PHYSICAL REVIEW C **83**, 065201 (2011)

Chiral symmetry restoration at finite density in large N_c QCD

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(Received 13 April 2011; published 13 June 2011)

At large N_c, cold nuclear matter is expected to form a crystal and thus spontaneously break translational symmetry. The description of chiral symmetry breaking and translational symmetry breaking can become intertwined. Here, the focus is on aspects of chiral symmetry breaking and its possible restoration that are by construction independent of the nature of translational symmetry breaking—namely spatial averages of chiral order parameters. A system will be considered to be chirally restored provided all spatially averaged chiral order parameters are zero. A critical question is whether chiral restoration in this sense is possible for phases in which chiral order parameters are locally nonzero but whose spatial averages all vanish. We show that this is not possible unless all chirally invariant observables are spatially uniform. This result is first derived for Skyrme-type models, which are based on a nonlinear sigma model and by construction break chiral symmetry on a point-by-point basis. A no-go theorem for chiral restoration (in the average sense) for all models of this type is obtained by showing that in these models there exist chirally symmetric order parameters that cannot be spatially uniform. Next, we will show that the no-go theorem applies to large N_c QCD in any phase that has a nonzero but spatially varying chiral condensate. The theorem is demonstrated by showing that in a putative chirally restored phase, the field configuration can be reduced to that of a nonlinear sigma model. It is also shown that this no-go theorem is fully consistent with the vanishing of the spatial average of the chiral condensate $\frac{1}{2}\mathrm{Tr}(U)$ (as happens in "half-skyrmion" configurations). This is because the chiral condensate is only one of an infinite set of chiral order parameters, *some* of which must be nonzero. It is also shown that while an approximation of a unit cell of a Skyrme crystal as a hypersphere does lead to a phase that is chirally restored (in the average sense), this is an artifact of the approximation.

DOI: 10.1103/PhysRevC.83.065201 PACS number(s): 21.65.−f, 11.15.Pg, 11.30.Rd, 12.38.Aw

I. INTRODUCTION

One of the long-standing problems in QCD is to understand finite-density nuclear matter. Such matter is only poorly understood in terms of QCD but is of fundamental importance in developing a theoretical understanding of the qualitative and quantitative features of QCD equations of state and phase diagrams. It is also important in studying astrophysical objects such as neutron stars, which are composed of dense nuclear matter, and in heavy ion collisions. While zero-density nuclear matter can be studied using lattice QCD, at finite densities or equivalently chemical potential and low temperatures, the fermion sign problem renders lattice studies intractable.

Given this situation, one is often forced to rely on QCD-inspired models to get insight. This is problematic. First of all, it is obvious that the models are *not* QCD and one needs to be cautious in interpreting their results. Second, many models rely at least implicitly on QCD being close to the large N_c limit [1,2], where mean-field methods can be justified. At large N_c, quantum effects become negligible and nuclear matter behaves classically in some important sense. In this limit of high densities and large N_c, it is generally believed that it is energetically favorable for nuclear matter to crystallize in its ground state.

The large N_c limit was recently used to motivate the existence of a new phase of finite density nuclear matter. McLerran and Pisarski [3] argued that at low temperatures nuclear matter condenses into a "quarkyonic phase" of confined but chirally restored nuclear matter dominated by baryon-baryon interactions. This idea was further developed in Ref. [4]. This phase is supposed to occur in large N_c QCD in the regime in which the quark chemical potential is of order N_c^0 large compared to Λ_{QCD}. The logic is simply that in this regime the quarks do not effect the gluons, and thus confinement described in terms of the Polyakov line is unaffected. Thus, provided the chemical potential is large enough to cause chiral restoration, one necessarily enters a quarkyonic phase. This analysis is, strictly speaking, only clean in the chiral limit of zero-quark masses, and for the purposes of this paper, we will ignore the effects of quark masses and set them to zero at the outset.

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There is a potential difficulty with the argument [3]. *A priori* it is not trivially obvious from the structure of large N_c QCD that at high baryon density the system necessarily becomes chirally restored. This issue is at the crux of this paper. The primary question is whether chiral symmetry is restored at high density. We have no answer to this critical question. However, there is a secondary question of importance that we can address, namely if chiral symmetry is restored, by what mechanism does this occur?

To set the context for this, let us consider a model-dependent argument used in Ref. [3] to justify the assumption that chiral symmetry is restored in large N_c QCD at sufficiently high density. The argument is based on the Skyrme model [5,6] treated at the classical level (which is justified at large N_c). The model, although phenomenological, does capture many aspects of large N_c QCD and of chiral symmetry. It has proven to be a reasonable, if crude way to describe static properties of baryons such as masses, charge radii, and magnetic moments [7]. It has been used to describe nuclear matter that at mean-field level is naturally described as a crystal [8–15].

At first sight, one might think that the Skyrme model, which is based on a nonlinear sigma model and hence by construction breaks chiral symmetry at every point in space, is unsuitable to study chiral restoration. However, at this stage, it is worth noting that at low but nonzero baryon density, two distinct types of symmetry are spontaneously broken: chiral symmetry, which is broken as in the vacuum, and translational symmetry, which is broken by the formation of a crystal. The two types of spontaneous symmetry breaking can become intertwined. Since our focus is on chiral symmetry breaking and *not* on translational symmetry breaking, it is reasonable to consider only those aspects of chiral symmetry breaking that do not depend on translational symmetry breaking.

It is easy to construct order parameters that are sensitive to chiral symmetry breaking but insensitive to any details of translational symmetry—namely by spatially averaging standard chiral order parameters. It is not unreasonable to consider a system chirally restored if all of the spatially averaged chiral order parameters vanish; this paper will focus on the question of chiral restoration in this spatially averaged sense. In the remainder of the paper, the phrase "chiral restoration" will often be used as a shorthand for "chiral restoration in the spatial averaged sense." One might hope that in Skyrme-type models, chiral symmetry, although broken at every point in space, could be restored in this spatially averaged sense.

In arguing that large N_c QCD should restore chiral symmetry at sufficiently high density, Ref. [3] presents evidence that the Skyrme model treated classically, in fact, exhibits chiral restoration in this spatially averaged sense and argues that large N_c QCD might similarly exhibit chiral restoration since the Skyrme model captures many aspects of large N_c QCD. The evidence that the Skyrme model restores chiral symmetry at high density is of two types: (i) the vanishing of the spatially averaged chiral condensate found in studies of different types of Skyrme crystals [7–10,12,13,15], and (ii) the vanishing of spatially averaged chiral order parameters as seen in analytical studies in which a Skymion in the crystal is approximated as a single Skyrmion on the compactified manifold of a hypersphere [8,16–18].

The evidence that chiral symmetry is restored in the spatially averaged sense at sufficiently high density in the Skyrme model appears to be quite strong. However, as will be shown in this paper, it is misleading. We prove a theorem that it is impossible for any Skyrme-type model to restore chiral symmetry in the spatially averaged sense at nonzero baryon density. We further show that the no-go theorem also applies to large N_c QCD with a nonzero, spatially varying chiral condensate. This is done by showing that in a chirally restored phase, expectation value of operators can be reduced to that of a generic nonlinear sigma model.

Before proving this no-go theorem, it is useful to discuss how it can possibly be correct given the evidence presented in Ref. [3]. First, let us consider the evidence of the vanishing of spatially averaged chiral condensate in Skyrme crystals above some critical density. It is certainly possible that the chiral condensate itself may vanish on spatial integration and indeed, it is expected to happen in so-called "half-skyrmion" configurations of the crystal's unit cell [5,11–13,15]. This occurs when isolated skyrmions (whose interaction energy at low density scales as r^3 [10]) are brought close together. It is believed that the lowest possible energy consistent with the topology of the model (a winding number of unity per unit cell of the crystal) is a new type of phase based on a configuration that has an additional axial symmetry. This constrains the field to have $\sigma = -1$ at the center of the unit cell. The configuration has a baryon number of $\frac{1}{2}$ and can be transformed into another half-skyrmion by the transformation $(\sigma, \boldsymbol{\pi}) \to (-\sigma, -\boldsymbol{\pi})$. The unit cell consists of two of these half-skyrmions. The chiral order parameter $\frac{1}{2}\text{Tr}(U) \equiv \sigma$, where σ (which is assumed to be proportional to $\overline{q}q$ of QCD), vanishes when integrated over the unit cell.

The key point is that while chiral restoration (in the average sense) necessarily implies the vanishing of the spatially averaged chiral condensate, the converse is not true. It is logically possible that chiral condensate could vanish due to the restoration of a discrete symmetry while other spatially averaged chiral order parameters remain nonzero. *A priori* it may seem implausible that such a scenario could be realized. Indeed, it is reminiscent of Stern's suggestion that in the vacuum, $\langle\overline{q}q\rangle$ vanishes but chiral symmetry remains broken due to higher dimensional condensates [19]. However, as shown by Kogan, Kovner, and Shifman (KKS) [20], using an elegant argument based on a rigorous QCD inequality for the Euclidean space functional integral, the Stern scenario is inconsistent with QCD. One might worry whether a similar scenario is also impossible for finite-density matter. However, the KKS argument depends on a real and positive functional determinant, which while true for the vacuum does not hold for finite-density matter in which the chemical potential leads to a functional determinant that is not manifestly real and positive. Thus, this argument is not applicable to finite-density matter. Moreover, as will be shown in this paper, this unlikely seeming scenario must occur for both Skyrme models and for large N_c QCD if there exists a regime where the average chiral condensate is not zero locally but vanishes under spatial averaging.

Now, let us consider the second class of evidence for chiral restoration in the average sense—namely the fact that when a Skyrmion in the crystal is approximated as a single Skyrmion on the compactified manifold of a hypersphere, all spatially averaged chiral order parameters vanish. In effect by using Skyrmions placed on the compactified manifold of a hypersphere "surface," $S^3(L)$ embedded in R^4 [8,16,17,21], one is effectively chopping off the "edges" of a unit cell in a real Skyrme crystal and somehow mapping this onto the hypersphere again with the winding number set to one. Of course, in doing this, one is making an uncontrolled approximation, but the virtue is simplicity with many quantities analytically calculable. The density of the Skyrme crystal is assumed to be the inverse volume of the hypersphere. For low densities (or large radius hypersphere), there are two solutions—a stable low-energy one, in which the winding number density (i.e., baryon density) is localized and spatially averaged chiral order parameters are nonzero, and an energetically unstable phase, in which the winding number density is spread uniformly and spatial averages of all chiral order parameters vanish. However, at some critical density (or equivalently radius), the two solutions merge, and above this density the lowest energy solution is the one with uniform baryon density and chiral restoration in the spatially averaged sense.

Of course, this evidence for chiral restoration depends on an uncontrolled approximation. One does not know at the outset how well one can approximate a Skyrmion in the crystal by a single Skyrmion on a hypersphere. Presumably, the logic underlying the approximation is that the principal effect of putting a Skyrmion into a crystal is to restrict the space over which it can spread. If this is the case, it is natural to assume that a Skyrmion in the restricted space of a hypersphere is likely to be qualitatively similar to a Skyrmion in a crystal. Thus, it is plausible that while details of such a precise equation of state or the density of the phase transition to the chirally restored phase would be affected by the approximation, qualitative features such as the existence of a chirally restored phase would not.

The preceding argument depends on a critical assumption, namely that using a hypersphere to restrict the volume of the Skyrmion acts generically like other restrictions on its volume. This need not be true. The hypersphere is a rather special geometrical structure. One might worry that the chiral restoration seen on the hypersphere is not a generic feature at all, but rather an accidental feature of the *ad hoc* choice of geometry. If this is the case, then the fact that Skyrmions exhibit chiral restoration at high density on the hypersphere gives no insight into what happens for Skyrme crystals. As will be shown, the vanishing of all spatially averaged chiral order parameters *is* a result of the peculiarities of the geometry of the hypersphere. This can be seen clearly by keeping the Skyrmion on a compact space, but one in which the geometry is distorted away from the hypersphere. When this is done, chiral restoration (in the average sense) disappears.

The paper is organized as follows. In the Sec. II, we prove a no-go theorem that prevents chiral symmetry restoration (in the average sense) for all Skyrme-type models, i.e., nonlinear sigma models in which the winding number density is identified as the baryon density. This proof consists of two parts. First, we derive as a necessary condition for chiral restoration (in the average sense) the condition that all chiral singlet observables must be spatially uniform. In Sec. III, we show that a particular chirally symmetric order parameter cannot be uniform while chiral symmetry is restored simultaneously in Skyrme-type models, thus establishing the theorem. Then we extend our no-go theorem from the realm of models to large N_c QCD itself. In particular, we prove that large N_c QCD at nonzero baryon density cannot have a chiral condensate that is nonzero but nonuniform spatially in such a way that all spatially averaged chiral order parameters vanish. In Sec. IV, we discuss in detail the case of a Skyrmion confined to a hypersphere in light of the no-go theorem. We show explicitly why the proof of the no-go theorem works for crystals in flat space but breaks down for a Skyrmion confined to a hypersphere. Next, we discuss why the insights gleaned from a Skyrmion confined from a hypersphere is misleading. We do this by considering Skyrmions confined to a compact manifold which is deformed from a hypersphere; chiral symmetry in the average sense is not restored. This demonstrates that chiral restoration observed on the hypersphere is an artifact of the (unphysical) choice of geometry. We end with a brief discussion of these results in Sec. V.

II. A NO-GO THEOREM IN THE SKYRME MODEL

In this section, we prove a no-go theorem for Skyrme-type models, which we define as chirally invariant nonlinear sigma models based on a matrix-valued field: $U(\mathbf{x})$ with $U \in SU(2)$ (and possibly other fields) with a Lagrangian rich enough to support stable topological solitons. The model is treated classically, in keeping with large N_c QCD. The classical field, $U(\mathbf{x})$, which we take to be continuous is a mapping from $R^3 \to SU(2)$ for flat space [and from $S^3(L) \to SU(2)$ for skyrmions on the hypersphere]. Physically, the field U encodes the dynamics of the pion. Such models automatically have an algebraically conserved winding number current, w^μ (that is $\partial_\mu w^\mu = 0$ for any field configuration) where

$$w^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[(U^\dagger\partial_\nu U)(U^\dagger\partial_\alpha U)(U^\dagger\partial_\beta U)]}{24\pi^2}. \quad (1)$$

In these models, w^μ is often taken to be to equal to the conserved baryon current, although in principle they can differ by a quantity that is conserved and has zero net charge.

In proving the no-go theorem, the precise details of the Lagrangian play no role, nor does the identification of the winding number current with baryon current. However, its chiral properties are essential. The Lagrangian must be invariant under a global $SU(2)_L \times SU(2)_R$ transformation where $U(\mathbf{x}) \to LU(\mathbf{x})R^\dagger$ and $U^\dagger(\mathbf{x}) \to RU^\dagger(\mathbf{x})L^\dagger$. If the theory has other fields, they must transform in a consistent way. Of course, the nonlinear sigma models build in spontaneous symmetry breaking at the outset; while the Lagrangian is chirally invariant, any field configuration at any point in space is not. That is $U(\mathbf{x}) \neq RU(\mathbf{x})L^\dagger$. Thus, by construction, chiral symmetry when evaluated point-by-point in space *cannot be restored* in any model in this class.

The notion of chiral symmetry restoration in the Skyrme model only makes sense as a global property and not a local one. We define chiral symmetry to be restored in the spatially averaged sense if *all* functions of the classical fields when spatially averaged over a unit cell equals the average of the same function averaged uniformly over the internal space of chiral rotations. One particularly interesting class of observables are those that depend on U (which already encodes spontaneous symmetry breaking) but not on spatial derivatives (or other possible fields in the problem), where one can project out the chiral singlet part by averaging uniformly over the $SU(2)$ field. Thus, for example, chiral restoration in this sense requires that any scalar function of $U(\mathbf{x})$ only (and not its derivatives) will satisfy

$$\frac{1}{V_{R^3}}\int_{R^3} dV\, F(U(\mathbf{x})) = \frac{1}{2\pi^2}\int_{SU(2)} d\mu\, F[U(\mu)], \tag{2}$$

where F is an arbitrary real-valued function of U and μ represents the three angles needed to specify a general $SU(2)$ matrix and $d\mu$ is the Haar measure of $SU(2)$. The logic of this is quite simple: averaging over the internal space ensures that all chiral order parameters will vanish while chiral singlets will not. Note that in Eq. (2), the average of the function of the field $U(\mathbf{x})$ in the internal space is independent of the dynamics of the model, but its spatial average depends on the equations of motion.

In this section, we prove that chiral restoration in the spatially averaged sense is impossible for the Skyrme model. This proof has two parts. The first is a demonstration that chiral restoration in this sense cannot occur unless all chiral singlet observables are spatially uniform. The second is that chiral restoration is inconsistent with the fact that all chiral order parameters must vanish (upon spatially averaging). This will be demonstrated via the explicit construction of a chiral singlet operator that cannot be spatially uniform in a regime where the spatial average of all chiral order parameters vanish.

A. Chiral restoration in the average sense requires spatial uniformity for all chiral singlets

The key to the first part of the proof is the intuition that the only natural way chiral restoration in the average sense can occur is if spatially integrating corresponds to integrating over the internal space with uniform weighting. The central part of the proof is a demonstration that this intuition is correct. Ultimately, this proves to be such a strong constraint that no field distributions in Skyrme-type models can satisfy it unless all chiral singlet observables are spatially uniform.

To begin the formal treatment of the first part of the proof, we note that chiral restoration requires *all* chiral order parameters vanish, or equivalently that the spatial average of *all* functions equals the uniform average over all chiral rotations. Thus, to show that chiral restoration cannot occur, it is sufficient to show that there exists some subset of chiral order parameters for which it is not true. Here, we will focus on observables constructed entirely from the field U, which is local and hence depends on a single point $\mathbf{x}$, as in Eq. (2). Examples of this class of observables include $F(U) = \mathrm{Tr}[U]$ and $F(U) = \mathrm{Tr}[U\tau_1]^2 + \mathrm{Tr}[U]^2$. Note that function U is completely specified by three Euler angles, denoted collectively as μ so that the most general F is simply the most general function μ consistent with periodicity conditions. The Wigner D matrices provide a complete basis of functions, which satisfy these boundary conditions so that we can always decompose $F[U(\mu)]$ into the following form:

$$F[U(\mu)] = \sum_{j,k} f^j_{m,k} D^j_{m,k}(\mu), \tag{3}$$

where the coefficients $f^j_{m,k}$ are constants that depend only on the function F. Note that the mapping from physical space onto U can be recast as a mapping from real space onto μ: $F[U(\mathbf{x})] = F\{U[\mu(\mathbf{x})]\}$. We will show that for any smooth field configuration, there must be members of this class of observable that do not satisfy Eq. (2) unless all chiral singlet observables are spatially uniform.

To proceed, it is useful to recall that the Wigner D matrices satisfy an orthogonality condition:

$$\int d\mu\, D^{j'}_{m',k'} D^{*j}_{m,k} = \frac{8\pi^2}{2j+1}\delta_{m'm}\delta_{j'j}\delta_{k'k}. \tag{4}$$

Since, the D matrix for $j = 0$ is a constant independent of μ, it follows from the orthogonality relation that for $j \neq 0$,

$$\int d\mu\, D^j_{m,k} = 0. \tag{5}$$

Thus, it is clear that F is a chiral order parameter (i.e., an observable that is necessarily zero if chiral symmetry is unbroken) if, and only if, when written in the form of the decomposition of Eq. (3), the coefficient $f^0_{0,0} = 0$.

Since the Wigner D matrices form a basis, Eq. (2) can only be satisfied for *all* choices of F if it is satisfied for each Wigner D matrix. A simple way to implement this is to simply choose $F = D^j_{m,k}$ $(j \neq 0)$. Equation (4) then implies that if chiral restoration in the average sense occurs, then

$$\int F\{U[\mu(\mathbf{x})]\}\, dV = \int D^j_{m,k} J(\mu) d\mu = 0, \tag{6}$$

where $J(\mu)$ is the determinant of the Jacobian for a transformation from the physical space onto the internal space, and the right-hand side of Eq. (4) is zero by orthogonality provided that $j \neq 0$. Note that the mapping of $\mathbf{x}$ onto μ need not be one-to-one, but one can always cast the integral into the preceding form by combining the Jacobian determinants for any regions in real space that are mapped into the same region of internal space.

The next step is simply to note that since the Jacobian determinant is a scalar function of μ, it too can be decomposed into a basis of Wigner D matrices

$$J(\mu) = \sum_{j',m',k'} c^{j'}_{m',k'} D^{*j'}_{m',k'}(\mu). \tag{7}$$

Here, we have used complex conjugates of the Wigner matrices to exploit the orthogonality condition of Eq. (4). Note that the coefficients $c^{j'}_{m',k'}$ characterize the field configuration and are independent of the observable F. Inserting this decomposition into Eq. (6) for $F = D^j_{m,k}$ and using orthogonality implies that chiral restoration in the average sense occurs only if $c^j_{m,k} = 0$ for all $j \neq 0$.

Note that the choice of F is arbitrary, and thus we can repeat the analysis for all j, m, and k and deduce that if chiral symmetry is restored, then *all* $c^j_{m,k}=0$, except when $j=0$ (which corresponds to a Jacobian independent of μ). Hence the determinant of the Jacobian is a constant. This in turn implies that chiral restoration in the average sense means that spatially averaging must be equivalent to a uniform averaging over the internal space, precisely as one would expect intuitively.

To proceed further, let us consider a broader class of order parameter: $G\{U[\mu(\mathbf{x})]\}S(\mathbf{x})$, where G is a chiral order parameter of the class considered previously, namely a function of U only and S is a chiral scalar of U (but may be nonlocal, e.g., $S(\mathbf{x})=U^\dagger(\mathbf{x})U(\mathbf{x}+\mathbf{d})$, with $\mathbf{d}$ a vector-valued fixed point, or may potentially also depend on derivatives or integrals, e.g., $\int d^3y\,\mathrm{Tr}[U^\dagger(\mathbf{x})U(\mathbf{x}+\mathbf{y})]\exp(-\frac{y^2}{L^2})$. It is easy to see that a necessary condition for chiral restoration in the average sense is

$$\frac{1}{V_{R^3}}\int_{R^3}G[\mu(\mathbf{x})]S(\mathbf{x})\,dV=\left[\frac{1}{2\pi^2}\int_{SU(2)}G(\mu)\,d\mu\right]\times\left[\frac{1}{V_{R^3}}\int_{R^3}S(\mathbf{x})\,dV\right]. \tag{8}$$

This follows since if the system is chirally restored in the average sense, chiral rotations do not affect the integral on the left-hand side. Thus, we can chirally average without affecting the result. On the other hand, $S(\mathbf{x})$ is unaffected by chirally averaging. Moreover, when averaging over chiral space for G, it does not matter where in chiral space one starts, one gets the same result; thus, the chiral average is independent of the value of $U(\mathbf{x})$ and hence of $\mathbf{x}$. One can therefore remove the chirally averaged G from the spatial average yielding Eq. (8).

Since G is an order parameter, the first factor on the right-hand side is necessarily zero and the integral on the left-hand side must vanish. Thus

$$\frac{1}{V_{R^3}}\int_{R^3}G\{U[\mu(\mathbf{x})]\}S(\mathbf{x})\,dV=\int G(\mu)S[\mathbf{x}(\mu)]\,J(\mu)d\mu=0, \tag{9}$$

where the second form follows from change variables of integration to μ; as before, we can always cast the integral into this form even if the mapping is not one-to-one with one region in μ corresponding to more than one region in $\mathbf{x}$ by summing over the contributions of the Jacobian time S for the different regions in $\mathbf{x}$. At this stage, we can simply repeat the argument used previously for F to show that chiral restoration in the average sense requires that $S[\mathbf{x}(\mu)]J(\mu)$ must be a constant independent of μ. On the other hand, we have already shown that J is a constant. Thus, we conclude that S also must be a constant.

Note that this is a very strong constraint. Since chiral restoration means that *all* chiral order parameters must vanish, and since the previous argument holds for any chiral-scalar operator, we conclude that in order for chiral restoration in the average sense to occur, *all chiral scalars* must be uniform in space. This completes the first part of the proof.

B. Spatial uniformity for all chiral singlets is incompatible with chiral restoration

The next step is to show that in Skyrme-type models it is not possible for all chiral singlets to be spatially uniform while simultaneously having chiral symmetry restored (in the average sense). We do so via an indirect proof: We will assume that the system is in a chirally restored phase and then show that if all chiral singlets are spatially uniform, there is a mathematical inconsistency.

A key point about chirally restored phases that was stated in the previous section is that spatially averaging is equivalent to averaging uniformly over the internal space. This in turn means that every value of U occurs at some point in space (and indeed in a crystal at least once per unit cell). Thus, there must exist four points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ with the property that

$$\begin{aligned}U(\mathbf{x}_0)&=I\\U(\mathbf{x}_1)&=i\tau_1\\U(\mathbf{x}_2)&=i\tau_2\\U(\mathbf{x}_3)&=i\tau_3,\end{aligned} \tag{10}$$

where I is the two-dimensional identity matrix.

To proceed further, we introduce three matrix-valued functions:

$$\begin{aligned}\theta_1(\mathbf{x})&=U(\mathbf{x})U^\dagger(\mathbf{x}+\mathbf{x}_1-\mathbf{x}_0),\\\theta_2(\mathbf{x})&=U(\mathbf{x})U^\dagger(\mathbf{x}+\mathbf{x}_2-\mathbf{x}_0),\\\theta_3(\mathbf{x})&=U(\mathbf{x})U^\dagger(\mathbf{x}+\mathbf{x}_3-\mathbf{x}_0).\end{aligned} \tag{11}$$

Next, observe that under chiral transformations $\theta_j(\mathbf{x})$ transforms according to $\theta_j(\mathbf{x})\to L\theta_j(\mathbf{x})L^\dagger$, where L is the SU(2) matrix generating left-chiral rotations. Note that by construction,

$$\theta_j(\mathbf{x}_0)=-i\tau_j. \tag{12}$$

The analysis in this section is based on the chiral properties of quantities constructed from θ_j.

It is obvious from the transformation properties that $\mathrm{Tr}[\theta_j(\mathbf{x})]$ is a chiral invariant. Since we are in a chirally restored phase in the average sense (by hypothesis), this requires $\mathrm{Tr}[\theta_j(\mathbf{x})]$ to be translationally invariant. However, by construction $\mathrm{Tr}[\theta_j(\mathbf{x}_0)]=0$, from which one sees that $\mathrm{Tr}[\theta_j(\mathbf{x})]=0$ for all $\mathbf{x}$. Since the $\theta_j(\mathbf{x})$ are generically traceless SU(2) matrices, they can be written in the following form:

$$\theta_j(\mathbf{x})=-i\sum_a n^a_j(\mathbf{x})\tau_a\quad\text{where}\quad n^a_j(\mathbf{x})=i\frac{\mathrm{Tr}[\theta_j(\mathbf{x})\tau^a]}{2}. \tag{13}$$

For our purposes, it is useful to consider the n^a_j as being components of three distinct unit isovectors each labeled by j. Thus $\hat{n}_j(\mathbf{x})\equiv[n^1_j(\mathbf{x}),n^2_j(\mathbf{x}),n^3_j(\mathbf{x})]$ so that $\theta_j(\mathbf{x})=-i\hat{n}_j(\mathbf{x})\cdot\vec{\tau}$, where arrows will indicate vectors in isospace and hats unit vectors in isospace. The fact that $\hat{n}_j(\mathbf{x})$ has unit norm follows from the fact that $\theta_j(\mathbf{x})\in\mathrm{SU}(2)$.

A very useful set of chirally invariant quantities can be constructed from the $\hat{n}_j(\mathbf{x})$:

$$W_{ij}^{\mathbf{d}}(\mathbf{x}) \equiv \hat{n}_i(\mathbf{x}) \cdot \hat{n}_j(\mathbf{x}+\mathbf{d}) = -\frac{\mathrm{Tr}[\theta_i(\mathbf{x})\,\theta_j(\mathbf{x}+\mathbf{d})]}{2}, \tag{14}$$

where $\mathbf{d}$ is a vector-valued parameter. It is quite straightforward to show that for any values of i, j, and $\mathbf{d}$, $W_{ij}^{\mathbf{d}}$ is, in fact, a chiral singlet. Note that given the result demonstrated in the previous section, *if* the system is in a chirally restored phase, then $W_{ij}^{\mathbf{d}}(\mathbf{x})$ must be independent of $\mathbf{x}$ for fixed i, j, and $\mathbf{d}$.

Clearly, by choosing different values of $\mathbf{d}$, one can probe spatial correlations among the L_i. One important case is where $\mathbf{d} = \mathbf{0}$. For notational convenience, we will introduce a special symbol for this:

$$Q_{ij}(\mathbf{x}) \equiv W_{ij}^{\mathbf{0}}(\mathbf{x}) = \frac{\mathrm{Tr}[\theta_i(\mathbf{x})\,\theta_j(\mathbf{x}+\mathbf{d})]}{2}. \tag{15}$$

It is trivial to see from Eq. (15) and Eq. (12) that

$$Q_{ij}(\mathbf{x}_0) = \delta_{ij}. \tag{16}$$

By hypothesis, the system is in a chirally restored phase, and thus $Q_{ij}(\mathbf{x})$ is a chiral singlet independent of $\mathbf{x}$. Since Q_{ij} is simply the inner product of a set of three unit isovectors, the most general way for it to be independent of position is if the set of $\hat{n}$ at one point are related to the set at another by the most general inner product conserving transformation, an element of SO(3). Thus we conclude that

$$\hat{n}_j(\mathbf{x}) = \overleftrightarrow{R}(\mathbf{x})\hat{n}_j(\mathbf{x}_0) \quad \text{with } \overleftrightarrow{R}(\mathbf{x}_0) \in \mathrm{SO}(3), \tag{17}$$

where $\overleftrightarrow{R}(\mathbf{x})$ is the same for all j. Note that assuming U describes a configuration, which is chirally restored in the average sense, $\overleftrightarrow{R}(\mathbf{x})$ is completely determined by the chiral field $U(\mathbf{x})$. Recall that $\theta_j(\mathbf{x})$ transforms according to $\theta_j(\mathbf{x}) \to L\theta_j(\mathbf{x})L^\dagger$ and are thus in the irreducible $(\frac{1}{2},\frac{1}{2})$ chiral representation. As such, they are chiral order parameters, and by definition their spatial averages must vanish in a chirally restored regime. This in turn means the spatial average of the $\hat{n}_j$ must be zero. Using Eq. (2), this means that in a chirally restored phase,

$$\frac{1}{V_{R^3}}\int_{R^3} dV\, \overleftrightarrow{R}(\mathbf{x})\hat{n}_j(\mathbf{x}_0) = 0, \tag{18}$$

for all j. However, since the $\hat{n}_j$ form a complete and orthonormal basis, this is possible only if the integral of the matrix itself vanishes:

$$\frac{1}{V_{R^3}}\int_{R^3} dV\, \overleftrightarrow{R}(\mathbf{x}) = 0. \tag{19}$$

The hypothesis that the system is in a chirally restored phase (in the spatially averaged sense) requires that all chiral singlet observables are independent of position. Since $W_{ij}^{\mathbf{d}}(\mathbf{x})$ is chiral singlet, it must be independent of $\mathbf{x}$. Using Eq. (17), and the definition of W from Eq. (14), we see that

$$W_{ij}^{\mathbf{d}}(\mathbf{x}) = \hat{n}_i^T(\mathbf{x}_0)\, \overleftrightarrow{R}^T(\mathbf{x})\, \overleftrightarrow{R}(\mathbf{x}+\mathbf{d})\hat{n}_j(\mathbf{x}_0). \tag{20}$$

Note that $W_{ij}^{\mathbf{d}}(\mathbf{x})$ can be expressed as the matrix $\overleftrightarrow{R}^T(\mathbf{x})\,\overleftrightarrow{R}(\mathbf{x}+\mathbf{d})$ evaluated between the vectors $\hat{n}_j(\mathbf{x}_0)$. However, since these vectors form a complete basis and are independent of $\mathbf{x}$, the only way for $W_{ij}^{\mathbf{d}}(\mathbf{x})$ to be independent of $\mathbf{x}$ is if the matrix $\overleftrightarrow{R}^T(\mathbf{x})\,\overleftrightarrow{R}(\mathbf{x}+\mathbf{d})$ itself is also independent of $\mathbf{x}$:

$$\overleftrightarrow{R}^T(\mathbf{x})\,\overleftrightarrow{R}(\mathbf{x}+\mathbf{d}) = \overleftrightarrow{R}(\mathbf{d}+\mathbf{x}_0). \tag{21}$$

But since the transpose of a rotation matrix is its inverse,

$$\overleftrightarrow{R}(\mathbf{x}+\mathbf{d}) = \overleftrightarrow{R}(\mathbf{x})\,\overleftrightarrow{R}(\mathbf{d}+\mathbf{x}_0). \tag{22}$$

Note that $\mathbf{x}$ and $\mathbf{d}$ are arbitrary three-dimensional vectors in space. Thus, the form of Eq. (22) holds under replacements of $\mathbf{x}$ and $\mathbf{d}$ by other vectors. Consider in particular the replacements $\mathbf{x} \to \mathbf{d}+\mathbf{x}_0$ and $\mathbf{d} \to \mathbf{x}-\mathbf{x}_0$. Thus, $\overleftrightarrow{R}(\mathbf{x}+\mathbf{d}) = \overleftrightarrow{R}(\mathbf{d}+\mathbf{x}_0)\,\overleftrightarrow{R}(\mathbf{x})$. Equating the two ways of writing $\overleftrightarrow{R}(\mathbf{x}+\mathbf{d})$ and relabeling $\mathbf{d}+\mathbf{x}_0$ as $\mathbf{y}$ yields

$$[\overleftrightarrow{R}(\mathbf{y}), \overleftrightarrow{R}(\mathbf{x})] = 0. \tag{23}$$

Note that since $\mathbf{d}$ was arbitrary so is $\mathbf{y}$, and Eq. (23) holds for all values of $\mathbf{x}$ and $\mathbf{y}$. Equation (23) strongly constrains the nature of $\overleftrightarrow{R}(\mathbf{x})$: $\overleftrightarrow{R}$ at any points commutes with $\overleftrightarrow{R}$ at any other point while general three-dimensional rotations do not. Indeed, the only way that Eq. (23) can be satisfied for all points is if the rotations at every point in space are all about a common spatially independent rotation axis in isospace. Let us denote the unit vector in the direction of this axis $\hat{n}_{\mathrm{rot}}$. It is a general property of rotation matrices in three dimensions that the axis of rotation is an eigenvector with an eigenvalue of unity. Since $\hat{n}_{\mathrm{rot}}$ is independent of $\mathbf{x}$, it is also an eigenvector of the spatially averaged rotation matrix again with an eignenvalue of unity:

$$\frac{1}{V_{R^3}}\int_{R^3} dV\, \overleftrightarrow{R}(\mathbf{x})\hat{n}_{\mathrm{rot}} = \hat{n}_{\mathrm{rot}}. \tag{24}$$

However, Eq. (24) is clearly inconsistent with Eq. (19): the spatially averaged matrix cannot simultaneously be zero and have an eigenvalue of one. Note that both Eq. (24) and Eq. (19) are direct mathematical consequences of our assumption that the system is in a chirally restored phase (in the spatially averaged sense). Equation (19) followed quite directly from this, as by definition all chiral order parameters, including θ_j, must vanish upon spatially averaging in this phase. Equation (24) followed from the requirement proved in the previous subsection that all chiral singlets, including $W_{ij}^{\mathbf{d}}$, must be spatially uniform in such a phase. Since these two are inconsistent, we have shown that the assumption underlying them—that the system is in a phase that is chirally restored in a spatially averaged sense—cannot be correct. Moreover, since the analysis was done for an arbitrary field configuration for the entire class of Skyrme-type models, we have proved that chiral restoration in the average sense cannot occur in models of this class.

Ultimately, this result should not be surprising. The only natural way to conceive of a situation in which chiral order parameters to be nonzero locally but zero under spatial averaging is for spatial averaging to uniformly cover all directions in the internal chiral space. This will occur if the field configurations map from real space to chiral space in such a way that uniform coverage in one is mapped to uniform coverage in the other. However, the internal chiral space is curved while the physical space is flat and hence such a mapping is not possible.

III. A NO-GO THEOREM FOR LARGE N_c QCD

In this section, we show that the result derived in the previous section for the Skyrme model holds for large N_c QCD itself. The precise statement is that if large N_c QCD is in a phase in which the chiral condensate $\langle\overline{q}q\rangle$ is generally nonzero but varies from point to point, then it is not possible for all chiral order parameters to vanish under spatial averaging. The strategy for doing this exploits the fact that the proof in the previous section holds for any field configurations describable as a nonlinear sigma model. Thus, our theorem will be established in general provided we can show rigorously that in a putative chirally restored phase with a spatially varying but nonzero chiral condensate, the expectation values for a key set of operators are reducible to those of a nonlinear sigma model.

To do this, we focus on the scalar-isoscalar and pseudoscalar-isovector quark bilinears $\overline{q}q$ and $\overline{q}\,\overrightarrow{\tau}\gamma^5 q$. These operators transform into one another as members of a $(\frac{1}{2},\frac{1}{2})$ representation of the $\mathrm{SU_L}(2)\times\mathrm{SU_R}(2)$ chiral group. Suppose that the system is in some known quantum mechanical state. Let us combine the expectation values of these operators in this state into a single, two-dimensional, matrix-valued function:

$$V(\mathbf{x}) \equiv \frac{\langle\overline{q}(\mathbf{x})q(\mathbf{x})\rangle\; I \;+\; \langle\,\overline{q}(\mathbf{x})\,\overrightarrow{\tau}\gamma^5 q(\mathbf{x})\cdot\,\overrightarrow{\tau}\,\rangle}{\langle\overline{q}q\rangle_{\mathrm{vac}}}, \tag{25}$$

where I is the identity matrix and the normalization factor, $\langle\overline{q}q\rangle_{\mathrm{vac}}$ is included so that in the vacuum state V simply becomes the identity matrix. Note that under chiral transformations on the operators, making up V means it transforms in precisely the same way as U does in the Skyrme model. One can construct order parameters from V. Since we are at large N_c,

$$\langle F[V(\mathbf{x})]\rangle = F[\langle V(\mathbf{x})\rangle], \tag{26}$$

where the angle brackets indicate quantum expectation values in the state. Thus we can treat the issue of chiral restoration in the spatially averaged sense as if V were a classical function.

To proceed further, let us introduce a chiral scalar function given by the norm of V:

$$v(\mathbf{x}) = \sqrt{\frac{\mathrm{Tr}[V^{\dagger}(\mathbf{x})V(\mathbf{x})]}{2}}. \tag{27}$$

By construction, $v(\mathbf{x})$ is real and nonnegative. For all points in space where $v(\mathbf{x}) \neq 0$, one can define

$$U(\mathbf{x}) \equiv \frac{V(\mathbf{x})}{v(\mathbf{x})}. \tag{28}$$

Note that this construction ensures that $U(\mathbf{x}) \in \mathrm{SU}(2)$. Thus, provided that $v(\mathbf{x})$ is nonzero throughout space and the expectation values vary smoothly through space, we have an associated smoothly varying SU(2) matrix $U(\mathbf{x})$, precisely as in nonlinear sigma models such as the Skyrme model. Note that the argument in Sec. II A does not depend on the dynamics but merely on the transformational properties plus assumptions about smoothness. Thus we conclude that *if* the system is in a phase that is chirally restored in the average sense, while simultaneously having $v(\mathbf{x}) \neq 0$ everywhere, all chiral scalars must be spatially uniform. Note, moreover, that $v(\mathbf{x})$ is a chiral scalar itself and therefore must itself be a constant in any putative chirally restored phase with v everywhere nonzero.

Next, we need to eliminate the possibility in large N_c QCD of $v(\mathbf{x})$ vanishing at some points in space while being in a phase that has a spatially varying chiral condensate while being chirally restored in the spatially averaged sense. To proceed, let us consider what happens if V is distorted slightly away from its physical value:

$$\tilde{V}(\mathbf{x}) = V(\mathbf{x}) + \lambda\,\Delta V(\mathbf{x}), \tag{29}$$

where λ is a small parameter and ΔV is an arbitrary smooth function with the property that it is nonzero in the neighbor of any point where V vanishes. We will use a tilde to denote a quantity perturbed away from its physical value due to the use of $\tilde{V}$ in place of V. Note that the change in V due to this distortion is small and smooth. So the changes in spatially averaged chiral order parameters obtained from $\tilde{V}$ are expandable as a Taylor series in λ. Thus, *if* the system was in a putative phase that was chirally restored in the average sense, then spatially averaged chiral order parameters computed with $\tilde{V}$ would be of order λ; $\tilde{V}$ would describe a nearly chirally restored regime.

To proceed, we exploit the fact that $\tilde{V}$ is constructed to be nonzero everywhere, implying that $\tilde{U}$ is well-defined everywhere. We will describe any putative nearly chirally restored regime associated with $\tilde{V}$ using $\tilde{U}$. Since the supposed regime is *nearly* chirally restored, the argument in Sec. II A goes through up to corrections associated with the fact that it is only nearly restored. Thus, we see that in such a regime all chiral singlets must be nearly constant in space: $\tilde{s}(\mathbf{x}) = \tilde{s}_0 + \lambda\,\delta\tilde{s}(\mathbf{x})$, where $\tilde{s}_0$ is a constant and $\lambda\,\delta\tilde{s}(\mathbf{x})$ describes the fluctuations away from it. Since $\tilde{v}(\mathbf{x})$ is a chiral scalar, we conclude that $\tilde{v}(\mathbf{x}) = \tilde{v}_0 + \lambda\,\delta\tilde{v}(\mathbf{x})$. On the other hand, by construction at any point in space, $\tilde{v}(\mathbf{x})$ can be expressed as a Taylor expansion in λ: $\tilde{v}(\mathbf{x}) = v(\mathbf{x}) + \lambda\,\Delta v(\mathbf{x})$. Equating these two forms yields

$$v(\mathbf{x}) = \tilde{v}_0 + \lambda\,[\delta\tilde{v}(\mathbf{x}) - \Delta v(\mathbf{x})]. \tag{30}$$

This implies that as $\lambda \to 0$, $v(\mathbf{x})$ approaches a constant value. However, a constant value of $v(\mathbf{x})$ is inconsistent with v going to zero at some points while generally being nonzero as a result of the nonzero chiral condensate. Thus, *if* a phase that is chirally restored in the spatially averaged sense exists and has a spatially varying chiral condensate, it must have a constant nonzero $v(\mathbf{x})$. Therefore, it is effectively reduced to a nonlinear sigma model and the proof in the preceding section applies. This completes the demonstration.

IV. SKYRMIONS ON A HYPERSPHERE

We have shown that chiral symmetry restoration in the spatially averaged sense is not possible for either the Skyrme model or for large N_c QCD. However, two important issues arise regarding the Skyrme model. As was noted in the Introduction, observed chiral restoration in the spatially averaged sense *is observed* for the Skyrme model on the hypersphere [8,16–18,21].

The first issue is quite straightforward. One might worry that since chiral symmetry does get restored in the average sense for Skyrmions on the hypersphere, this might indicate there is a flaw in the proof that Skyrmions cannot have chiral restoration at finite density. Of course, it need not indicate a flaw, since the geometry for which the no-go theorem was derived—extended nuclear matter in flat space—is different from the curved space of the hypersphere. Nevertheless, it is important to pin down how the proof can be valid for three-dimensional flat space but fail for the hypersphere.

The second issue concerns intuition. As noted in the Introduction, the approximation of a Skyrmion crystal by a single Skyrmion on a hypersphere is *ad hoc*. The notion, though, was that the main qualitative effect of putting a Skyrmion into a crystal is to restrict the space over which it can spread. This effect is certainly also present with a single Skyrmion on a hypersphere. To the extent that qualitative questions such as whether or not chiral restoration (in a spatially averaged sense) occurs is a generic feature that depends on whether a Skyrmion is confined to a sufficiently small volume, one might think that a Skyrmion on a hypersphere would be a good way to discover this. However, this is clearly wrong. Chiral restoration in the average sense occurs on the hypersphere but not for crystals in flat space. It is important to understand why the intuition gained from the hypersphere fails.

Before addressing these issues, let us briefly review the geometry of the hypersphere. This can be characterized as a three-dimensional curved surface in a four-dimensional Euclidean space with fixed radius:

$$x^2 + y^2 + z^2 + w^2 = L^2, \tag{31}$$

where L is the radius of the hypersphere. It is useful to parametrize this in terms of three angular variables, $0 \leqslant \mu, \theta \leqslant \pi$, and $0 \leqslant \phi \leqslant 2\pi$, with

$$\begin{aligned} x &= L \sin\mu \sin\theta \cos\phi, \\ y &= L \sin\mu \sin\theta \sin\phi, \\ z &= L \sin\mu \cos\theta, \\ w &= L \cos\mu. \end{aligned} \tag{32}$$

We denote the metric tensor for this geometry as $g^{\rm hs}$. With parametrization used here, it is diagonal, and the diagonal matrix elements are:

$$\begin{aligned} g_{\mu\mu} &= L^2, \\ g_{\theta\theta} &= L^2 \sin^2\mu, \\ g_{\phi\phi} &= L^2 \sin^2\mu \sin^2\theta. \end{aligned} \tag{33}$$

The volume element is given by

$$dV = \sqrt{\det g}\, d\mu\, d\theta\, d\phi = L^3 \sin^2\mu\, d\mu \sin\theta\, d\theta\, d\phi \tag{34}$$

Consider the configuration

$$U_0(\mu,\theta,\phi) = \begin{bmatrix} \cos(\mu) + \sin(\mu)\cos(\theta) & \sin(\mu)\sin(\theta)\exp(-i\phi) \\ \sin(\mu)\sin(\theta)\exp(i\phi) & \cos(\mu) - \sin(\mu)\cos(\theta) \end{bmatrix}. \tag{35}$$

It is straightforward to verify that this configuration has winding number unity and that any chiral order parameter constructed from U_0 vanishes when integrated uniformly over the hypersphere. Thus the configuration U_0 corresponds to chiral restoration in the spatially averaged sense on the hypersphere. The explicit example of U_0 shows that the hyperspherical geometry does allow configurations with chiral restoration in the spatially averaged sense. The theorem proved in Sec. II showed such a configuration—at least for the case of flat space. This means either the proof of the theorem is wrong or some aspect of the proof holds for flat space but fails for the case of the hypersphere.

Fortunately, it is easy to see how the proof of the no-go theorem can hold for flat space and not for the hypersphere. At several critical points in the derivation, the fact that the space is flat plays a critical role. Consider as an example the relation $\overleftrightarrow{R}(\mathbf{x}+\mathbf{d}) = \overleftrightarrow{R}(\mathbf{x})\,\overleftrightarrow{R}(\mathbf{d}+\mathbf{x}_0)$ from Eq. (22); this relation is central in the analysis leading to Eq. (23), which is at the core of the theorem. However, as written, this result is not meaningful on the hypersphere: there is no notion of linearly adding the vectors associated with two points in the space to obtain the vector associated with a third point in the space. Thus, the notion of the point $\mathbf{x}+\mathbf{d}$ is simply ill-posed on the hypersphere and the proof developed in flat space does not go through.

Let us now turn to the issue of the failure of the intuition that a single Skyrmion in the hypersphere should be qualitatively similar to the Skyrme crystal as the principal effects of both should be to confine a Skyrmion to a limited spatial region. The intuition can break down for one of two reasons: either limiting the space in which a Skyrmion extends by a single Skyrmion in a compact geometry is *not* qualitatively similar to limiting it by placing it in a crystal in flat space, or because the hypersphere is an atypical compact geometry. It is instructive to understand which is the cause. There are good reasons to suspect that it is due to the atypical properties of the hypersphere. The hypersphere has a much higher symmetry than typical geometries one can consider. It is plausible that these symmetries rather than generic properties are responsible for the chiral restoration (in the spatially averaged sense) seen for Skyrmions on the hypersphere. This becomes particularly plausible when one considers the internal space associated with the field U. It can be parameterized as

$$U = sI + i\vec{p}\cdot\vec{\tau}, \tag{36}$$

where s and $\vec{p}$ can be extracted from U using the relations

$$s = \frac{\mathrm{Tr}\,(U)}{2}\ ,\ p_j = -\frac{-i\mathrm{Tr}(\tau_j U)}{2}. \tag{37}$$

Note that by construction, s and $\vec{p}$ satisfy the constraint

$$s^2 + p_1^2 + p_2^2 + p_3^2 = 1. \tag{38}$$

This constraint in the internal geometry in Eq. (38) is of the same form as the constraint in Eq. (31) describing the hypersphere. Indeed, it is easy to see that the configuration in Eq. (35), which restores chiral symmetry on the hypersphere (in the spatially averaged sense), has the property that it maps points on the hypersphere to the analogous points in the SU(2) matrix U:

$$s = \frac{w}{L}, \quad p_1 = \frac{x}{L}, \quad p_2 = \frac{y}{L}, \quad p_3 = \frac{z}{L}. \tag{39}$$

Thus, averaging over the hypersphere uniformly automatically leads to averaging uniformly over the internal space, forcing all chiral order parameters constructed from U to vanish.

One way to test whether the observed chiral restoration seen for a Skyrmion on a sufficiently small hypersphere is due to the special geometrical properties of the hypersphere is to ask what happens to spatially averaged chiral order parameters in this regime if the geometry is distorted slightly away from a hypersphere. If chiral symmetry is generically restored for single Skyrmions in small compact spaces, these spatially averaged order parameters should remain zero. If, instead, chiral restoration is a result of special features associated with the hyperspherical geometry, one would expect these to become nonzero.

To make this analysis concrete, we need to impose dynamics associated with a particular variant of the Skyrme model. Here, we will use the simplest one, which contains field U and up to four-derivative terms:

$$\mathcal{L} = \frac{f_\pi^2}{4} Tr(\partial_\mu U^\dagger \partial^\mu U) + \frac{\epsilon^2}{4} Tr(U^\dagger \partial_\mu U,\, U^\dagger \partial_\nu U)^2, \tag{40}$$

where $f_\pi \approx 93$ MeV is the pion decay constant, and ϵ is a dimensionless parameter necessary to stabilize the soliton. In the remainder of this paper, we will use dimensionless units by setting f_π and ϵ to unity. One can use dimensional analysis to reinsert factors of f_π and ϵ at the end of the problem, if desired. To do so, simply multiply lengths by $2\sqrt{2}\epsilon/f_\pi$ and energies by $\sqrt{2}\epsilon f_\pi$ [16]. For the purposes of determining whether or not chiral symmetry is restored, these rescaling factors are irrelevant.

For the case of an unperturbed hyperspherical geometry, the minimum energy winding number unity configuration is a hedgehog localized in one region of the hypersphere, providing the radius L is smaller than the critical value of $\sqrt{2}$ [17]. For $L < \sqrt{2}$, the minimum energy configuration is given by U_0 in Eq. (35) or configurations obtained from it by a global SU(2) rotation in internal space. Our strategy is to start with the lowest energy configuration for a value of L well into the restored phase (for concreteness we take $L = 1$). We then consider a small perturbation on the geometry, compute the shift in the configuration due to this to lowest order in the perturbation, and use this perturbed configuration to compute the spatially averaged chiral order parameters at first order in the shift. If chiral restoration in the spatially averaged sense were a generic feature seen when a Skyrmion is confined by a compact geometry of sufficiently small size, one expects spatially averaged order parameters to remain zero.

The most general perturbation of the hypersphere can be formulated as a shift in the contravariant metric perturbation away from that of the hypersphere parametrized in terms of an overall expansion parameter λ:

$$g^{ij}_{\rm pert} = g^{ij}_{\rm hs} + \frac{\lambda}{L^2}\gamma^{ij}, \tag{41}$$

where γ^{ij} is dimensionless. We will consider a very simple class of perturbations to the geometry: those in which all matrix elements of γ vanish except the ($i = 1$, $j = 1$) component. Further, we will take γ^{11} to be a sinusoidal function of μ only, which ensures that the perturbation does not induce discontinuities in derivatives:

$$\gamma^{ij} = \delta^{i1}\delta^{j1}\sin(p\mu), \tag{42}$$

where p is a nonzero integer. Of course, this is a very restricted class of perturbation. If spatially averaged chiral order parameters remain zero with it, it would tell us very little since this might be a result of the highly constrained and symmetric form of the metric perturbation. However, if they do not remain zero, then it is sufficient to show that the hyperspherical geometry does not behave generically.

We now solve the Skyrme model for the lowest energy unit winding number solution with this geometry working to first order in λ. To do so we consider configurations of the most symmetric sort consistent with a winding number of unity. These are hedgehog configurations of the form

$$U(\mu,\theta,\phi) = \exp[i\vec{\tau}\cdot\hat{r}\, f(\mu)], \quad \text{where } \hat{r} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}, \quad \text{with } f(0) = 0 \text{ and } f(\pi) = \pi. \tag{43}$$

We note that for the unperturbed geometry, U_0, the configuration in Eq. (35) associated with chiral restoration is of this form with $f(\mu) = \mu$. For the perturbed case, we exploit the fact that the most symmetric form remains consistent with the topological constraints. It is guaranteed that the configuration of this form, which minimizes the energy, will also be a solution of the full Euler-Lagrange equations. As a general rule, a solution of the Euler-Lagrange equations obtained from a highly symmetric ansatz need not correspond to a global minimum of the energy. However, in this case it should, as in the unperturbed case where the hedgehog is known to be a global minimum for that problem, and this problem differs from that only to first order in λ. Next, we take the ansatz in Eq. (43), using

$$f(\mu) = \mu + \lambda\delta f(\mu), \quad \text{with } \delta f(0) = \delta f(\pi) = 0. \tag{44}$$

Using this form, we compute the energy to second order in λ (noting that first-order perturbations in the field correspond to second-order perturbations in the energy) and vary this form to get differential equations for δf, which are then solved numerically. Using these numerical solutions, the spatial averages of various chiral order parameters are computed.

A useful set of chiral order parameters to consider are local functions constructed from U or, equivalently, from s and $\vec{p}$ from Eq. (37). We will focus here on three representative

TABLE I. Spatially averaged chiral parameters at order λ. These are computed using geometries of the perturbed hypersphere given in Eq. (41). The configurations used to compute these are minimum-energy configurations of the hedgehog form based on the Lagrangian in Eq. (40).

Metric perturbation	s	$\frac{3s^2}{4} - \frac{p_1^2+p_2^2+p_3^2}{4}$	s^3
$\gamma^{ij} = \delta^{i1}\,\delta^{j1}\sin(\mu)$	0	$-0.0954931\,\lambda$	0
$\gamma^{ij} = \delta^{i1}\,\delta^{j1}\sin(2\mu)$	0	0	$-0.0314619\,\lambda$
$\gamma^{ij} = \delta^{i1}\,\delta^{j1}\sin(3\mu)$	0	$-0.0197049\,\lambda$	0
$\gamma^{ij} = \delta^{i1}\,\delta^{j1}\sin(4\mu)$	0	0	$-0.00699138\,\lambda$
$\gamma^{ij} = \delta^{i1}\,\delta^{j1}\sin(5\mu)$	0	$0.0156628\,\lambda$	0

samples from this list:

$$s, \quad \frac{3s^2}{4} - \frac{p_1^2+p_2^2+p_3^2}{4}, \quad \text{and } s^3. \tag{45}$$

It is clear that s and s^3 are chiral order parameters since $s \to -s$ is a chiral transformation. Similarly, it should be clear that $\frac{3s^2}{4} - \frac{p_1^2+p_2^2+p_3^2}{4}$ is a chiral order parameter, since in a chirally restored phase one necessarily has $\langle s^2\rangle = \langle p_1^2\rangle = \langle p_2^2\rangle = \langle p_3^2\rangle$ and thus the quantity vanishes. In Table I, we show the spatial averages of these order parameters computed to leading order in λ.

It is noteworthy that not all entries in this table are zero. Thus making small perturbations away from the hyperspherical geometry yields nonzero order parameters. From this we see that chiral restoration seen for Skyrmions on the hypersphere is not generic but rather is a consequence of the special geometric properties of the hypersphere. Of course, this geometry has no physical significance and was used for ease of computation. While it was hoped that a calculation in this simple situation would give useful physical intuition about the more complicated situation of Skyrmion crystals, it appears that the opposite is true with regard to chiral symmetry. Indeed, the special properties of the geometry that makes analytic calculations of the small-radius (high-density) phase simple also make the intuition totally unreliable, even for qualitative issues associated with chiral symmetry breaking and its possible restoration in the average sense.

V. DISCUSSION

The question of whether dense, cold nuclear matter is chirally restored above some critical baryon chemical potential remains a problem of central importance to nuclear physics. It remains unanswered. The answer is also potentially unknown in the simpler case of large N_c QCD.

This paper dealt with a complication at large N_c: the fact that nuclear matter is likely to crystallize, and thus the breaking of chiral symmetry and of translational symmetry may get entangled. One could imagine a situation in which the breaking of translational symmetry can introduce nonzero but spatially averaged chiral order parameters that integrate to zero over space. To deal with this complication, it was suggested here that the natural focus should be on spatially averaged chiral order parameters that by construction are insensitive to details of how translational symmetry is broken. We demonstrated here that while it is possible for some spatially averaged chiral order parameters to vanish, it is not possible for all of them to if the chiral condensate is generally nonzero but spatially varying. Thus, chiral symmetry restoration in a spatially averaged sense is not possible at large N_c unless the chiral condensate is zero everywhere.

This result is of some significance in connection to the question of whether chiral restoration occurs at high baryon density in large N_c QCD. It has been argued on the basis of Skyrme models that chiral restoration does occur at sufficiently high density. The fact that at high densities the lowest energy configurations in Skyrme crystals have a vanishing spatially averaged chiral condensate has been taken as support for chiral restoration [3]. As shown here, however, there must be other chiral order parameters that are nonzero upon spatial averaging. It has similarly been argued that the vanishing of chiral order parameters upon spatial averaging for a Skyrmion on a sufficiently small hypersphere has also been taken as evidence for chiral restoration at high density [3]. However, as shown here, this was an artifact of the hyperspherical geometry and is thus unconnected to the question of what happens in crystals in flat space.

ACKNOWLEDGMENTS

T.D.C. and P.A. acknowledge the support of the US Department of Energy through Grant No. DEFG02-93ER-40762. M.S. is grateful for the support from the University of Maryland Office of Undergraduate Studies, and R.A. acknowledges the support of the Theoretical Quarks, Hadrons and Nuclei (TQHN) group at the University of Maryland.

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