fig. 1, here the familiar density-dependent operator \hat{M}'_{com} is used. Following Eq. (35), \hat{M}'_{com} is less energy weighted than \hat{M}_{com} and so gives a more comparable CM strength in LEB and HEB. As discussed above, the CM is determined by $\nabla \cdot \vec{j}_{\text{nuc}}$ and so is purely irrotational. It has no any contribution from j_m and thus is fully convective.

In Fig. 3, the VM, TM, and CM are shown in the $T = 1$ channel. As compared to the previous $T = 0$ case, we see dramatic changes in the magnitude and composition of VM and TM. In the LEB, these modes become stronger and dominated by the j_m contribution. The reason for the changes is obvious. The isovector spin factors, $g_s^{n,p}(T = 1) = -4.70\zeta$, are much larger than the isoscalar ones, $g_s^{n,p}(T = 0) = 0.88\zeta$. So, the $T = 1$ spin contribution grows about

FIG. 3. The same as in Fig. 2 but for the $T = 1$ modes.

$[g_s^{n,p}(T=1)/g_s^{n,p}(T=0)]^2 \sim 29$ times. It becomes dominant and significantly increases the total VM and TM strengths. Note that this effect does not concern the HEB that remains purely convective. Besides, the j_m effect is zero for the CM.

The next point to be considered is collectivity of the modes. To this end, Figs. 4 and 5 compare RPA and unperturbed particle-hole ($1ph$) strengths. It is seen that the RPA residual interaction noticeably down-shifts the strength for $T = 0$ and up-shifts it for $T = 1$. The maximal collective shifts (defined as the difference between RPA and $1ph$ peaks) take place in the CM, where they reach 1–2 MeV in LEB and 2–4 MeV in HEB. The HEB shift is comparable to that of the $E1(T = 1)$ giant dipole resonance (GDR) exhibited in Fig. 6 and so is indeed very large. This indicates that HEB modes, VM and CM, are collective. The LEB modes, for exception of a few high peaks, are less collective. The LEB almost coincides with the region of the unperturbed $1ph$ dipole strength depicted in Fig. 6 and so for the LEB the single-particle aspect is also important. These observations are confirmed by an inspection of the detailed structure of the RPA states and agree with the previous studies [15] for the high-energy CM and low-energy TM.

FIG. 4. Comparison of SRPA and $1ph$ strengths for the $T = 0$ modes. For VM and TM, the strengths are computed with the total nuclear current j_{nuc} .

The next figures take a closer look at the detailed structures of a few most collective LEB modes. In Figs. 7 through 9, the VM, TM, and CM neutron and proton velocity fields (48) for particular $T = 0$ and $T = 1$ RPA ($\mu = 0$) states are shown. The states in the region 6–10 MeV with the maximal VM, TM, and CM responses are considered: 8.3 MeV (VM), 8.7 MeV (TM), 7.1 MeV (CM) for $T = 0$ and 9.1 MeV (VM), 9.8 MeV (TM), 8.8 MeV (CM) for $T = 1$. These states are easily recognized in Figs. 2 through 5 as the highest peaks at the left side of the LEB. The states combine collectivity and single-particle effects: their structure is a coherent superposition of many $1ph$ components with maximal contributions 20–40%. The velocities are plotted in cylindrical coordinates (z, ρ) and, for simplicity, only the quadrant $(z > 0, \rho > 0)$ is presented.

Figures 7 through 9 show that velocity fields are rather involved, indicating a complex structure of the RPA states. The clear imprints of the vortex motion are seen as local curls. Sometimes, the fields well reproduce a typical toroidal picture [see, e.g., the $T = 0$ proton velocities for VM and CM in Figs. 7(b) and 9(b)]. The curls for the CM may be explained by its strong coupling to TM in LEB. In Fig. 9(a), the strong dipole component is also seen. However, in most of the panels,

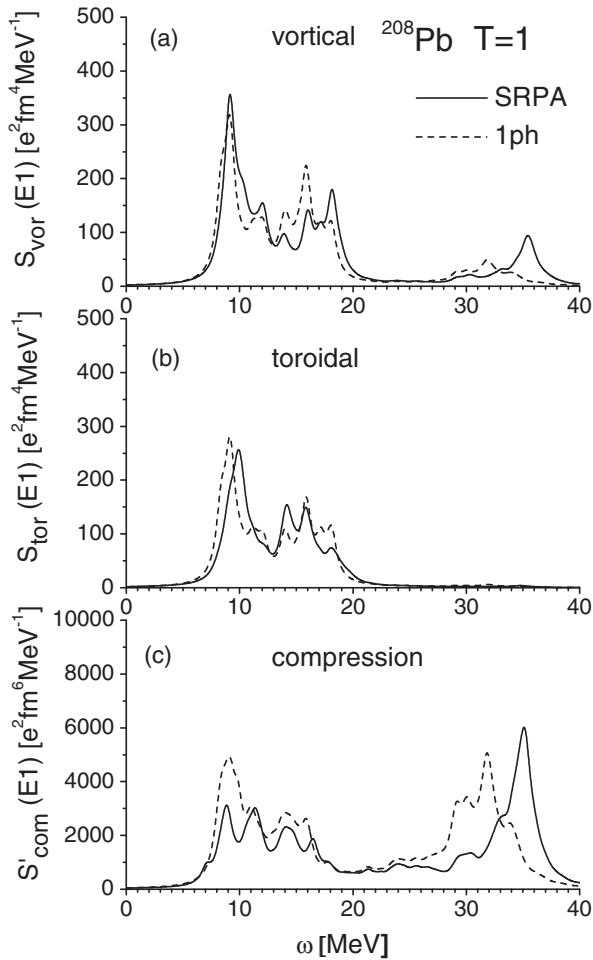


FIG. 5. The same as in Fig. 4 but for the $T = 1$ modes.

the large impact of the single-particle motion (e.g., of the nodal structure of the leading $1ph$ components) distorts the familiar collective TM and CM flows and considerably complicates the picture.

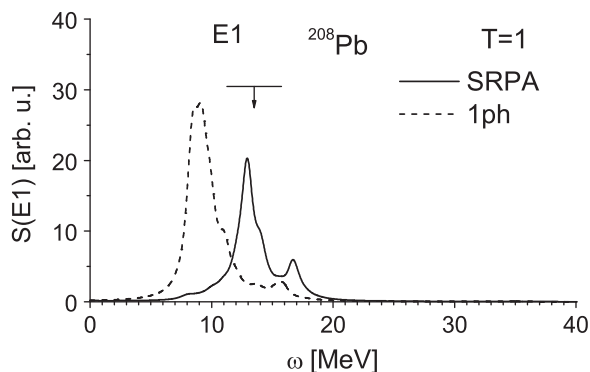


FIG. 6. SRPA and $1ph$ strengths for the $E1(T = 1)$ GDR. Like for other modes, the strength functions are plotted without the energy weight. The experimental width and energy [54] are shown by the horizontal line and arrow, respectively.

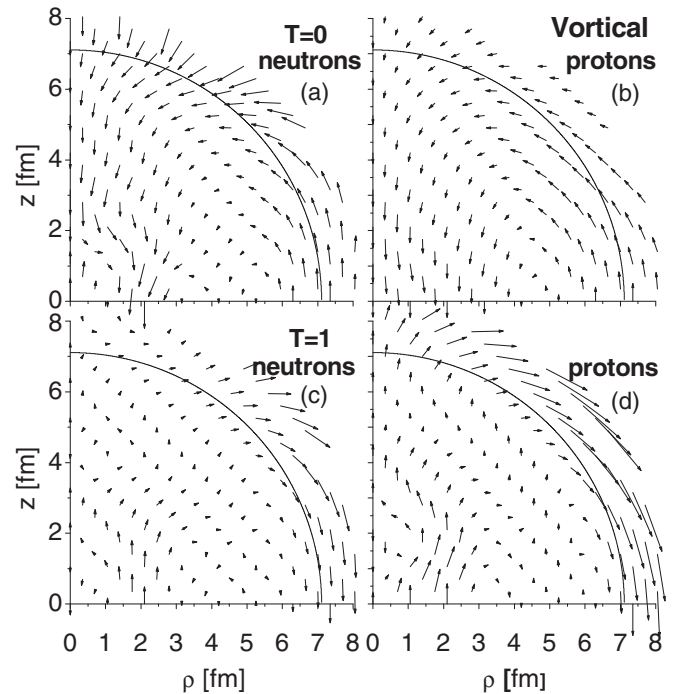


FIG. 7. (a–b) Neutron and proton vortical velocity fields $\vec{v}_v^q(z, \rho)$ for the state $\omega_v = 8.3$ MeV with a maximal $T = 0$ vortical response. (c–d) The same for the state $\omega_v = 9.1$ MeV with a maximal $T = 1$ vortical response. For a better view, the velocities are amplified by the factors 50 (a–b) and 20 (c–d).

VI. CONCLUSION

The nuclear vorticity and relevant vortical, toroidal, and compression modes (VM, TM, and CM) were explored on general formal grounds. The operators of all three modes were derived as second-order terms in the long-wavelength expansion of the electrical multipole operator and its subsidiary counterpart built following the concept in Ref. [7]. For the first time, the vortical operator was constructed and related to its toroidal and compression counterparts. The detailed comparison of the modes and their operators was done. It was explicitly shown that, while VM (CM) is vortical (irrotational) by construction, the TM is of a mixed character. The vorticity criteria from HD and arguments based on the decoupling to the charge conservation [7] were inspected. It was shown that the latter deviates from the HD definition and may lead occasionally to misleading conclusions like, e.g., a vorticity of the CM.

The electric dipole VM, TM, and CM were computed and investigated in ^{208}Pb within the self-consistent Skyrme RPA approach. Unlike most of the previous studies, both convection and magnetization (spin) parts of the nuclear current were taken into account and both isoscalar ($T = 0$) and isovector ($T = 1$) channels of the modes were analyzed. It was shown that VM and CM have low-energy and high-energy branches while TM mainly appears in the low-energy branch. The CM strictly dominates in a high-energy branch. In the $T = 0$ channel, the VM and TM are almost completely determined by the nuclear convection current while in the $T = 1$ channel, their low-energy branches are strictly dominated by the spin

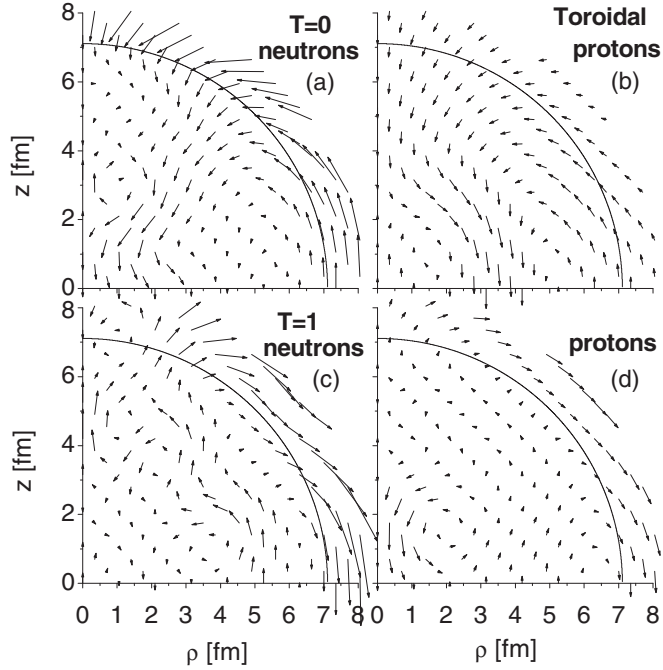


FIG. 8. The same as in Fig. 7 but for the states $\omega_v = 8.7$ MeV ($T = 0$) and $\omega_v = 9.8$ MeV ($T = 1$) with a maximal toroidal strength.

current. This strong isospin effect is straightforwardly explained by low (high) values of $T = 0$ ($T = 1$) spin g factors, which drastically change the ratio between the convective and spin contributions of the current. The effect cannot appear

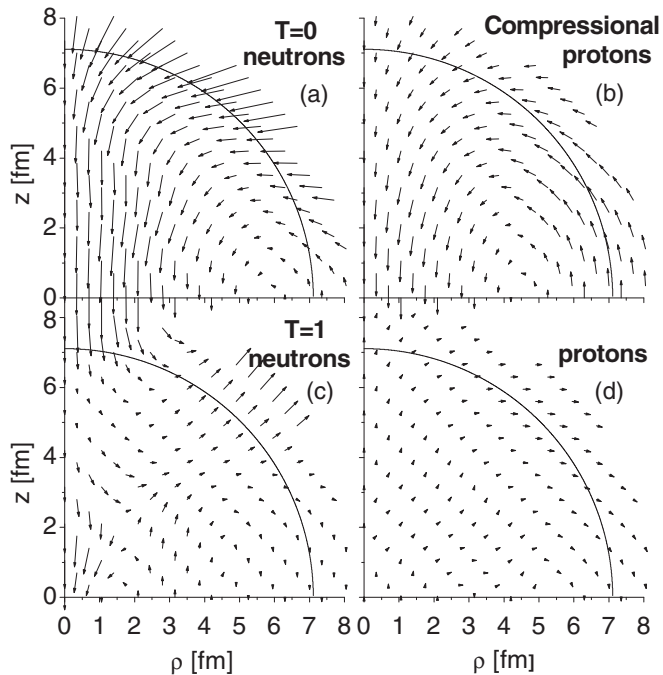


FIG. 9. The same as in Fig. 7 but for the states $\omega_v = 7.1$ MeV ($T = 0$) and $\omega_v = 8.8$ MeV ($T = 1$) with a maximal compression strength in the low-energy branch.

in the CM since this irrotational mode has vanishing spin contribution.

The collectivity was found strong for the high-energy VM and CM and rather weak for the low-energy VM, TM, and CM. In the latter case, the velocity fields of the modes are rather involved. This is partly a consequence of the complex structure of the RPA states mixing the collective with detailed $1ph$ contributions. This holds, in particular, for the vorticity, which seems to be of both single-particle and collective origin.

The VM, TM, and CM were shown to be closely related and, at the same time, display considerable differences in their detailed strength distribution. These modes seem to represent one family with complementing aspects. It will be interesting to analyze the results of our study more deeply and use them to disentangle the modes in the (e, e') and hadron reactions. This is in our next plans.

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APPENDIX A: REMOVAL OF THE CHARGE-CONSERVATION CONSTRAINT

In Sec. III A, the vortical operator $\hat{M}_{\text{vor}}(E\lambda\mu, k)$, completely unconstrained by the continuity equation, is built from the $\hat{M}(E\lambda\mu, k)$ operator by replacing the curl of the nuclear current $\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})$ by the vortical quantity

$$\hat{\omega}_\lambda = [\vec{\nabla} \times \hat{j}_{\text{nuc}}(\vec{r})]_\lambda - \frac{i}{\lambda} kc [\vec{\nabla} \hat{\rho}(\vec{r}) \times \vec{r}]_\lambda, \quad (\text{A1})$$

where the terms $[\dots]_\lambda$ assume λ components of the multipole expansion of the values inside the brackets. Below we present arguments that motivate Eq. (A1) and compare it with the prescription in Ref. [7]. For simplicity, we consider only the convection current and neglect isospin.

1. Simple arguments

Using the HD definition of the velocity field (3), we may write the *truly vortical* quantity

$$\rho_0(\vec{r}) \vec{\nabla} \times \vec{v}_v(\vec{r}) = \vec{\nabla} \times \delta \vec{j}_v(\vec{r}) - \vec{\nabla} \rho_0(\vec{r}) \times \hat{v}_v(\vec{r}). \quad (\text{A2})$$

Except for the second velocity-dependent term, the r.h.s. of Eq. (A2) reminds the operator construction (A1) and thus may be used for the justification of $\hat{\omega}_\lambda$ as a vortical quantity.

To make Eq. (A2) closer to Eq. (A1), it is worth expressing \vec{v} in terms of the (global) velocity operator

$$\hat{v} = \hat{r} = \frac{i}{\hbar} [\hat{H}, \hat{r}] = ikc\hat{r}. \quad (\text{A3})$$

This suggests the replacement

$$\hat{v} \rightarrow ikc\hat{r}, \quad (\text{A4})$$

in Eq. (A2) and thus casts Eq. (A2) to a form similar to Eq. (A1) (up to the multiplier $1/\lambda$). Note, however, that the velocities in Eqs. (A2) and (A3) are not the same. The replacement of the velocity field $\vec{v}_v(\vec{r})$ by a global velocity becomes strictly valid only in the sum-rule limit when all the strength is concentrated in one collective state. It probably remains acceptable for distributed spectra that often gather around a strongly collective mode.

2. Correspondence of the recipes

It is also worth relating the recipe (10) for the transition densities [7] with our recipe (15) and its analog (A1) for the operators. As was mentioned in Sec. III A, both recipes serve to build the vortical quantities, though by different ways. While Eq. (10) excludes from the current the $j_{\lambda\lambda-1}(r)$ terms, the recipe (15) leads to exact compensation of the lowest-order k terms in the vortical operator.

One may show that the recipes (10) and (15) actually correspond to each other. This may be done by treating (15) in terms of the transition densities (6) and currents (7). Using the relation between the density and current expansion multipoles

$$kc\rho_\lambda(r) = -\sqrt{\frac{\lambda}{2\lambda+1}} \left\langle \frac{d}{dr} - \frac{\lambda-1}{r} \right\rangle j_{\lambda\lambda-1}(r) + \sqrt{\frac{\lambda+1}{2\lambda+1}} \left\langle \frac{d}{dr} + \frac{\lambda+2}{r} \right\rangle j_{\lambda\lambda+1}(r), \quad (\text{A5})$$

one may show that (15) indeed leads to the unconstrained vortical transition density (10). In this case, the second r.h.s. term of Eq. (A1) has the form

$$ikc [\vec{\nabla} \delta\rho_{fi}(\vec{r}) \times \vec{r}] = \sum_{\lambda\mu} a_{\lambda\mu}^{fi} S_{\lambda\lambda}(r) \vec{Y}_{\lambda\lambda\mu}(\hat{r}), \quad (\text{A6})$$

with the multipoles

$$S_{\lambda\lambda}(r) = \sqrt{\lambda(\lambda+1)} kc\rho_\lambda(r). \quad (\text{A7})$$

Being scaled by $1/\lambda$, these multipoles coincide with the expansion coefficients of $\delta\vec{S}_{fi}(\vec{r})$ in Eq. (9).

APPENDIX B: EXTRACTION OF SPURIOUS ADMIXTURES

The isoscalar electric dipole VM, TM, and CM may have spurious admixtures related to the center-of-mass motion of the nucleus. There are various methods to derive the center-of-mass corrections (c.m.c.) (see, e.g., Refs. [4,15,55–57]). Most of the methods neglect the velocity-dependent, spin-dependent, and spin-orbit-dependent terms in the nuclear interaction and assume a closure relation where all excitation is contained in one single collective state. Hence the methods are

approximate. Nevertheless, being simple and convenient, they are widely used in the Skyrme-Hartree-Fock calculations [15]. Below we use the method [56] to derive the c.m.c. for isoscalar TM and CM. For the isoscalar VM, the c.m.c. is shown to be zero.

Let us consider a system with the Hamiltonian $\hat{H} = \hat{T} + \hat{V}$ whose interaction \hat{V} does not depend on spin and velocities. Then, in the isoscalar ($T = 0$) case, for any one-body external field $\hat{F} = \sum_i^A f(\vec{r}_i)$, where $f(\vec{r}_i)$ is an arbitrary function of nucleon coordinates, one may write the sum rules [46]

$$\sum_\nu \omega_\nu \langle 0 | \hat{\rho}(\vec{r}) | \nu \rangle \langle \nu | \hat{F} | 0 \rangle = -\frac{1}{2m} \vec{\nabla} \cdot [\rho_0(\vec{r}) \vec{\nabla} f(\vec{r})], \quad (\text{B1})$$

$$\sum_\nu \langle 0 | \hat{j}(\vec{r}) | \nu \rangle \langle \nu | \hat{F} | 0 \rangle = \frac{1}{2mi} \rho_0(\vec{r}) \vec{\nabla} f(\vec{r}), \quad (\text{B2})$$

for the isoscalar transition density $\langle 0 | \hat{\rho}(\vec{r}) | \nu \rangle$ and current $\langle 0 | \hat{j}(\vec{r}) | \nu \rangle$. The sum runs through the full set of the excitation eigenstates $|\nu\rangle$ ($\hat{H}|\nu\rangle = \omega_\nu|\nu\rangle$) with the eigenvalues ω_ν . Further, $\rho_0(\vec{r}) = \langle 0 | \sum_i^A \delta(\vec{r} - \vec{r}_i) | 0 \rangle$ is the nuclear ground-state density and m is the nucleon mass.

If the external field mainly excites a single collective state $\bar{\nu}$, then only the term ($\nu = \bar{\nu}$) survives in Eqs. (B1) and (B2) and the corresponding transition density and current are uniquely specified through $\rho_0(\vec{r})$ and $f(\vec{r})$ [46]

$$\langle 0 | \hat{\rho}(\vec{r}) | \bar{\nu} \rangle = -\frac{1}{2m} \frac{1}{\omega_{\bar{\nu}} \langle \bar{\nu} | \hat{F} | 0 \rangle} \vec{\nabla} \cdot [\rho_0(\vec{r}) \vec{\nabla} f(\vec{r})], \quad (\text{B3})$$

$$\langle 0 | \hat{j}(\vec{r}) | \bar{\nu} \rangle = \frac{1}{2mi} \frac{1}{\langle \bar{\nu} | \hat{F} | 0 \rangle} \rho_0(\vec{r}) \vec{\nabla} f(\vec{r}). \quad (\text{B4})$$

Hence we get simple recipes for the transition densities and currents

$$\delta\rho(\vec{r}) \propto \vec{\nabla} \cdot [\rho_0(\vec{r}) \vec{\nabla} f(\vec{r})], \quad (\text{B5})$$

$$\delta\vec{j}(\vec{r}) \propto \rho_0(\vec{r}) \vec{\nabla} f(\vec{r}) \propto \rho_0(\vec{r}) \vec{v}(\vec{r}), \quad (\text{B6})$$

to be used in further c.m.c. derivation. Here Eq. (B6) actually defines an irrotational flow with the velocity $\vec{v}(\vec{r}) \sim \vec{\nabla} f(\vec{r})$.

A change of the expectation value $\langle \hat{O} \rangle$ of any one-body operator $\hat{O} = \sum_i^A o(\vec{r}_i)$, caused by an external field \hat{F} , is

$$\delta\langle \hat{O} \rangle = \int d^3r \delta\rho(\vec{r}) o(\vec{r}) \quad (\text{B7})$$

$$\propto \int d^3r o(\vec{r}) \vec{\nabla} \cdot [\rho_0(\vec{r}) \vec{\nabla} f(\vec{r})] \quad (\text{B8})$$

$$= - \int d^3r \rho_0(\vec{r}) \vec{\nabla} f(\vec{r}) \cdot \vec{\nabla} o(\vec{r}) \quad (\text{B9})$$

$$\propto - \int d^3r \delta\vec{j}(\vec{r}) \cdot \vec{\nabla} o(\vec{r}), \quad (\text{B10})$$

where we use the relations (B5) and (B6). Both $\delta\rho(\vec{r})$ and $\delta\vec{j}(\vec{r})$ may be applied to determine $\delta\langle \hat{O} \rangle$. These cases are suitable for the modes determined by the density-dependent and current-dependent operators, respectively. Note that Eq. (B10) with $\delta\vec{j}(\vec{r})$ is general and may be derived and used by itself, regardless to the formalism in Eqs. (B1)–(B9) and character of the flow. The quantity $\delta\vec{j}(\vec{r})$ is then determined by Eq. (3) and, unlike Eq. (B6), the velocity $\vec{v}(\vec{r})$ of the flow can be not only

irrotational but also vortical or mixed. Altogether, the relations (B7) through (B10) may be reduced to an expression

$$\delta\langle\hat{O}\rangle \approx \int d^3r \rho_0(\vec{r}) \vec{v}(\vec{r}) \cdot \vec{\nabla}o(\vec{r}), \quad (\text{B11})$$

with $\vec{v}(\vec{r})$ covering both irrotational (B6) and more general cases.

If $\langle\hat{O}\rangle$ is invariant with respect to the field \hat{F} , then the requirement $\delta\langle\hat{O}\rangle = 0$ is kept and Eq. (B11) gives

$$\int d^3r \rho_0(\vec{r}) \vec{v}(\vec{r}) \cdot \vec{\nabla}o(\vec{r}) = 0. \quad (\text{B12})$$

For

$$o(\vec{r}) = rY_{1\mu}^*(\hat{r}), \quad (\text{B13})$$

the operator $\hat{O} = \sum_i^A o(\vec{r}_i)$ describes the c.m. coordinate and Eq. (B12) means that this coordinate is not affected by the field \hat{F} . By using

$$\vec{\nabla}o(\vec{r}) = \vec{\nabla}rY_{1\mu}^*(\hat{r}) = \sqrt{3} \vec{Y}_{10\mu}^*(\hat{r}), \quad (\text{B14})$$

the condition (B12) is cast into

$$\int d^3r \rho_0(\vec{r}) \vec{v}(\vec{r}) \cdot \vec{Y}_{10\mu}^*(\hat{r}) = 0. \quad (\text{B15})$$

The condition (B15) may be applied to the density-dependent CM operator $\hat{M}'_{\text{com}}(E1\mu)$ by imposing the irrotational velocity

$$\vec{v}_{\text{com}}(\vec{r}) = \vec{\nabla}f_{\text{com}}(\vec{r}) = \vec{\nabla}Y_{1\mu}(\hat{r})(r^3 - \eta r), \quad (\text{B16})$$

where the second term is the actual c.m.c. with the coefficient η to be determined from Eq. (B15). Then, assuming spherical nuclei ($\rho_0(\vec{r}) = \rho_0(r)$) and using

$$\int d^3r \vec{Y}_{\lambda\mu}^*(\hat{r}) \cdot \vec{Y}_{\lambda'\mu'}(\hat{r}) = \delta_{\lambda\lambda'} \delta_{\mu\mu'}, \quad (\text{B17})$$

we get

$$\eta = \frac{5}{3} \langle r^2 \rangle_0, \quad (\text{B18})$$

with

$$\langle r^2 \rangle_0 = \frac{\int_0^\infty r^4 \rho_0(r) dr}{\int_0^\infty r^2 \rho_0(r) dr}, \quad (\text{B19})$$

and finally the expression (39) for $\hat{M}'_{\text{com}}(E1\mu)$.

For the current-dependent VM, TM, and CM operators (36)–(38), we put to Eq. (B15) the velocities

$$\vec{v}_{\text{vor}}(\vec{r}) = r^2 \vec{Y}_{12\mu}(\hat{r}) - \eta \vec{Y}_{10\mu}(\hat{r}), \quad (\text{B20})$$

$$\vec{v}_{\text{tor}}(\vec{r}) = \frac{\sqrt{2}}{5} r^2 \vec{Y}_{12\mu}(\hat{r}) + \vec{Y}_{10\mu}(\hat{r})(r^2 - \eta), \quad (\text{B21})$$

$$\vec{v}_{\text{com}}(\vec{r}) = \frac{\sqrt{2}}{5} r^2 \vec{Y}_{12\mu}(\hat{r}) - \vec{Y}_{10\mu}(\hat{r})(r^2 - \eta), \quad (\text{B22})$$

where the second terms with η are the relevant c.m.c.. For the exception of $\vec{v}_{\text{com}}(\vec{r})$, these velocities are not reduced to the gradient form. Actually, they are taken in the form of the external fields involved in the operators (36)–(38) and generating the corresponding modes. Such a presentation is in accordance with the self-consistent treatment of nuclear

excitations [1], which is done here in terms of small variations $\delta\vec{j}(\vec{r})$ of the nuclear current.

By using Eqs. (B20)–(B22), the requirement (B15) gives

$$\eta = 0, \quad (\text{B23})$$

for VM and

$$\eta = \langle r^2 \rangle_0, \quad (\text{B24})$$

for TM and CM. Thus we get the corrected expression (37) and (38) for $\hat{M}'_{\text{tor}}(E1\mu)$ and $\hat{M}'_{\text{com}}(E1\mu)$. Note that vector harmonics $\vec{Y}_{12\mu}(\hat{r})$ related to the vorticity do not contribute to the c.m.c. This reflects the physical fact that vorticity, being a curl flow, must be fully decoupled from the c.m. translation motion. Hence the c.m.c. is zero for the VM. However, the TM is not completely vortical and so its c.m.c. does not vanish.

Note that the above c.m.c. are approximate. Indeed, the calculations give for the VM, TM, and CM responses two broad structures, which actually do not meet the sum-rule condition of excitation of a single collective state. Besides, the prescription [46] uses the commutator $[\hat{H}, \hat{\rho}]$ where \hat{H} is assumed not to include the terms with velocity, spin, and spin-orbit dependence. However, the effect of spin-dependent terms in the commutator is obviously zero. The momentum (velocity)-dependent interaction does not matter for the Galilean-invariant Skyrme functional (the most common case) but may be important if this invariance is violated. The spin-orbit interaction may affect the c.m.c.

APPENDIX C: NUCLEAR DENSITY AND CURRENT OPERATORS

The density operator reads

$$\hat{\rho}(\vec{r}) = \sum_{q=n,p} e_{\text{eff}}^q \sum_{k \in q} [\delta(\vec{r} - \vec{r}_k)], \quad (\text{C1})$$

where e_{eff}^q are proton and neutron effective charges.

The operator of the full nuclear current consists of the convective and magnetic (spin) parts [48]

$$\hat{j}_{\text{nuc}}(\vec{r}) = \hat{j}_c(\vec{r}) + \hat{j}_m(\vec{r}) = \frac{e\hbar}{m} \sum_{q=n,p} [\hat{j}_c^q(\vec{r}) + \hat{j}_m^q(\vec{r})], \quad (\text{C2})$$

where

$$\hat{j}_c^q(\vec{r}) = -ie_{\text{eff}}^q \sum_{k \in q} [\delta(\vec{r} - \vec{r}_k) \vec{\nabla}_k + \vec{\nabla}_k \delta(\vec{r} - \vec{r}_k)], \quad (\text{C3})$$

$$\hat{j}_m^q(\vec{r}) = \frac{g_s^q}{2} \sum_{k \in q} \vec{\nabla} \times \hat{s}_{qk} \delta(\vec{r} - \vec{r}_k), \quad (\text{C4})$$

and \hat{s}_q is the spin operator, μ_N is the nuclear magneton, g_s^q is the spin g factor, k numerates the nucleons.

The $T = 0$ modes use the values

$$e_{\text{eff}}^n = e_{\text{eff}}^p = 1, \quad g_s^{n,p}(T = 0) = \frac{1}{2}(g_s^n + g_s^p), \quad (\text{C5})$$

while the $T = 1$ modes employ

$$e_{\text{eff}}^n = -e_{\text{eff}}^p = -1, \quad g_s^{n,p}(T = 1) = \frac{1}{2}(g_s^n - g_s^p). \quad (\text{C6})$$

APPENDIX D: SRPA EQUATIONS AND GENERATOR OPERATORS

The SRPA Hamiltonian is self-consistently derived [37,38] from the functional

$$\mathcal{E} = \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{Sk}} + \mathcal{E}_{\text{pair}} + \mathcal{E}_{\text{Coul}}, \quad (\text{D1})$$

involving kinetic-energy, Skyrme, pairing, and Coulomb terms. The Skyrme functional $\mathcal{E}_{\text{Sk}}(\rho, \tau, \vec{J}, \vec{j}, \vec{s}, \vec{T})$ depends on time-even (nucleon ρ , kinetic energy τ , spin-orbit J) and time-odd (current \vec{j} , spin \vec{s} , vector kinetic energy \vec{T}) densities. The Hamiltonian reads [37,38]

$$\hat{H} = \hat{h}_{\text{HFB}} + \hat{V}_{\text{res}}, \quad (\text{D2})$$

where \hat{h}_{HFB} is the HFB mean field

$$\hat{h}_{\text{HFB}} = \int d^3r \sum_{\alpha_+} \left[\frac{\delta \mathcal{E}}{\delta J_{\alpha_+}(\vec{r})} \right] \hat{J}_{\alpha_+}, \quad (\text{D3})$$

and \hat{V}_{res} is the separable residual interaction

$$\hat{V}_{\text{res}} = \frac{1}{2} \sum_{k,k'=1}^K (\kappa_{kk'} \hat{X}_k \hat{X}_{k'} + \eta_{kk'} \hat{Y}_k \hat{Y}_{k'}), \quad (\text{D4})$$

with one-body operators

$$\hat{X}_k = i \int d^3r d^3r' \sum_{\alpha_+, \alpha'_+} \frac{\delta^2 \mathcal{E}}{\delta J_{\alpha_+} \delta J_{\alpha'_+}} \langle [\hat{P}_k, \hat{J}_{\alpha_+}] \hat{J}_{\alpha'_+},$$

$$\hat{Y}_k = i \int d^3r d^3r' \sum_{\alpha_-, \alpha'_-} \frac{\delta^2 \mathcal{E}}{\delta J_{\alpha_-} \delta J_{\alpha'_-}} \langle [\hat{Q}_k, \hat{J}_{\alpha_-}] \hat{J}_{\alpha'_-},$$

and inverse strength matrices

$$\kappa_{kk'}^{-1} = -i \langle [\hat{P}_k, \hat{X}_{k'}] \rangle, \quad \eta_{kk'}^{-1} = -i \langle [\hat{Q}_k, \hat{Y}_{k'}] \rangle. \quad (\text{D5})$$

Here α_+ and α_- enumerate time-even J_{α_+} and time-odd J_{α_-} densities, respectively; $\hat{J}_{\alpha_{\pm}}$ are the density operators; \hat{Q}_k and $\hat{P}_k = i[\hat{H}, \hat{Q}_k]$ are time-even and time-odd hermitian generator operators. The operators of the residual interaction \hat{X}_k and \hat{Y}_k are time-even and time-odd by construction, respectively.

The single-particle Hamiltonian \hat{h}_{HFB} is determined by the first functional derivatives of the initial functional (D1) while operators \hat{X}_k and \hat{Y}_k are driven by the second functional derivatives of the same functional. The residual interaction includes all the possible terms arising from Eq. (D1). Hence the model is fully self-consistent. The number K of separable terms in Eq. (D4) is determined by the number of the generator (input) operators \hat{Q}_k . Usually we have $K = 3-5$. This results in a low rank of the RPA matrix and so in an efficient calculation scheme.

The SRPA formalism itself does not prescribe the form of the generators \hat{Q}_k and \hat{P}_k . At the same time, their choice is important for a fast convergence of the factorized residual interaction \hat{V}_{res} to the true one with a minimal number of separable terms. The set of generating operators is introduced so as to initiate in the nucleus all the relevant kinds of motion for the considered modes. For time-even modes, the initial generators \hat{Q}_k are chosen first and then their time-odd counterparts are determined from $\hat{P}_k = i[\hat{H}, \hat{Q}_k]$. Instead for time-odd modes, the initial generators \hat{P}_k are inserted and then their time-even counterparts $\hat{Q}_k = i[\hat{H}, \hat{P}_k]$ are determined. The generators may be arbitrarily and separately scaled, which does not influence the results. The coupling of the modes (e.g., of electric and magnetic ones in deformed nuclei) may request both time-even and time-odd generators in the set. The optimal sets of the generators were developed for $E1(T=1)$ [37,38] and spin-flip $M1$ [43-45] GR.

Here we use the minimal sets of the generators suitable for the description of VM, TM, and CM. The generators cover the main parts of the corresponding operators and take into account the coupling between the modes. For the VM, they are

$$\hat{P}_1 = \int d^3r \hat{j}_c(\vec{r}) r^2 \vec{Y}_{12\mu}(\hat{r}),$$

$$\hat{P}_2 = \int d^3r \hat{j}_m(\vec{r}) r^2 \vec{Y}_{12\mu}(\hat{r}), \quad (\text{D6})$$

$$\hat{Q}_3 = \int d^3r \rho(\vec{r}) [r^3 - \frac{5}{3} \langle r^2 \rangle_0 r] Y_{1\mu}(\hat{r}),$$

i.e., cover the time-odd parts of the vortical operator with the convection and magnetization currents as well as the time-even compression operator with the c.m.c. (to prevent generation of the spurious motion).

For the TM, the generators read

$$\hat{P}_1 = \int d^3r \hat{j}_c(\vec{r}) \left[r^2 \frac{\sqrt{2}}{5} \vec{Y}_{12\mu}(\hat{r}) + \vec{Y}_{10\mu}(\hat{r}) (r^2 - \langle r^2 \rangle_0) \right],$$

$$\hat{P}_2 = \int d^3r \hat{j}_m(\vec{r}) \left[r^2 \frac{\sqrt{2}}{5} \vec{Y}_{12\mu}(\hat{r}) + \vec{Y}_{10\mu}(\hat{r}) (r^2 - \langle r^2 \rangle_0) \right],$$

$$\hat{Q}_3 = \int d^3r \rho(\vec{r}) \left[r^3 - \frac{5}{3} \langle r^2 \rangle_0 r \right] Y_{1\mu}(\hat{r}), \quad (\text{D7})$$

(i.e., cover the time-odd parts of the toroidal operator with the convection and magnetization currents as well as the time-even compression operator). The TM needs the c.m.c. and so now this correction is included to all the generators.

Finally the set for the CM includes the compression operator \hat{Q}_3 itself and the convective toroidal generator \hat{P}_1 . The generators with the magnetic current $\hat{j}_m(\vec{r})$ are not involved since their effect on the CM is zero.

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