Reexamination of the α-α "fishbone" potential

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The fishbone potential of composite particles simulates the Pauli effect by nonlocal terms. We determine the α-α fishbone potential by simultaneously fitting to two-α resonance energies, experimental phase shifts, and three-α binding energies. We found that, essentially, a simple Gaussian can provide a good description of two-α and three-α experimental data without invoking three-body potentials.

I. INTRODUCTION

The potential is the most important nonobservable physical quantity in quantum mechanics. It is not an observable, yet it determines all the physical observables. There are two ways of determining the quantum mechanical potential. If the quantum system has a classical counterpart, one can use the correspondence principle. Otherwise, we determine the potential by calculating observables and comparing them to experiments.

Almost all, so-called elementary, particles are in fact composite particles made of even more elementary particles. These constituents are fermions that obey the Pauli principle (i.e., they cannot occupy the same quantum state). The Pauli principle and the internal structure and dynamics of the constituent fermions can lead to a very complicated potential for the composite particles.

The simplest way to model the Pauli effect is to use a repulsive short-range potential, which suppresses the wave function at short distances. The parameters of this phenomenological potential are determined either by the inverse scattering method or by fitting models to experiments. In most models local potentials are used. Generally, these models cannot provide an acceptable description of three-body data, which results in the need for three-body potentials.

An alternative approach is that we try to incorporate all the information about the internal structure and dynamics of composite particles into their mutual interactions. Several potentials based on the nuclear cluster model [1] have been proposed. One possibility is to derive the interactions from the cluster model in the framework of the resonating group method (see, e.g., Ref. [2] for a recent review). The other approach is more phenomenological. It uses some information on the structure of composite particles but incorporates some phenomenological potential whose parameters are determined by fitting the results to experiments. The pioneer of this type of model is the orthogonality-condition model by Saito [3], where the states are orthogonal to predefined Pauli-forbidden states. In the method of Buck, Friedrich, and Wheatley [4], a deep potential is adopted, and it is assumed that the few lowest states are forbidden by the Pauli principle. The fishbone model by Schmid [5,6] goes beyond previous models as it introduces the concept of partially Pauli-forbidden states.

It was a common belief that, if we incorporate all the information about the internal structure of the particles into their mutual interactions, then the three-body potential would be small, perhaps negligible. However, for α particles, while phenomenological shallow local potentials under bind the three-α system, cluster-model-inspired phenomenological potentials considerably over bind them.

In this work we revisit the problem of interactions of composite particles. We consider the fishbone model of the α-α interaction. We have chosen the α-α potential because the α particle has an exceptionally strong binding energy. We adopted the fishbone model because, in our opinion, this is the most elaborated phenomenological cluster-model-motivated potential. The variant of the fishbone potential has been designed to minimize and to neglect the three-body potential. Therefore, we can try to determine the interaction by a simultaneous fit to two- and three-body data.

Previously, we studied the α-α fishbone model [7] and proposed a new parametrization of the fishbone α-α interaction. We fit the two-body phase shifts and the three-α ground-state energy. Later we found that the results are not stable against varying the parameters. Here, besides two-body phase shifts and the three-α ground state, we include the three-α $L = 0$ ground state, the $L = 0$ resonant state, and the $L = 2$ bound state.

In Sec. II we outline the fishbone model for the composite particles. In Sec. III we determine the α-α potential by using two-α and three-α data. Finally, we draw some conclusions.

II. FISHBONE OPTICAL MODEL

The fishbone model is motivated by the cluster model. In the resonating group model the total wave function is an antisymmetrized product of the cluster Φ and the intercluster χ relative states:

$$|\Psi\rangle = |\{\mathcal{A}\Phi\chi\}\rangle. \tag{1}$$

The state Φ, which is supposed to be known in this model, describes the internal properties of the clusters, including spin and isospin structure. The unknown relative motion state χ is

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determined from the variational ansatz

$$\langle\Phi\delta\chi|\mathcal{A}(H - E)\mathcal{A}|\Phi\chi\rangle = 0. \tag{2}$$

This ansatz results in a rather complicated equation for χ, which was possible to solve only by using serious approximations on Φ and on the interaction of the particles. In a typical example, Ψ describes fermions in harmonic oscillator potential wells located some distance apart and χ is the relative motion of the oscillator wells. We can easily see that, if we express χ in terms of harmonic oscillator states, some of the lowest states in the relative motion space are not allowed due to the Pauli principle.

The Schrödinger equation for the two-body fishbone model is given by [5]

$$(h^0 + \mathcal{V}_l)|\chi\rangle = E|\chi\rangle, \tag{3}$$

where h^0 is the kinetic energy operator. Our knowledge of the internal structure and the Pauli principle are incorporated in the fishbone potential:

$$\mathcal{V}_l = v_l - \sum_{i,j} |u_{l,i}\rangle\langle u_{l,i}|(h^0 + v_l - \epsilon_{l,i})|u_{l,j}\rangle \bar{M}_{l,ij}\langle u_{l,j}|, \tag{4}$$

where l refers to a partial wave and v_l is a local potential. The states $|u_{l,i}\rangle$ are eigenstates of the norm operator,

$$\langle\Phi\vec{r}|\mathcal{A}|\Phi u_{l,i}\rangle = (1 - \eta_{l,i})\langle\vec{r}|u_{l,i}\rangle, \tag{5}$$

where $\vec{r}$ is the center-of-mass distance of the two clusters. If the relative motion is forbidden by the Pauli principle, then $\langle\Phi\vec{r}|\mathcal{A}|\Phi u_{l,i}\rangle = 0$ and $\eta_{l,i} = 1$. The $\eta_{l,i}$ eigenvalues are ordered such that $|\eta_{l,i}| \geqslant |\eta_{l,i+1}|$. The matrix $\bar{M}$ is then given by

$$\bar{M}_{ij} = \begin{cases} 1 - \dfrac{1 - \eta_{l,i}}{[(1 - \bar{\eta}_{l,i})(1 - \bar{\eta}_{l,i})]^{1/2}} & \text{if } i \leqslant j \\ 1 - \dfrac{1 - \eta_{l,j}}{[(1 - \bar{\eta}_{l,j})(1 - \bar{\eta}_{l,i})]^{1/2}} & \text{if } i > j, \end{cases} \tag{6}$$

where $\bar{\eta}_{l,i} = 0$ if $\eta_{l,i} = 1$ and $\bar{\eta}_{l,i} = \eta_{l,i}$ otherwise. In matrix form, if we have one Pauli-forbidden state, we have

$$\bar{M}_l = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,3}}} & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,4}}} & \cdots \\ 1 & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,3}}} & 0 & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,4}}} & \cdots \\ 1 & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,3}}} & 1 - \sqrt{\frac{1-\eta_{l,2}}{1-\eta_{l,4}}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7}$$

The matrix elements of $\bar{M}$ exhibit a fishbone-like structure; hence the name of the model. In this model the Pauli-forbidden states become eigenstates at energy ϵ. By choosing a large positive ϵ, they become bound states at large positive energy and thus disappear from the physically relevant part of the spectrum. There are several versions of the fishbone model which differ in off-shell transformations; that is, in transformations which effect the internal part of the wave function and leave the asymptotic part, and consequently the spectrum, unchanged. This version of the model minimizes the three-body potential.

We can extend the two-body fishbone model to three clusters by embedding the two-body fishbone potential into the three-body Hilbert space [6]. We use the usual configuration space Jacobi coordinates. For example, the coordinate x_1 denotes the vector between particles 2 and 3, while y_1 connects the center of mass of the subsystem $(2, 3)$ with particle 1. The three-body fishbone Hamiltonian is given by

$$H = H^0 + \mathcal{V}_{x_1}\mathbf{1}_{y_1} + \mathcal{V}_{x_2}\mathbf{1}_{y_2} + \mathcal{V}_{x_3}\mathbf{1}_{y_3}, \tag{8}$$

where $H^0 = h^0_{x_i} + h^0_{y_i}$ (with $i = 1, 2, 3$) is the kinetic energy operator. Here we have omitted the three-body potential.

The fishbone potential is rather complicated. It has a local Coulomb-like part augmented by a nonlocal short-range potential. The numerical treatment is also nontrivial. However, in the past couple of years, in a series of publications, we developed a method for dealing with potentials of this type. We solve the Lippmann-Schwinger integral equation for two-body problems and Faddeev integral equations for three-body problems. We approximate the short-range parts of the potential in the Coulomb-Sturmian basis. This basis allows an exact analytic evaluation of the Coulomb Green's operator in terms of a continued fraction for the two-body case and in terms of a contour integral for the three-body case. For details, see, for example, Ref. [7] and references therein.

III. FISHBONE MODEL OF α-α INTERACTION

We adopt a model that, in the α particles, the nucleons are in $0s$ states in a harmonic oscillator well with width parameter a. The norm kernel eigenvalues are also harmonic oscillator states with the same width parameter and the eigenvalues are known [8]: $\eta_{0,i} = 1, 1, 1/4, 1/16, 1/64, \ldots, \eta_{2,i} = 1, 1/4, 1/16, 1/64, \ldots$, and $\eta_{4,i} = 1/4, 1/16, 1/64, \ldots$ This shows that, in the $l = 0$ relative motion channel, there are two Pauli-forbidden states; in $l = 2$ there is one and in $l = 4$ and higher channels there are none. The decreasing value of η indicates that, in the relative motion the harmonic oscillator, states with higher quantum number are less and less suppressed by the Pauli principle. For the ϵ parameter of the fishbone model, which aims to remove the Pauli-forbidden states, we took $\epsilon = 60\,000$ MeV. In this range of ϵ, the dependence of the results was beyond the fifth significant digit. We used the experimental phase shifts from Ref. [9].

A fishbone potential of the α-α system was determined by Kircher and Schmid [10]. The harmonic oscillator width parameter was fixed to $a = 0.55$ fm^{-2}, which leads to the length parameter $r_0 = (2a_0)^{-1/2} = 0.9535$ fm. The local potential was taken in the form

$$v_l(r) = v_0 \exp(-\beta r^2) + \frac{4e^2}{r}\text{erf}\left(\sqrt{\frac{2a}{3}}r\right), \tag{9}$$

where v_0 and β are fitting parameters. They were determined by fitting to experimental phase shifts. The values $v_0 = -108.419\,98$ MeV and $\beta = 0.188\,98$ fm^{-2} were found to provide the best fit.
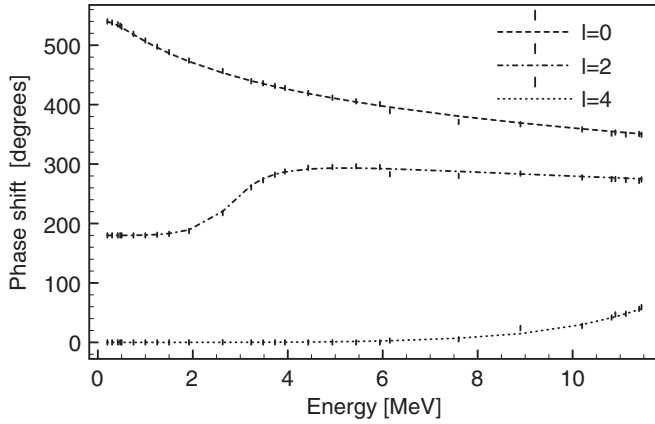
FIG. 1. Fit to experimental $l = 0$, $l = 2$, and $l = 4$ phase shifts from two-body data.

While this potential provides a reasonably good fit to $l = 0$ and $l = 2$ phase shifts, it results in too much binding for the three-α system. It gives $E^{3\alpha}_{L=0} \sim -15.5$ MeV, while the experimental three-α binding energy is $E^{3\alpha}_{\text{expt}} = -7.275$ MeV. One may conclude that, similarly to conventional local-potential models, there is a need for a three-body potential. This was the choice that Oryu and Kamada [11] adopted. They added a phenomenological three-body potential to the fishbone potential of Kircher and Schmid and found that a huge three-body potential is needed to reproduce the experimental data. So, the conclusion was that, although the fishbone potential provides a good fit to two-body data, it needs a strong three-body force to reproduce the three-body data. This is with a potential that was designed such that the three-body force could be neglected. This is certainly not the case. So, although the model has some good features with this parametrization, it does not live up to its promise.

IV. RESULTS

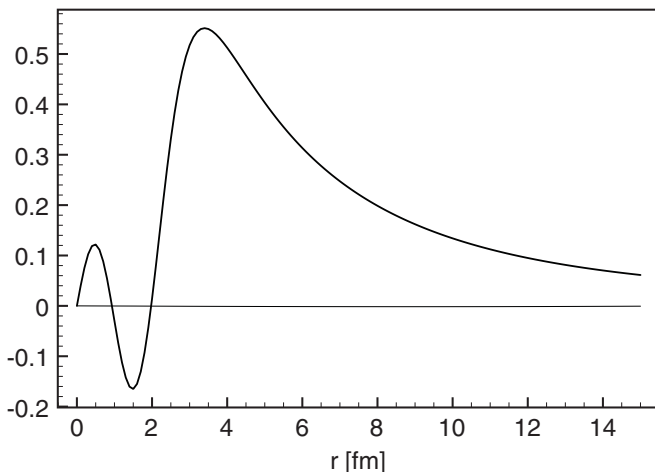We refit the two-α experimental data with the potential (9). Besides the phase-shift data, we also incorporated

FIG. 2. Wave function of $l = 0$ α-α resonant state at energy $E_{\alpha\text{-}\alpha} = 0.091\,58 - 0.000\,003i$ MeV.
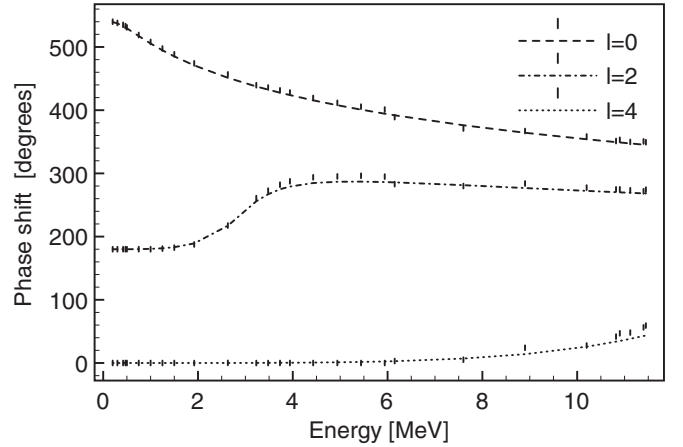

FIG. 3. Fit to experimental $l = 0$, $l = 2$, and $l = 4$ phase shifts from two- and three-body data.

the famous $l = 0$ two-α resonance state at $E_{\alpha\text{-}\alpha} = 0.0916 - 0.000\,003i$ MeV. We found that the parameters $a = 0.5838$ fm^{-2}, $v_0 = -109.97$ MeV, and $\beta = 0.194\,17$ fm^{-2} provide the best fit. The fit to the α-α phase shift is given in Fig. 1. By calculating three-α states we get $E^{3\alpha}_{L=0} = -13.6$ MeV, $E^{3\alpha}_{L=0} = -0.2$ MeV, and $E^{3\alpha}_{L=2} = -11.3$ MeV. So, the fishbone model with parameters extracted from only two-body data still cannot provide a good description of three-body experimental binding energies.

On the other hand, two-body data are sensitive mostly on the outer part of the potential and are less sensitive on the inner part. We may use this fact to find a fishbone potential which fits simultaneously two- and three-body data. So, besides the two-body phase shifts and the $l = 0$ two-α resonant state, we incorporated the $L = 0$ three-α ground state at $E = -7.275$ MeV, the $L = 0$ three-α excited state at $E = 0.375$ MeV, and the $L = 2$ three-α bound state at $E = -2.836$ MeV.

We achieved the best fit to experiments with parameters $a = 0.6266$ fm^{-2}, $v_0 = -101.78$ MeV, and $\beta = 0.1881$ fm^{-2}. This set of parameters provides an $l = 0$ two-α resonant state at $E_{\alpha\text{-}\alpha} = 0.091\,58 - 0.000\,003i$ MeV. The corresponding wave function is shown in Fig. 2. Note the peculiarity of the fishbone model in that the ground-state wave function has nodes due to the orthogonality to the fully Pauli-forbidden states. Figure 3 shows the α-α phase shifts. We can see that the agreement with experiments is almost as good as before. For the three-α states we get $E^{3\alpha}_{L=0} = -7.01$ MeV, $E^{3\alpha}_{L=0} = 0.51$ MeV, and $E^{3\alpha}_{L=2} = -4.5$ MeV. The $L = 0$ states are almost in perfect agreement with the experimental values. The $L = 2$ bound state is slightly over bounded.

V. SUMMARY AND CONCLUSION

In this work we propose a unified parametrization of the fishbone α-α potential. We determined the potential by a simultaneous fit to two-α and three-α data. We found that, with three fitting parameters, one can describe the two-body resonance and phase shifts in all partial waves. The potential

also provides a reasonably good description of three-body data without invoking any three-body potential.

We can learn from this study that, if we incorporate our knowledge of the structure of composite particles into their interaction, like we do in the fishbone model, we can achieve a substantial simplification of the potential. Here, in the α-α case, we have only three parameters, while in the conventional Ali-Bodmer-type potentials, we have a couple of independent parameters in each partial wave plus additional parameters for the three-body potential. We believe that cluster-model-based models for the interaction of composite particles deserve further consideration. The fishbone model of Schmid is especially appealing because it faithfully represents the Pauli principle and uses the concept of partly Pauli-forbidden states. It could also serve as a framework for the nucleon-nucleon potential.

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