Duality symmetry of the Balitsky-Fadin-Kuraev-Lipatov equation: Reggeized gluons versus color dipoles

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It is shown that the duality symmetry of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation can be interpreted as a symmetry under rotation of the BFKL kernel in the transverse space from s channel (color-dipole model) to t channel (Reggeized gluon formulation). It is argued that the duality symmetry also holds in the nonforward case due to a very special structure of the nonforward BFKL kernel, which can be written as a sum of three forward BFKL kernels. The duality symmetry is established by identifying the dual coordinates with the transverse coordinates of a nondiagonal dipole scattered off the target.

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I. INTRODUCTION

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [1] describes the amplitude of scattering at very high center-of-mass energy $\sqrt{s}$ with $|t/s| \ll 1$, where t is the square of the transferred momentum. The leading-order BFKL is obtained by summing terms $(\alpha_s \ln\ s)^n$, where each power of the coupling constant α_s is accompanied by the corresponding power of the logarithm of energy. This kinematic regime is called multi-Regge kinematics. In the multi-Regge kinematics, the transverse degrees of freedom fully decouple from the longitudinal ones. This allows the formulation of the BFKL equation as evolution in complex time (rapidity) with the integral kernel operating in the transverse space. The BFKL equation was originally formulated using the fact that t-channel gluons Reggeize and the production vertices of the s-channel gluons factorize in the Regge kinematics. In this picture, the BFKL equation describes a compound state of two Reggeized gluons. An alternative derivation of the BFKL evolution was proposed by Mueller [2] using s-channel unitarity for the evolution of colorless dipoles in the limit of the large number of colors. The BFKL equation was solved [3] using the conformal invariance of the BFKL kernel. It was also noticed that the BFKL kernel has another interesting property called the duality symmetry, found by Lipatov [4]. This symmetry means that the form of the BFKL equation does not change if the gluon momentum k is replaced by its conjugate coordinate. It was shown that this symmetry can explain the integrability of the BFKL equation. However, it was also suggested that the duality symmetry should hold only for the case of zero-momentum transfer for a system of two Reggeized gluons.

The objective of the present study is to show that the duality symmetry of the BFKL equation also holds in the nonforward case, though not in an explicit way. The analysis started by the present author [5] is continued and the duality symmetry is established as a symmetry between Reggeized gluon formulation and the dipole picture of the BFKL evolution. In particular, it is shown that the evolution equation for a dipole with different sizes to the left and to the right of the unitarity cut can be written in the form of the BFKL equation in the dual coordinates. The dual momenta coordinates and the conjugate coordinates are not *a priori* related objects. The fact that they can be identified is to be understood as a sign for the duality symmetry. However, there seems to be no obvious choice of the Fourier transform (at least of a single variable) that can take one picture to another. This is the reason why this symmetry is referred to as the hidden duality symmetry. The hidden duality symmetry can also be interpreted as a symmetry under rotation of the BFKL kernel in the transverse space from the t channel (Reggeized gluons) to the s channel (color dipoles) and back.

II. DUALITY SYMMETRY OF THE BFKL EQUATION

This section explains briefly how the duality symmetry appears in the leading-order BFKL equation and shows why it is related to the integrability. The duality symmetry of the system of interacting Reggeons in the limit of a large number of colors was formulated by Lipatov [4]. The following briefly outlines the major relevant points of this study.

The outline starts with a general description of the BFKL approach and presents its formulation in terms of the holomorphic Hamiltonian in the Schrödinger-like equation. The BFKL equation describes the behavior of the scattering amplitude in the limit of the center-of-mass energy $\sqrt{s}$ being much larger than the typical transferred momentum $|t/s| \ll 1$ (the Regge kinematics). The leading-order BFKL evolution equation is obtained by summing the powers of the parameter $\alpha_s \ln s$, where each power of the strong coupling constant α_s is accompanied by the corresponding power of the logarithm of energy. In this picture, the BFKL Pomeron appears as a compound state of two Reggeized gluons of transverse momenta $\vec{k}$ and $\vec{k}-\vec{q}$ as illustrated in Fig. 1. The color singlet BFKL in the limit of large number of color N_c reads

$$\left(\frac{\partial}{\partial y} - \epsilon(-\vec{k}^2) - \epsilon[-(\vec{k}-\vec{q})^2]\right)\mathcal{F}(\vec{k},\vec{k}-\vec{q}) = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\chi}\,\frac{K(\vec{k},\vec{\chi})}{\vec{\chi}^2(\vec{\chi}-\vec{q})^2}\mathcal{F}(\vec{\chi},\vec{\chi}-\vec{q}), \qquad (1)$$

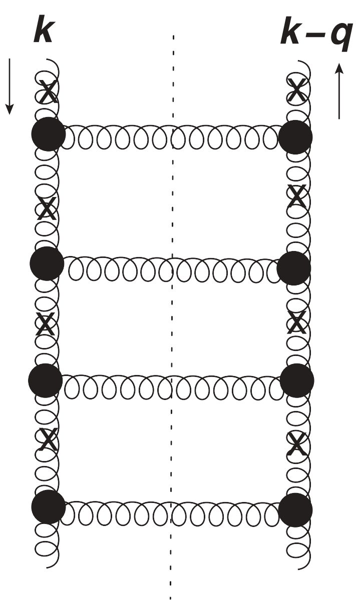

FIG. 1. The BFKL evolution equation describes high-energy scattering as a compound state of two t-channel Reggeized gluons, with real s-channel gluon emissions crossing the dashed line of the unitarity cut. The effective real production vertices are denoted by the dark blobs, and the fact that t-channel gluons are Reggeized is reflected by crosses.

where the gluon Reggeization enters the equation through the Regge gluon trajectory

$$\epsilon(-\vec{k}^2) = \frac{\alpha_s N_c}{4\pi^2} \int d^2\vec{\chi}\, \frac{-\vec{k}^2}{\vec{\chi}^2(\vec{\chi}-\vec{k})^2}. \tag{2}$$

The real emission part of the kernel is given by

$$K(\vec{k},\vec{\chi}) = \vec{q}^{\,2} - \frac{\vec{k}^2(\vec{\chi}-\vec{q})^2}{(\vec{\chi}-\vec{k})^2} - \frac{\vec{\chi}^2(\vec{k}-\vec{q})^2}{(\vec{\chi}-\vec{k})^2}. \tag{3}$$

In the leading-order BFKL, the transverse momenta components decouple from the longitudinal ones (rapidity). Due to this factorization, the BFKL Pomeron can be written as a state in the two-dimensional transverse space that evolves with rapidity, which plays the role of an imaginary time. This fact makes it possible to formulate the color singlet BFKL dynamics in the form of the Schrödinger equation for the wave function $f_{m,\tilde{m}}(\vec{\rho}_1,\vec{\rho}_2,\ldots,\vec{\rho}_n;\vec{\rho}_0)$ for a system of n Reggeized gluons [6–8]. The BFKL equation is obtained for $n=2$. The vectors $\vec{\rho}_k$ are two-dimensional coordinates of the Reggeized gluons and m and $\tilde{m}$ are the conformal weights

$$m = \frac{1}{2} + i\nu + \frac{n}{2}, \qquad \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}, \tag{4}$$

which are expressed in terms of the anomalous dimension $\gamma = 1 + 2i\nu$ and the integer conformal spin n. The anomalous dimension and the conformal spin in this context were introduced when solving the BFKL equation in the complex coordinates

$$\rho_k = x_k + iy_k, \quad \rho_k^* = x_k - iy_k \tag{5}$$

using the conformal properties of the BFKL kernel.

The BFKL wave function $f_{m,\tilde{m}}$ satisfies the Schrödinger equation

$$E_{m,\tilde{m}} f_{m,\tilde{m}} = H f_{m,\tilde{m}} \tag{6}$$

with the energy $E_{m,\tilde{m}}$ being proportional to the position of the singularity in the complex angular momentum j plane. In the multicolor limit, the Hamiltonian possesses a property of holomorphic separability

$$H = \tfrac{1}{2}(h + h^*), \tag{7}$$

where the holomorphic and the antiholomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1}, \qquad h^* = \sum_{k=1}^{n} h^*_{k,k+1} \tag{8}$$

are expressed through the BFKL operator [9]

$$h_{k,k+1} = \ln(p_k) + \ln(p_{k+1}) + \frac{1}{p_k}\ln(\rho_{k+1})p_k + \frac{1}{p_{k+1}}\ln(\rho_{k+1})p_{k+1} + 2\gamma. \tag{9}$$

In Eq. (9) one defines $\rho_{k,k+1} = \rho_k - \rho_{k+1}$, $p_k = i\partial/(\partial\rho_k)$, $p_k^* = i\partial/(\partial\rho_k^*)$, and $\gamma = -\psi(1)$ (the Euler constant). The holomorphic separability of the Hamiltonian means the holomorphic factorization of the wave function

$$f_{m,\tilde{m}}(\vec{\rho}_1,\vec{\rho}_2,\ldots,\vec{\rho}_n;\vec{\rho}_0) = \sum_{r,l} c_{r,l}\, f_m^r(\rho_1,\rho_2,\ldots,\rho_n;\rho_0) \times f_{\tilde{m}}^l(\rho_1^*,\rho_2^*,\ldots,\rho_n^*;\rho_0^*) \tag{10}$$

and the Schrödinger equations in the holomorphic and the antiholomorphic spaces

$$\epsilon_m f_m = h f_m, \quad \epsilon_{\tilde{m}} f_{\tilde{m}} = h^* f_{\tilde{m}}, \quad E_{m,\tilde{m}} = \epsilon_m + \epsilon_{\tilde{m}}. \tag{11}$$

The degenerate solutions are accounted for by the coefficients $c_{r,l}$ in Eq. (10), which are fixed by the single-valuedness condition for the wave function in the two-dimensional space.

It is interesting to note that the BFKL wave function can be normalized in two different ways,

$$\| f \|_1^2 = \int \prod_{r=1}^{n} d^2\rho_r \left| \prod_{r=1}^{n} \rho_{r,r+1}^{-1} f \right|^2 \quad \text{and} \quad \times \| f \|_2^2 = \int \prod_{r=1}^{n} d^2\rho_r \left| \prod_{r=1}^{n} p_r f \right|^2. \tag{12}$$

This is in an agreement with the hermicity properties of the Hamiltonian, since the transposed Hamiltonian h^t can be obtained by two different similarity transformations [10]

$$h^t = \prod_{r=1}^{n} p_r h \prod_{r=1}^{n} p_r^{-1} = \prod_{r=1}^{n} \rho_{r,r+1}^{-1} h \prod_{r=1}^{n} \rho_{r,r+1}. \tag{13}$$

The BFKL Hamiltonian is invariant under cyclic permutations corresponding to the Bose symmetry of the Reggeon wave function $i \to i+1$ $(i = 1,2,\ldots,n)$ in multicolor limit. It was noticed by Lipatov [4] that the Hamiltonian is also invariant under the canonical transformation

$$\rho_{k-1,k} \to p_k \to \rho_{k,k+1} \tag{14}$$

accompanied by the change of the operator ordering. This property becomes obvious if the Hamiltonian Eq. (8) is rewritten in the form of

$$h = h_p + h_\rho \tag{15}$$

with

$$h_p = \sum_{k=1}^{n}\left(\ln(p_k) + \frac{1}{2}\sum_{\lambda=\pm1}\rho_{k,k+\lambda}\ln(p_k)\rho_{k,k+\lambda}^{-1} + \gamma\right) \tag{16}$$

and

$$h_\rho = \sum_{k=1}^{n}\left(\ln(\rho_{k,k+1}) + \frac{1}{2}\sum_{\lambda=\pm1} p_{k+(1+\lambda)/2}^{-1}\ln(\rho_{k,k+1}) \times p_{k+(1+\lambda)/2} + \gamma\right). \tag{17}$$

The invariance of the BFKL Hamiltonian under the change of the variables in Eq. (14) together with the change of the operator ordering was called the duality symmetry. The duality symmetry implies that the BFKL Hamiltonian commutes $[h, A] = 0$ with the differential operator

$$A = \rho_{12}\rho_{23}\ldots\rho_{n1}p_1p_2\ldots p_n, \tag{18}$$

or, more generally, there is a family of mutually commuting integrals of motion [10]

$$[q_r, q_s] = 0, \quad [q_r, h] = 0 \tag{19}$$

and they are given by

$$q_r = \sum_{i_1<i_2<\cdots<i_r}\rho_{i_1,i_2}\rho_{i_2,i_3}\ldots\rho_{i_r,i_1}p_{i_1}p_{i_2}\ldots p_{i_r}. \tag{20}$$

The operators q_r build a complete set of the invariants of the transformation. Therefore the Hamiltonian h is their function

$$h = h(q_2, q_3, \ldots, q_n) \tag{21}$$

and a common eigenfunction of q_r is simultaneously a solution to the Schrödinger equation. This fact explains why the duality symmetry is related to the integrability of a system of Reggeons in the limit of the large number of colors N_c. In the the multicolor limit, only nearest-neighbor interactions are not suppressed and the BFKL dynamics are similar to that of the Ising spin-chain model.

The transformation Eq. (14) of the holomorphic BFKL Hamiltonian is a unitary transformation only for a vanishing total momentum

$$\vec{p} = \sum_{r=1}^{n}\vec{p}_r \tag{22}$$

which guarantees the cyclicity of the momenta p_r important for their representation by the difference of coordinates $\rho_{r,r+1}$. For the compound state of two Reggeized gluons (usual BFKL case) for $n = 2$, this can be achieved only for the zero-transferred momentum $\vec{q} = 0$. Only in this case can one really identify the dual coordinates $\vec{\rho}_{r,r+1}$ of the momenta $\vec{p}_r$ with their conjugate coordinates. In a more general case, these two are not the same object. However, the integrability of the nonforward BFKL suggests that the duality symmetry should also be present in the case of $\vec{q} \neq 0$, but in an implicit way. The main objective of the present study is to show that the dipole formulation of the BFKL evolution can provide a suitable framework for studying the duality symmetry of the nonforward BFKL. The evolution equation for the scattering of a nondiagonal dipole is shown to coincide with the nonforward BFKL equation in the dual space, provided some dual condition to the so-called BFKL condition is imposed on the dipole scattering amplitude. The BFKL condition is a result of the unitarity and the multi-Regge kinematics used for deriving the leading-order BFKL as discussed below. In this formalism, the duality symmetry of the nonforward BFKL equation appears in an implicit way due to the fact that the set of coordinates of the scattered dipole can be identified with a set of dual coordinates of the Reggeized gluons' momenta. However, it appears that there is no obvious choice of the Fourier transform that can relate the dipole coordinates to the Reggeized gluon momenta individually. This is the reason why the duality symmetry is established implicitly. The duality symmetry also holds in the nonforward case because of the special structure of the nonforward BFKL kernel, which can be viewed as the sum of the three forward kernels. As it was already mentioned, the duality symmetry of the forward BFKL can be shown explicitly, which suggests that the sum of the three forward kernels should also possess this property.

The main motivation of the present study is to investigate properties of the BFKL equation in the Reggeized gluon and color-dipole formulations, and to establish a connection between them. In particular, the duality between the two formulations considered here naturally explains the mixing of the real and virtual contributions in the two pictures as discussed in Sec. IV. The duality symmetry of the multicolor system of many Reggeized gluons was used by Lipatov to find an analytic solution for the Odderon wave function. It is the present author's belief that the analysis presented in this study can be generalized to a nonlinear case and will be helpful in finding analytic solution to the Balitsky-Kovchegov (BK) equation.

Another possible implementation of the analysis presented here is the use of uncut-cut-uncut (UCU) representation of the BFKL equation for the calculation of not fully inclusive processes with different multiplicities of produced particles, similar to the Abramovsky-Gribov-Kancheli (AGK) cutting rules in the old fashioned Regge theory.

One remark is in order. A system may possess another symmetry with a similar name, called the dual conformal symmetry. The dual conformal symmetry is a usual conformal symmetry in the dual coordinates ($k_i = x_i - x_{i+1}$) and, generally, is not related to the duality symmetry. The dual conformal symmetry is now successfully implemented in calculating multileg planar amplitudes in the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory (for an up-to-date discussion, see Ref. [11] and references therein) and it was also recently considered in connection with the BFKL equation [12]. This symmetry is beyond the scope of the present study.

In the next section, the BFKL equation is written in the dual coordinates and its structure is analyzed. The argument is made that the nonforward BFKL equation can be represented as a three-point amplitude, due to the BFKL condition associated with the lack of the crossing symmetry in the BFKL approach.

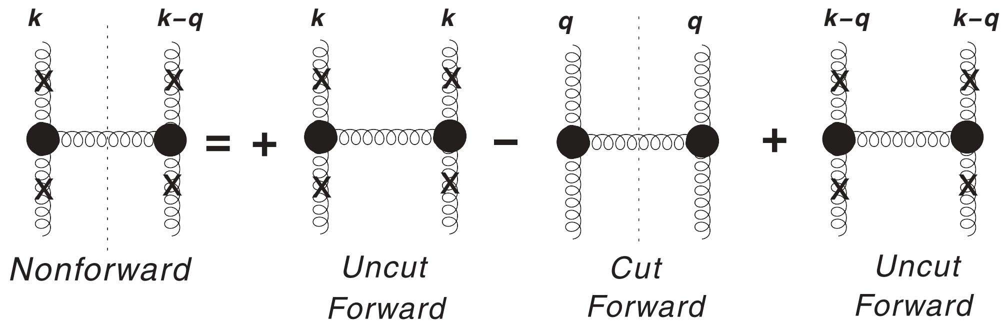

FIG. 2. The uncut-cut-uncut (UCU) structure of the nonforward BFKL. The nonforward BFKL kernel can be written as a linear combination of three forward kernels, two uncut (for two gluon pairs k and $k-q$) and one cut (for the gluon pair q). The cut kernel does not possess virtual contributions, which is reflected by the absence of crosses on gluons q.

III. BFKL EQUATION IN DUAL COORDINATES

This section discusses the structure of the BFKL equation and writes it in the dual coordinates. It is shown that the nonforward BFKL kernel $q \neq 0$ can be written as a sum of three forward kernels, which can be interpreted as two uncut and one cut kernel (UCU structure). The UCU structure of the BFKL equation is crucial for establishing the duality symmetry in the nonforward case as well.

To start, the BFKL equation is recast into a form useful for this discussion.[1] The direct substitution of Eq. (2) and Eq. (3) in Eq. (1) gives

$$\begin{aligned}\frac{\partial\mathcal{F}(k,k-q)}{\partial y} =& +\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ k^2}{\chi^2(\chi-k)^2}\mathcal{F}(\chi,\chi-q)\\ &-\frac{\alpha_s N_c}{4\pi^2}\int\frac{d^2\chi\ k^2}{\chi^2(\chi-k)^2}\mathcal{F}(k,k-q)\\ &+\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ (k-q)^2}{(\chi-q)^2(\chi-k)^2}\mathcal{F}(\chi,\chi-q)\\ &-\frac{\alpha_s N_c}{4\pi^2}\int\frac{d^2\chi\ (k-q)^2}{(\chi-q)^2(\chi-k)^2}\mathcal{F}(k,k-q)\\ &-\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ q^2}{\chi^2(\chi-q)^2}\mathcal{F}(\chi,\chi-q). \end{aligned} \tag{23}$$

For the present purpose, it is convenient to write the second line of Eq. (23) in a slightly different form, changing the integration variable $\chi \to \chi - q$

$$\begin{aligned}\frac{\partial\mathcal{F}(k,k-q)}{\partial y} =& +\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ k^2}{\chi^2(\chi-k)^2}\mathcal{F}(\chi,\chi-q)\\ &-\frac{\alpha_s N_c}{4\pi^2}\int\frac{d^2\chi\ k^2}{\chi^2(\chi-k)^2}\mathcal{F}(k,k-q)\\ &+\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ (k-q)^2}{\chi^2(\chi-k+q)^2}\mathcal{F}(\chi+q,\chi)\\ &-\frac{\alpha_s N_c}{4\pi^2}\int\frac{d^2\chi\ (k-q)^2}{\chi^2(\chi-k+q)^2}\mathcal{F}(k,k-q)\\ &-\frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2\chi\ q^2}{\chi^2(\chi-q)^2}\mathcal{F}(\chi,\chi-q). \end{aligned} \tag{24}$$

One can see that the kernel of the nonforward BFKL equation can be written as a sum of the three forward kernels, where two of them are the usual forward BFKL kernels given by the first and the second lines of Eq. (24), while the third line has only the real emission part. This interpretation is better understood from Fig. 2, where the unitarity cut is denoted by the vertical dashed line. The BFKL kernel that describes the bound states of two Reggeized gluons k and $k-q$ can be viewed as the sum of the uncut forward kernels for the gluon pair k and k and the gluon pair $k-q$ and $k-q$ (the first term and the third term on the right-hand side in Fig. 2) and the cut forward kernel for the scattering of the pair of fictitious gluons q and q (the second term on the right-hand side in Fig. 2). The last contribution seats exactly on the unitarity cut and thus does not possess any virtual contribution. This UCU structure of the nonforward BFKL kernel plays an important role in showing the duality symmetry of the BFKL equation and in finding the dual equation in the dipole picture as shown below. To see this, Eq. (24) is first written in the dual coordinates properly chosen by making the following observations.

The duality symmetry also holds for a case of the multi-Reggeon exchange, and is not limited to the system of two Reggeized gluons, as in the BFKL equation Eq. (24). Another important point is that only the upper momenta of the Reggeized gluons are to be taken into account. In particular, this means that in the case of the BFKL Pomeron, there are only three momenta for the duality symmetry, because the Regge kinematics selects the t channel for a propagation of the BFKL state breaking the crossing symmetry. Together with the unitarity condition and the strong ordering of the produced particles (multi-Regge kinematics), this results into some constraint on the form of the leading-order BFKL amplitude, which is called the BFKL condition. This condition is implicitly written in the leading-order BFKL as explained below.

By inspecting the arguments of the BFKL amplitude in Eq. (24) of both the real and the virtual parts, one can deduce that their difference is always equal to the transferred momentum, namely, for $\mathcal{F}(k_i,k_j)$, $k_i - k_j = q$. This is a consequence of the use of the t-channel unitarity together with

[1]From now on, only the two-dimensional transverse momenta is dealt with and the vector sign is omitted to make the presentation clear.

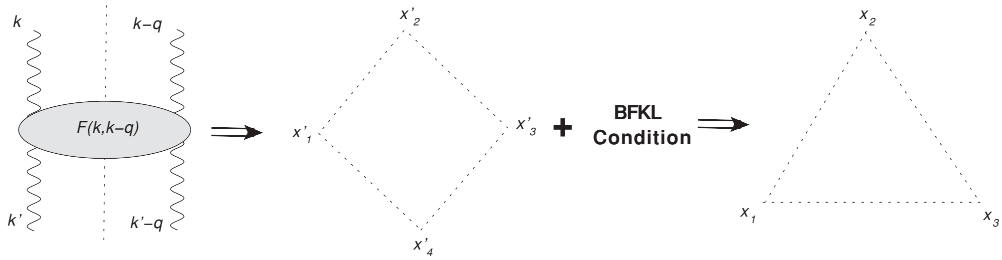

FIG. 3. The BFKL condition constrains the representation of the BFKL scattering amplitude as a function of only three dual coordinates instead of four external points.

a special kinematics in the BFKL approach. This condition is the BFKL condition. It suggests that the BFKL amplitude should be treated as a three-point function with external momenta

$$k_1 = k; \quad k_2 = q - k; \quad k_3 = -q. \tag{25}$$

At first sight, the BFKL amplitude is a four-point scattering amplitude with four external (transverse) momenta k, $k-q$, k', and $k'-q$, but the BFKL condition removes the necessity in the fourth external momentum, leaving three momenta, which obey the conservation law. This means that the BFKL amplitude is in fact a function of only two external transverse momenta (i.e., k and q or k and $k-q$). In other words, the duality symmetry deals with only the upper gluon momenta or only the lower gluon momenta, but never with both of them. This observation suggests the need to pick up only three dual coordinates. For our purposes we define the dual coordinates

$$\begin{aligned} k &= k_1 = x_1 - x_2 = x_{12}; \\ q - k &= k_2 = x_2 - x_3 = x_{23}; \\ -q &= k_3 = x_3 - x_1 = x_{31} \end{aligned} \tag{26}$$

so that the overall momenta conservation $k_1 + k_2 + k_3 = 0$ is automatically satisfied and the BFKL amplitude can be represented as a three-point function in the dual space as illustrated in Fig. 3. Using this definition, the BFKL equation Eq. (24) can be written as follows:

$$\begin{aligned} \frac{\partial \mathcal{F}(x_{12}, x_{23})}{\partial y} &= +\frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z\, x_{12}^2}{z^2(z - x_{12})^2} \left[\mathcal{F}(z, z + x_{31}) - \frac{1}{2}\mathcal{F}(x_{12}, -x_{23}) \right] \\ &\quad + \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z\, x_{23}^2}{z^2(z + x_{23})^2} \left[\mathcal{F}(z - x_{31}, z) - \frac{1}{2}\mathcal{F}(x_{12}, -x_{23}) \right] \\ &\quad - \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z\, x_{31}^2}{z^2(z + x_{31})^2} \mathcal{F}(z, z + x_{31}). \end{aligned} \tag{27}$$

The dual coordinates defined in Eq. (26) are nothing but a useful parametrization of the transverse momenta of the Reggeized gluons. However, if a representation of the BFKL equation is found in the coordinates, canonically conjugated to the transferred momenta, which is identical to the BFKL equation in the dual coordinates of Eq. (27), it would imply that there exists some symmetry. This situation is similar to one considered in the previous section, where the Hamiltonian for a system on n Reggeized gluons can be written in two alternative ways given by Eq. (16) and Eq. (17), resulting in a set of the integral of motion leading to the integrability of the system. It is this author's claim that this also holds for the nonforward BFKL, as shown later.

The next section discusses the evolution equation for the nondiagonal dipole scattering and shows that it can be written in the form of the BFKL in the dual coordinates of Eq. (27) by imposing on it the dual condition to the BFKL condition.

IV. SCATTERING OF NONDIAGONAL DIPOLE

In this section it is shown that the evolution equation for the scattering of the nondiagonal dipole depicted in Fig. 4 can be brought to the form of the BFKL equation in the dual space Eq. (27). The scattering of the nondiagonal dipole with different coordinates in the amplitude and the conjugate amplitude (to the left and to the right of the unitarity cut) was considered by Levin and the present author [13] as an auxiliary problem in proving the single inclusive production formula in the dipole formulation. Such a dipole can be constructed if one fixes the momentum of the antiquark line and thus keeps its coordinates different to the left and to the right of the unitarity cut, whereas the lower quark line momentum is integrated over, resulting in $\delta^{(2)}(\rho_1 - \rho_{1'})$. The nonlinear evolution equation was derived and solved using the notion of the generalized optical theorem. The function $M(12|12')$, for which the equation was derived, is an auxiliary function for proving the single inclusive production formula for the dipole

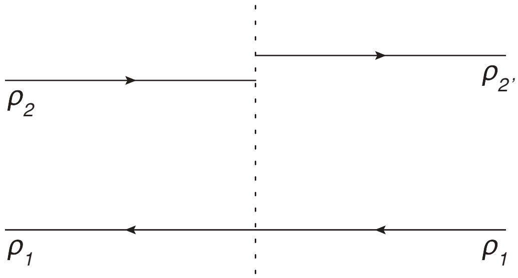

FIG. 4. The schematic representation of a color dipole, which has different transverse sizes to the left and to the right of the unitarity cut denoted by the vertical dashed line. For the present purposes, it is enough to consider only the difference in the coordinates of the upper (antiquark) line, keeping the the coordinates of the lower (quark) the same. The broken antiquark line illustrates only the fact that the sizes are different. There is no discontinuity in the charge, flow, etc.

model. It has a meaning of the nondiagonal dipole total cross section since for $\rho_2 = \rho_{2'}$ it reduces to $M(12|12) = 2N(12)$ (the optical theorem in the coordinate space), where $N(12)$ is the BFKL amplitude in the Möbius representation. This property was imposed by the definition of $M(12|12')$, since for $\rho_2 = \rho_{2'}$ the nondiagonal dipole takes the form of a usual dipole, which is described by the BFKL equation.

The evolution equation for the nondiagonal dipole is derived using real-virtual noncancellations (i.e., including the interactions in the final state). The final-state interactions fully cancel in the inclusive case, but as far as gluon production is concerned such cancellation does not happen and this fact is crucial for obtaining the closed form of the single-gluon production cross section with evolution effects included. For more details about the way this was derived, one is referred to Levin and the present author [13]. Here, the discussion only involves the linear version of this evolution equation, its properties, and showing that it can be written as a nonforward BFKL in the dual space. This result would mean that there exists a hidden duality symmetry of the nonforward BFKL that appears implicitly from this analysis due to the fact that the set of dual coordinates (with dimensions of mass) can be associated with the set of transverse coordinates of the dipoles. This extends the duality symmetry shown by Lipatov for the forward case to a nonzero-momentum transfer, which can potentially explain the integrability of the BFKL equation.

For the purposes of this paper, only the linear part of the resulting nonlinear evolution equation for a nondiagonal dipole scattering is retained. [13] It reads

$$\begin{aligned}\frac{\partial M(12|12')}{\partial y} = \frac{\bar{\alpha}_s}{2\pi}\int d^2\rho_3 \Bigg\{ &-\frac{1}{2}\left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{23}}{\rho_{23}^2}\right)^2 M(12|12') - \frac{1}{2}\left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{2'3}}{\rho_{2'3}^2}\right)^2 M(12|12') \\ &+\left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{23}}{\rho_{23}^2}\right)\left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{2'3}}{\rho_{2'3}^2}\right)[2N(13)+M(32|32')] - \left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{23}}{\rho_{23}^2}\right)\left(\frac{\rho_{23}}{\rho_{23}^2}-\frac{\rho_{2'3}}{\rho_{2'3}^2}\right) M(13|12') \\ &-\left(\frac{\rho_{2'3}}{\rho_{2'3}^2}-\frac{\rho_{23}}{\rho_{23}^2}\right)\left(\frac{\rho_{13}}{\rho_{13}^2}-\frac{\rho_{2'3}}{\rho_{2'3}^2}\right) M(12|13) - \frac{1}{2}\left(\frac{\rho_{23}}{\rho_{23}^2}-\frac{\rho_{2'3}}{\rho_{2'3}^2}\right)^2 M(12|12')\Bigg\}. \end{aligned} \tag{28}$$

As it was already mentioned, the function $M(12|12')$ is defined such that $M(12|12) = 2N(12)$.[2] This definition follows from the fact that, in the simple case of equal dipole sizes $\rho_2 = \rho_{2'}$, all necessary real-virtual cancellations take place removing all final-state interactions and one deals with the scattering of an usual color dipole described by the BFKL equation in the coordinate space. Indeed, it is easy to see that Eq. (28) reduces to the BFKL equation for $\rho_2 = \rho_{2'}$ (see Ref. [13]). This definition and the properties of the initial condition suggested the possible form of the solution to the nondiagonal dipole evolution equation

$$M(12|12') = N(12) + N(12') - N(22'). \tag{29}$$

It was checked by the explicit substitution that this form of the solution also keeps in the nonlinear case of the generalized Balitsky-Kovchegov (BK) [14,15] equation considered in Ref. [13]. The nonlinear equation is a generalization of the Balitsky-Kovchegov equation and coincides with it for $\rho_2 = \rho_{2'}$ similar to the linear case. Using this form of solution in Eq. (28) obtains

$$\begin{aligned}&\frac{\partial[N(12)+N(12')-N(22')]}{\partial y} \\ &\quad = \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12}^2 d^2\rho_3}{\rho_{13}^2\rho_{23}^2}[N(13)+N(32)-N(12)] \\ &\qquad +\frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12'}^2 d^2\rho_3}{\rho_{13}^2\rho_{2'3}^2}[N(13)+N(32')-N(12')] \\ &\qquad -\frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{22'}^2 d^2\rho_3}{\rho_{23}^2\rho_{2'3}^2}[N(32)+N(32')-N(22')], \end{aligned} \tag{30}$$

which is just a linear combination of three BFKL equations for initial dipoles with coordinates 12, 12′, and 22′. This recalls the uncut-cut-uncut (UCU) structure of the BFKL equation in the momentum space mentioned in the previous section. It is

[2] Here, indices of the argument stand for the transverse coordinates of the quark ρ_1 and the antiquark ρ_2 ($\rho_{2'}$) lines and not only for the dipole size $\rho_{12} = \rho_1 - \rho_2$, in contrast to the common notation.

worthwhile mentioning that the generalized BK equation also has the UCU structure, which fully corresponds to the picture drawn by Ciafaloni, Marchesini, and Veneziano deriving the cut Reggeon calculus [16,17]. They found that the Pomeron can be described as a linear combination of three propagating states, which correspond to one cut and two uncut Pomerons $\phi^+ + \phi^- - \phi^c$. The Reggeon field ϕ^+ stands for the Pomeron to the left of the unitarity cut, ϕ^- for the Pomeron to the right of the unitarity cut, and ϕ^c represents the Pomeron living on the cut. The introduction of the triple Pomeron splitting vertex (fan diagrams) preserves this structure, while the Pomeron loops break it explicitly. The same result was also obtained by Levin and the present author [18] using generating functional approach to the analysis of the multiparticle states in the dipole model based on Abramovski-Gribov-Kancheli (AGK) cutting rules [19].

The present goal is to show that the evolution equation for the nondiagonal dipole reproduces the nonforward BFKL equation in the dual coordinates. It is not difficult to see that with the help of the solution Eq. (29) recast Eq. (30) can be recast into the form of

$$\begin{aligned}\frac{\partial M(12|12')}{\partial y} &= \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12}^2 d^2\rho_3}{\rho_{13}^2\rho_{23}^2}\left[M(32|32') - \frac{1}{2}M(12|12')\right.\\ &\left.\quad + M(32|22') - \frac{1}{2}M(12|22')\right]\\ &\quad + \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12'}^2 d^2\rho_3}{\rho_{13}^2\rho_{2'3}^2}\left[M(32|32') - \frac{1}{2}M(12|12')\right.\\ &\left.\quad + M(32'|22') - \frac{1}{2}M(12'|22')\right]\\ &\quad - \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{22'}^2 d^2\rho_3}{\rho_{23}^2\rho_{2'3}^2} M(32|32'). \end{aligned} \tag{31}$$

We immediately notice that Eq. (31) is very similar to Eq. (27) except for the last two terms in brackets of the first two lines. This is despite the fact that the functions $M(12|12')$ are defined in a very much different way than the BFKL amplitude. The main difference is that the BFKL amplitude $\mathcal{F}(k, k-q)$ accounts for the requirement of the BFKL condition. In particular, this means that for $\mathcal{F}(k, k-q) = \mathcal{F}(k_1, k_2)$ the arguments should satisfy

$$k_1 - k_2 = q, \tag{32}$$

which is translated in terms of the dual coordinates Eq. (26), as the function $\mathcal{F}(z_1, z_2)$ should satisfy $z_1 - z_2 = -x_{31}$. It is now possible to identify the dual coordinates of Eq. (26) with the transverse dipole coordinates as follows:

$$\rho_{12} = x_{12}; \quad \rho_{12'} = -x_{23}; \quad \rho_{22'} = x_{31}. \tag{33}$$

The dual condition of the BFKL in Eq. (32) for $M(ij|ik)$ reads

$$\rho_{ij} - \rho_{ik} = -\rho_{22'}. \tag{34}$$

Imposing the dual condition of the BFKL in Eq. (34) on the evolution equation for the nondiagonal dipole removes undesired terms in Eq. (31) and leaves

$$\begin{aligned}\frac{\partial \tilde{M}(\rho_{12}|\rho_{12'})}{\partial y} &= \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12}^2 d^2\rho_3}{\rho_{13}^2\rho_{23}^2}\left[\tilde{M}(\rho_{32}|\rho_{32'})\right.\\ &\left.\quad - \frac{1}{2}\tilde{M}(\rho_{12}|\rho_{12'})\right] + \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12'}^2 d^2\rho_3}{\rho_{13}^2\rho_{2'3}^2}\\ &\quad \times\left[\tilde{M}(\rho_{32}|\rho_{32'}) - \frac{1}{2}\tilde{M}(\rho_{12}|\rho_{12'})\right]\\ &\quad - \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{22'}^2 d^2\rho_3}{\rho_{23}^2\rho_{2'3}^2}\tilde{M}(\rho_{32}|\rho_{32'}). \end{aligned} \tag{35}$$

Recasting Eq. (35) in a more transparent form gives

$$\begin{aligned}\frac{\partial \tilde{M}(\rho_{12}|\rho_{12'})}{\partial y} &= \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12}^2 d^2\rho_{32}}{\rho_{32}^2(\rho_{32} - \rho_{12})^2}\left[\tilde{M}(\rho_{32}|\rho_{32} + \rho_{22'})\right.\\ &\left.\quad - \frac{1}{2}\tilde{M}(\rho_{12}|\rho_{12'})\right] + \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{12'}^2 d^2\rho_{32'}}{\rho_{32'}^2(\rho_{32'} - \rho_{12'})^2}\\ &\quad \times\left[\tilde{M}(\rho_{32'} - \rho_{22'}|\rho_{32'}) - \frac{1}{2}\tilde{M}(\rho_{12}|\rho_{12'})\right]\\ &\quad - \frac{\bar{\alpha}_s}{2\pi}\int \frac{\rho_{22'}^2 d^2\rho_{32}}{\rho_{32}^2(\rho_{32} + \rho_{22'})^2}\tilde{M}(\rho_{32}|\rho_{32} + \rho_{22'}), \end{aligned} \tag{36}$$

which is identical to the nonforward BFKL equation in the dual space Eq. (27) provided the dipole coordinates and the dual coordinates are identified, as in Eq. (33). It is not surprising that the equation for $M(12|12')$ includes more terms than the BFKL for $\mathcal{F}(k, k-q)$, since $M(12|12')$ was defined without any additional constraints except to reproduce the dipole BFKL for $\rho_2 = \rho_{2'}$, in contrast to the BFKL amplitude.

By construction of the dipole model, the coordinates ρ_{ij} are conjugate to the momenta k_i of the Reggeized gluons. The fact that the dual momenta coordinates of Eq. (26) can be idetified with the dipole coordinates indicates that the duality symmetry is also preserved in the nonforward case. However, there seems to be no obvious way to introduce the Fourier transform that connects them and thus the duality symmetry is hidden.

In this discussion, the issue of the initial condition and the impact parameter $b_{12} = (\rho_1 + \rho_2)/2$ dependence is ignored. These two are related to each other since the impact parameter defines a reference point that connects the evolution to the target. Any fixed reference point explicitly breaks the translational symmetry and, thus, the impact parameter cannot be related to the set of the dual coordinates in Eq. (26). For a similar reason, the lower momenta k' and $k' - q$ of the amputated BFKL amplitude, shown in Fig. 3, are not considered. More accurately, the upper and the lower momenta are not considered simultaneously. The evolution is assigned to the upper momenta, while the lower momenta enter through the initial condition (or vice versa). Any attempt to include the initial condition to duality picture would contradict the lack of the impact parameter dependence in the dipole picture, but, as it has already been pointed out, the b dependence is incompatible with the requirement of the translational invariance.

The hidden duality symmetry is related only to the pure evolution, without any reference to the initial condition. As

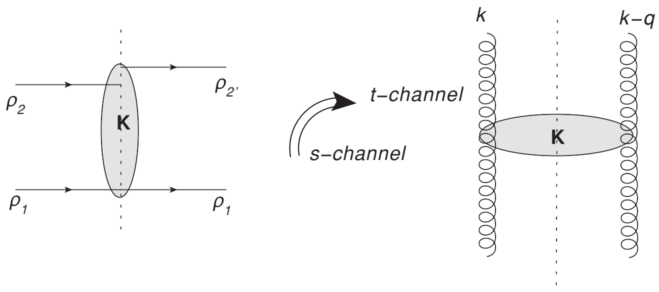

FIG. 5. The duality symmetry can be interpreted as a symmetry under rotation of the BFKL kernel in the transverse space from the s channel (color dipoles) to the t channel (Reggeized gluons). The unitarity cut is denoted by a dashed vertical line.

it was anticipated in Ref. [5], the duality symmetry is the symmetry under rotation of the BFKL kernel in the transverse space from the s channel to the t channel and back, as illustrated in Fig. 5. This rotation is, in fact, a rotation between the Reggeized gluon formulation of the BFKL evolution and the dipole picture. The connection between the two pictures is certainly not complete without matching the initial condition. The proper matching is formulated as follows. At the first stage, one makes a suitable choice of the dual coordinates, then the physical picture is changed by rotating the kernel of the evolution equation in the transverse space and the function is given the proper interpretation (either Reggeized gluon or dipole-scattering amplitude). Finally, at the second stage, the initial conditions are chosen in accordance to the physical picture. The second stage obviously has nothing to do with the duality-symmetry property of the BFKL evolution. This point does not seem to be particularly important in the case of the linear evolution considered here, but it becomes crucial for clarifying the meaning of the duality symmetry of the Balitsky-Kovchegov equation.

V. CONCLUSION

The duality symmetry of the leading-prder BFKL equation was discussed. The duality symmetry of the BFKL equation was formulated by Lipatov [4] for a system of n Reggeized gluons and, in the case of the color singlet BFKL equation ($n = 2$), the duality symmetry was shown to hold only in the forward ($q = 0$) case. In the present study, it is argued that the duality symmetry is also valid in the nonforward case, though in an implicit way. The hidden duality symmetry is established by identifying the dual coordinates (with dimension of mass) of the BFKL in the momentum space with the transverse sizes of a nondiagonal dipole scattered off the target. The evolution equation for the nondiagonal dipole having different sizes to the right and to the left of the unitarity cut was derived by Levin and the present author [13]. Its analytical solution was also found: a linear combination of three amplitudes of usual dipoles. This structure is similar to the structure of the nonforward BFKL, which can also be decomposed in three pieces, each corresponding to the forward BFKL. Two of the pieces can be viewed as uncut BFKL, while one piece does not have virtual contribution and is interpreted as a cut BFKL. The uncut-cut-uncut structure of the BFKL kernel uncovered in the present study is consistent with the picture drawn by Ciafaloni, Marchesini, and Veneziano [16,17] in cut Reggeon calculus, where the Pomeron is represented by three fields, which denote one cut and two uncut Pomerons.

It is argued that the duality symmetry can be viewed as a symmetry under rotation of the BFKL kernel in the transverse space from the s channel (color dipoles) to the t channel (Reggeized gluons) and back, as illustrated in Fig. 5. This provides a natural explanation of the mixing of the real-virtual contributions in matching between the Reggeized gluon approach and the color dipole picture.

The present author believes that this analysis can be extended to the nonlinear case and will be useful in finding the full analytic solution to the BK equation.

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