Solutions of the bound-state Faddeev-Yakubovsky equations in three dimensions by using *NN* and 3*N* potential models

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A recently developed three-dimensional approach (without partial-wave decomposition) is considered to investigate solutions of Faddeev-Yakubovsky integral equations in momentum space for three- and four-body bound states, with the inclusion of three-body forces. In the calculations of the binding energies, spin-dependent nucleon-nucleon (*NN*) potential models [soft-core potential S3, Malfliet-Tjon (MT) I-III, Yamaguchi-type potentials (YS), and $P_{5.5}$-model of Gibson-Lehman (P_{55}GL)] are considered along with the scalar two-meson exchange three-body potential. The presently reported results agree well with the ones obtained by other techniques, demonstrating the advantage of an approach in which the formalism is much more simplified and easy to manage for direct computation.

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I. INTRODUCTION

In recent years, calculations of three- and four-body bound and scattering states based on the Faddeev-Yakubovsky (FY) scheme are performed in a three-dimensional (3D) approach, which avoids truncation problems and the necessity of complicated recoupling algebra that accompanies calculations based on partial-wave (PW) decomposition [1–9]. Instead, in the 3D approach, the equations and amplitudes are formulated directly as functions of momentum-vector variables. This is a straightforward procedure that is convenient for obtaining final observables such as the total energy. For a PW observable, one can easily project the final state onto the specific required partial-wave channel.

For three-nucleon (3*N*) and four-nucleon (4*N*) bound states, the FY equations with two- and three-nucleon interactions have been recently formulated in a realistic 3D approach [10]. The formalism, according to the number of spin-isospin states that are taken into account, leads to a finite number of coupled three-dimensional integral equations to be solved. It has been shown that considering the continuous-angle variables instead of the discrete-angular momentum quantum numbers in the evaluation of the transition and permutation operators, coordinate transformations, as well as the three-nucleon forces (3*N*Fs), leads to less complicated expressions in comparison with the PW representation. However, it should be mentioned that, with respect to the PW representation, the present formalism with the smaller number of equations leads to higher dimensionality of integral equations. In other words, the price for the smaller number of equations in the 3D representation is the higher dimensionality of the integral equations. It should be clear that by switching off the spin-isospin quantum numbers, one can easily reach the bosonic type of three-dimensional FY integral equations which are solved in Refs. [11–14].

In view of the above, we can observe that one real advantage in using a non-PW approach rather than PW-based methods lies in a simplified computational algorithm, which is obtained in a straightforward manner from the original equations. For interacting systems with two and three particles, the procedure was already shown to be quite reliable and easy to be implemented. The advantage of the 3D approach is more evident in the formulation of 4*N* interacting systems, where it completely avoids the extremely complicated algebra of coupling of spin-angular momentum quantum numbers. However, it is clear that this advantage of the 3D approach, when dealing with the formalism and the corresponding computations, comes at the expense of possible numerical precision when considering more than two Jacobi momentum vector variables. In such a case, by working with the non-PW approach, after the momentum-variable discretization, one may have to deal with matrices larger than the ones that occur in the case of PW-based calculations, making the latter procedure preferable.

By considering previous numerical comparisons between 3D and PW-based results, we should note the perfect agreement between the obtained full-wave function of the three-nucleon system, as well as the corresponding momentum distribution functions [11]. In view of these results, in the case of a four-nucleon interacting system, the numerical accuracy obtained by the 3D approach is expected to be about the same as the accuracy verified in PW-based calculations. This agreement should show up in the analysis of the corresponding observables, which is partially done by considering bound-state solutions of three- and four-nucleon systems with 3*N*Fs in the present approach.

The 3D approach has been shown to be efficient in solving the Faddeev equations for the 3*N* scattering calculations, especially at intermediate and higher energies [15]. Also, the recent proton-deuteron elastic and breakup calculations show that the 3D approach has the potential to provide a more rigorous treatment of Coulomb effects [16].

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In the case of continuum problems, as for example when obtaining scattering observables, where partial-wave summation can be problematic, the 3D approach is expected to be particularly more efficient than a method using PW decomposition. Clearly, the intrinsic limitations of the PW-based calculations are not only due to the complexity of deriving the necessary equations, but also due to the limitations in computer resources requiring a very large number of angular-momentum states in order to achieve convergence for the scattering observables. By increasing the energy, the number of PW channels strongly proliferates and consequently leads to more numerical difficulties with respect to accuracy and storage requirements. However, as shown in Ref. [15], relativistic three-body scattering calculations at energies up to 1-GeV laboratory kinetic energy have been done successfully by using direct vector-variable calculations, avoiding PW decomposition. Since the 3D approach does not use partial wave decomposition, carrying all the PW channels automatically, the same numerical effort is spent in observable calculations at higher or lower energies. Essentially, the 3D technique is not only shown to be a viable alternative to the well-established PW-based calculations at low-energy regions, but it also appears to be a necessary approach at higher energies where the PW approach is no longer feasible.

One should also note that channel-independent observables, such as the total differential cross section, can be obtained using the 3D formalism and, consequently, be compared to experimental data. Since experimental data are not always available, one needs to extract a channel-dependent observable from this 3D approach, such as the NN phase shifts. To achieve this aim, one can easily project the obtained final state onto the specific PW channel, as was done by Fachruddin, leading to very accurate results in excellent agreement with established PW results [17].

It is useful to mention a recent alternative 3D representation for $3N$ bound states where the spin-isospin couplings are not explicitly carried out [18]. The unique aspect of this formalism is the evaluation of NN t matrices, the $3N$Fs, and the Faddeev components as products of scalar functions with scalar products of spin operators and momentum vectors. The spin operators have been removed and the final formalism leads to scalar functions of momentum vectors only.

In the present paper, our purpose is to calculate FY bound-state solutions using nucleon-nucleon potential models with three-nucleon forces, following the non-PW 3D approach as shown in Ref. [10]. We report results obtained for three- and four-nucleon binding energies by employing spin-isospin-dependent NN potential models along with a scalar two-meson exchange $3N$F. The main goal of the present work is to demonstrate advantages of the 3D approach in few-body systems by testing the 3D representation of the FY integral equations with several potential models not previously considered in 3D approach studies.

The current paper is organized as follows: In Sec. II, we briefly review the coupled three-dimensional FY integral equations for the $4N$ bound state. In Sec. III, we present our numerical results for three- and four-nucleon binding energies and compare them to the results obtained from other techniques. Finally, our summary and conclusions are in Sec. IV.

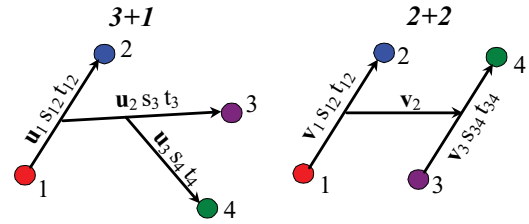

FIG. 1. (Color online) Definition of the $3+1$ and $2+2$ type of Jacobi coordinates of a $4N$ system.

II. A BRIEF REVIEW OF FY EQUATIONS IN THREE DIMENSIONS

In the FY formalism, the bound state of four nucleons in the presence of $3N$Fs is described by the following coupled equations [7]:

$$|\psi_1\rangle = G_0 t\, P[(1-P_{34})|\psi_1\rangle + |\psi_2\rangle] + (1+G_0 t) G_0 V^3_{123} |\Psi\rangle,$$
$$|\psi_2\rangle = G_0 t\, \tilde{P}[(1-P_{34})|\psi_1\rangle + |\psi_2\rangle], \qquad (1)$$

where the Yakubovsky components $|\psi_1\rangle$ and $|\psi_2\rangle$ stand for $3+1$ (K type or 1 2 3, 4) and $2+2$ (H type or 1 2, 3 4) partitions of the four nucleons, respectively. G_0 is the free $4N$ propagator, the operator t is the NN transition matrix, and P, P_{34}, and $\tilde{P}$ are permutation operators. The quantity V^3_{123} defines a part of the $3N$F in the cluster (1 2 3), which is symmetric under the exchange of particles 1 and 2. As shown in Fig. 1, for non-PW momentum space representation of the coupled Yakubovsky components [i.e., Eq. (1)], two different sets of basis states are needed,

$$|\mathbf{u};\alpha\rangle \equiv \left|\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; \alpha^S_{1234}\, \alpha^T_{1234}\right\rangle \equiv \left|\mathbf{u}_1, \mathbf{u}_2,\ \mathbf{u}_3;\right.$$
$$\left. \times \left(\left(s_{12}\ \tfrac{1}{2}\right) s_{123}\ \tfrac{1}{2}\right) S M_S \left(\left(t_{12}\tfrac{1}{2}\right) t_{123}\ \tfrac{1}{2}\right) T\ M_T\right\rangle,$$
$$|\mathbf{v};\beta\rangle \equiv \left|\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3; \beta^S_{1234}\, \beta^T_{1234}\right\rangle$$
$$\equiv |\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3; (s_{12}\ s_{34}) S\, M_S\ (t_{12}\ t_{34}) T\ M_T\rangle, \qquad (2)$$

where these basis states are complete in the $4N$ Hilbert space,

$$\sum_{\xi}^{\mathbf{A}} |\mathbf{A};\xi\rangle\langle\mathbf{A};\xi| = \mathbf{1}, \quad \sum_{\xi}^{\mathbf{A}} \equiv \sum_{\xi} \int D^3A$$
$$\equiv \sum_{\xi} \int d^3A_1 \int d^3A_2 \int d^3A_3, \qquad (3)$$

where $\mathbf{A}$ indicates each one of the $\mathbf{u}$ and $\mathbf{v}$ vector sets and ξ indicates α and β quantum number sets. Representation of the coupled equations (1) in the introduced basis states, Eq. (2),

leads to two sets of coupled integral equations:

$$\langle \mathbf{u}\,;\alpha\,|\psi_1\rangle = \sum_{\alpha'}^{\mathbf{u}'}\sum_{\alpha''}^{\mathbf{u}''}\langle \mathbf{u}\,;\alpha\,|G_0 t|\,\mathbf{u}'\,;\alpha'\,\rangle\langle \mathbf{u}'\,;\alpha'\,|P|\,\mathbf{u}''\,;\alpha''\rangle\left(\sum_{\alpha'''}^{\mathbf{u}'''}\langle \mathbf{u}''\,;\alpha''|1-P_{34}|\,\mathbf{u}'''\,;\alpha'''\,\rangle\,\langle\,\mathbf{u}'''\,;\alpha'''\,|\psi_1\rangle\right.$$
$$\left.+\sum_{\beta'}^{\mathbf{v}'}\langle\,\mathbf{u}''\,;\alpha''\,|\,\mathbf{v}'\,;\beta'\,\rangle\,\langle\,\mathbf{v}'\,;\beta'\,|\psi_2\rangle\right)+\sum_{\alpha'}^{\mathbf{u}'}\sum_{\alpha''}^{\mathbf{u}''}\langle\,\mathbf{u}\,;\alpha\,|(1+G_0 t)G_0|\,\mathbf{u}'\,;\alpha'\,\rangle\,\langle\,\mathbf{u}'\,;\alpha'\,|V_{123}^{(3)}|\,\mathbf{u}''\,;\alpha''\,\rangle\langle\,\mathbf{u}''\,;\alpha''\,|\Psi\rangle,$$

$$\langle \mathbf{v}\,;\beta\,|\psi_2\rangle = \sum_{\beta'}^{\mathbf{v}'}\sum_{\beta''}^{\mathbf{v}''}\langle \mathbf{v}\,;\beta\,|G_0 t|\,\mathbf{v}'\,;\beta'\,\rangle\,\langle \mathbf{v}'\,;\beta'\,|\tilde{P}|\,\mathbf{v}''\,;\beta''\rangle$$
$$\times\left(\sum_{\alpha'}^{\mathbf{u}'}\sum_{\alpha''}^{\mathbf{u}''}\langle \mathbf{v}''\,;\beta''\,|\,\mathbf{u}'\,;\alpha'\rangle\,\langle \mathbf{u}'\,;\alpha'\,|1+P_{34}|\,\mathbf{u}''\,;\alpha''\rangle\,\langle \mathbf{u}''\,;\alpha''\,|\psi_1\rangle+\langle \mathbf{v}''\,;\beta''\,|\psi_2\rangle\right). \tag{4}$$

To evaluate the above coupled integral equations, it is necessary to evaluate the matrix elements of two-body t matrices, permutation operators, as well as the coordinate transformations. These have been evaluated in detail in Ref. [10]. It is useful to mention that one needs the free $4N$ basis states $|\mathbf{A};\gamma\rangle$, where the spin-isospin parts γ are given as $|\gamma\rangle \equiv |\gamma_S\ \gamma_T\rangle \equiv |m_{s_1}\,m_{s_2}\,m_{s_3}\,m_{s_4}\ m_{t_1}\,m_{t_2}\,m_{t_3}\,m_{t_4}\rangle$. In changing the $4N$ basis states (i.e., $|\alpha\rangle$ and $|\beta\rangle$) to the free $4N$ basis states $|\gamma\rangle$, it is necessary to calculate the usual Clebsch-Gordan coefficients $\langle\gamma|\alpha\rangle = g_{\gamma\alpha} \equiv g^S_{\gamma\alpha}\,g^T_{\gamma\alpha}$ and $\langle\gamma|\beta\rangle = g_{\gamma\beta} \equiv g^S_{\gamma\beta}\,g^T_{\gamma\beta}$ (see Ref. [10]). After the above-mentioned operators and coordinate transformations are carried out, the coupled Yakubovsky equations can be obtained explicitly:

$$\langle \mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3;\alpha|\psi_1\rangle = \frac{1}{E-\frac{u_1^2}{m}-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m}}\left[\int d^3u_2'\sum_{\gamma',\gamma'''}g_{\alpha\gamma'''}\,\delta_{m'''_{s_4}m'_{s_4}}\delta_{m'''_{s_3}m'_{s_1}}\,\delta_{m'''_{t_4}m'_{t_4}}\delta_{m'''_{t_3}m'_{t_1}}{}_a\left\langle \mathbf{u}_1;m'''_{s_1}m'''_{s_2}\,m'''_{t_1}m'''_{t_2}|t(\epsilon)|\frac{-1}{2}\mathbf{u}_2\right.\right.$$
$$\left.-\mathbf{u}_2';m'_{s_2}m'_{s_3}\,m'_{t_2}m'_{t_3}\right\rangle_a\left\{\sum_{\alpha''}g_{\gamma'\alpha''}\left\langle \mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\ \mathbf{u}_2',\ \mathbf{u}_3;\alpha''\,|\psi_1\right\rangle\right.$$
$$-\sum_{\alpha''}g_{\gamma'_{1243}\alpha''}\left\langle \mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\ \frac{1}{3}\mathbf{u}_2'+\frac{8}{9}\mathbf{u}_3,\ \mathbf{u}_2'-\frac{1}{3}\mathbf{u}_3;\alpha''|\psi_1\right\rangle$$
$$\left.+\sum_{\beta'}g_{\gamma'\beta'}\left\langle \mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\ -\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3,\ \frac{1}{2}\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3;\beta'|\psi_2\right\rangle\right\}$$
$$+\left\{\left\langle \mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3;\alpha\left|V_{123}^{(3)}\right|\Psi\right\rangle+\frac{1}{2}\sum_{\gamma',\gamma'',\alpha'''}g_{\alpha\gamma'}\,g_{\gamma''\alpha'''}\int d^3u_1'\,\frac{\delta_{m'_{s_3}m''_{s_3}}\delta_{m'_{s_4}m''_{s_4}}\delta_{m'_{t_3}m''_{t_3}}\delta_{m'_{t_4}m''_{t_4}}}{E-\frac{u_{1'}^2}{m}-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m}}\right.$$
$$\left.\left.\times{}_a\left\langle \mathbf{u}_1;m'_{s_1}m'_{s_2}\,m'_{t_1}m'_{t_2}|t(\epsilon)|\mathbf{u}_1';m''_{s_1}m''_{s_2}\,m''_{t_1}m''_{t_1}\right\rangle_a\left\langle \mathbf{u}_1',\ \mathbf{u}_2,\ \mathbf{u}_3;\alpha'''\left|V_{123}^{(3)}\right|\Psi\right\rangle\right\}\right],$$

$$\langle \mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\beta\,|\psi_2\rangle = \frac{1}{E-\frac{v_1^2}{m}-\frac{v_2^2}{2m}-\frac{v_3^2}{m}}\int d^3v_3'\sum_{\gamma',\gamma'''}g_{\beta\gamma'''}\,\delta_{m'''_{s_3}m'_{s_1}}\,\delta_{m'''_{s_4}m'_{s_2}}\,\delta_{m'''_{t_3}m'_{t_1}}\,\delta_{m'''_{t_4}m'_{t_2}}$$
$$\times{}_a\left\langle \mathbf{v}_1;m'''_{s_1}m'''_{s_2}\,m'''_{t_1}m'''_{t_2}\,|t(\epsilon^*)|\mathbf{v}_3';m'_{s_3}m'_{s_4}\,m'_{t_3}m'_{t_4}\right\rangle_a\left\{\sum_{\alpha'}g_{\gamma'\alpha'}\left\langle \mathbf{v}_3,\ \frac{2}{3}\mathbf{v}_2+\frac{2}{3}\mathbf{v}_3',\ \frac{1}{2}\mathbf{v}_2-\mathbf{v}_3';\alpha'|\psi_1\right\rangle\right.$$
$$\left.-\sum_{\alpha'}g_{\gamma'_{1243}\alpha'}\left\langle \mathbf{v}_3,\ \frac{2}{3}\mathbf{v}_2-\frac{2}{3}\mathbf{v}_3',\ \frac{1}{2}\mathbf{v}_2+\mathbf{v}_3';\alpha'|\psi_1\right\rangle+\sum_{\beta'}g_{\gamma'\beta'}\,\langle \mathbf{v}_3,\ -\mathbf{v}_2,\ \mathbf{v}_3';\beta'|\psi_2\rangle\right\}, \tag{5}$$

where ${}_a\langle|t(\epsilon)|\rangle_a$ and ${}_a\langle|t(\epsilon^*)|\rangle_a$ are antisymmetrized NN t matrices. This spin-isospin 3D formalism can be simplified to the bosonic case by switching off the spin-isospin quantum numbers (see Refs. [13,14]).

TABLE I. List of parameters of the *NN* potentials used in this work. Each potential contains two parts, V_0 and V_1, where the indices 0 and 1 denote the spin of the $2N$ subsystem. Each part is written as a sum of a few terms; each is expressed as $V_{si}\, f[\mu_{si}, r(p, p')]$, where $f(\mu_{si}, r) = \exp(-\mu_{si}\, r^2)$ for a Gauss-type potential, $f(\mu_{si}, r) = \exp(-\mu_{si}\, r)/r$ for a Yukawa-type potential, and $f(\mu_{si}, p, p')^{-1} = \frac{\xi_i^2}{m} \frac{(p\,p')^{2i-2}}{(p^2+\mu_{si}^2)^i (p'^2+\mu_{si}^2)^i}$ for separable potentials. The potential strengths V_{si} are in MeV for S3, in fm^{-3} for YS and $P_{5.5}GL$, and dimensionless for MT I-III. The range parameters, exchanged masses for MT I-III, μ_{si} are in fm^{-2} for S3 and in fm^{-1} for others. For separable potentials $\xi_1 = 1.0000$ and $\xi_2 = 2.9499$.

Potential	Type	i	V_{0i}	μ_{0i}	V_{1i}	μ_{1i}
S3	Gauss	1	1000.0	3.00	1000.0	3.00
		2	−326.7	1.05	−166.0	0.80
		3	43.0	0.60	23.0	0.40
YS	Separable					
YS I		1	−0.1490	1.165	−0.4160	1.450
YS II		1	−0.1430	1.150	−0.3815	1.406
YS III		1	−0.1323	1.130	−0.3815	1.406
YS IV		1	−0.1323	1.130	−0.3628	1.406
MT I-III	Yukawa	1	7.39	3.110	7.39	3.110
		2	−2.64	−1.555	−3.22	−1.555
$P_{5.5}GL$	Separable	1	0.13230	1.130	−0.18752	1.2766
		2			−0.18752	1.7610

III. NUMERICAL RESULTS FOR THE THREE- AND FOUR-NUCLEON BINDING ENERGIES

In this section, we present numerical results for the three- and four-nucleon binding energies. The details of the numerical algorithm for solving the coupled three-dimensional integral equations can be found in Refs. [10,13,14].

A. Results for *NN* potential models

In order to check our proposed 3D formulation for the three- and four-nucleon bound states, we apply the formalism to the following spin-dependent *NN* potential models: soft-core potential S3 [19], Yamaguchi-type potentials (YS) [20], Malfliet-Tjon (MT) I-III [21] and $P_{5.5}$-model of Gibson-Lehman (P_{55}GL) [22]. We are aware that realistic *NN* potentials have already been used even for nuclei with $A > 4$, but the main goal of the present work is the test of the 3D representation of the FY equations for more realistic potentials than have been used before in such four-body calculations. The parameters of the above potentials are given in Table I.

Our results will be compared to several techniques: the variational method (VAR) [23], the Hyperspherical Harmonics expansion (HH) [24–27], several types of approximations for the subsystem kernels of the four-body problem by operators of finite rank (SKFR) [28–32], the integrodifferential equation approaches (IDEA) and the S-wave projected integrodifferential equation (SIDE) [33,34], the Coupled-Rearrangement-Channel (CRC) [35], the differential Faddeev-Yakubovsky (DFY) [1,36], the FY (PW) [3], and the coupled two-dimensional integral equations (2DI) [20]. Our results for the triton and α-particle binding energies are shown in Tables II–V in comparison to the results of other techniques. Table II collects the binding energies for the S3 potential, Table III for the YS-type potentials, Table IV for the MT I-III potential, and Table V for the $P_{5.5}GL$ potential.

As shown in Table II, our result for the α-particle binding energy for the spin-dependent (spin-averaged) S3 potential with value −28.8 (−25.7) MeV is in good agreement with the results of the HHE, SIDE, and DFY techniques and especially with the FY result in PW decomposition. Also, our result for the triton binding energy with values −8.20 and −6.41 MeV, corresponding to spin-dependent and spin-averaged versions of this potential, respectively, are in excellent agreement with the FY results in PW decomposition. It should be pointed out that the results with the spin-averaged version of the potentials differ from previous results where the original version of the potentials was used. The difference between the obtained results of the original and averaged versions of the potentials is to be expected and it is quite natural.

The calculated triton and α-particle binding energies for separable, spin-dependent Yamaguchi-type potentials with different methods are listed in Table III. Our results for the

TABLE II. Triton and α-particle binding energies for the S3 potential (in MeV).

Method	E_t	E_α
VAR [23]		−26.47
HHE [25]		−26.01
SIDE [33]	−8.20	−27.93
CRC [35]		−28.74
DFY [1]		−28.79
FY(PW) [3]	−8.20	−28.80
FY(3D)	−8.20	−28.8
SIDE^{av} [33]		−25.38
DFY^{av} [1]		−25.50
HHE^{av} [24]		−25.97
DFY^{av} [36]		−25.68
FY(PW)^{av} [3]	−6.41	−25.69
FY(3D)^{av}	−6.41	−25.7
Expt.	−8.48	−28.30

TABLE III. α-particle binding energy for YS-type potentials (in MeV). The numbers in parentheses correspond to triton binding energies.

Method	YS I	YS II	YS III	YS IV
FY(PW) [3]	−45.87(−11.05)			
SKFR [31]	−45.73			
SKFR [29]	−45.59			
SKFR [30]	−45.32			
2DI [20]	−45.7(−11.05)	−44.2(−10.71)	−42.3 (−10.13)	−37.7 (−8.48)
FY (3D)	−45.9(−11.05)	−44.4(−10.70)	−42.4 (−10.13)	−37.8 (−8.47)
Expt.		−28.30(−8.48)		

α-particle (triton) binding energy for YS I, II, III, and IV with values −45.9 (−11.05), −44.4 (−10.70), −42.4 (−10.13), and −37.8 (−8.47) MeV, respectively, are in excellent agreement with the 2DI results.

As demonstrated in Table IV, the calculation of the α-particle binding energy by using the spin-dependent and spin-averaged version of the MT I-III potential in the FY (PW) scheme converges to values of −30.29 and −28.83 MeV, while the triton binding energy converges to values −8.54 and −7.55 MeV, correspondingly. As shown in this table, our calculations for spin-dependent versions of this potential yields the values −8.54 and −30.3 MeV for triton and α-particle binding energies, correspondingly, which are in good agreement with the FY (PW) results. Also, our results for the triton and α-particle binding energies with the spin-averaged version of this potential with values −7.57 and −28.8 MeV are also in excellent agreement with the corresponding FY (PW) results.

In Table V, we present the triton and α-particle binding energies for the $P_{5,5}GL$ potential calculated with the SKFR and FY methods. Our results for triton and α-particle binding energies with values −8.04 and −28.9 MeV are in excellent agreement with the corresponding PW results. In the next section, we present our results for binding energies with the inclusion of 3*N*Fs.

TABLE IV. Triton and α-particle binding energies for the MT I-III potential (in MeV).

Method	E_t	E_α
SKFR [32]		−29.6
SKFR [28]		−30.36
SIDE [33]	−8.54	−29.74
DFY [1]	−8.54	−30.31
IDEA [34]	−8.86	−30.20
HH [27]		−30.33
EIHH [26]	−8.72	−30.71
DFY(PW) [1]		−30.312
FY(PW) [3]	−8.54	−30.29
FY(3D)	−8.54	−30.3
FY(PW)^{av} [3]	−7.55	−28.83
FY(3D)^{av}	−7.55	−28.8
Expt.	−8.48	−28.30

B. Results for *NN* with 3*N* potential models

In our calculations with a 3*N*F, we use a model of the 3*N*F that is based on multimeson exchanges. We study two different types of 3*N*Fs, a purely attractive and a superposition of attractive and repulsive (MT 3-I and MT 3-II, respectively) (Ref. [12]). The parameters of these 3*N*Fs are chosen so that the correction due to the triton binding energy calculated with the modified MT II-II *NN* potential is small, and they lead to binding energies near to the experimental triton binding energy.

As shown in Table VI, our results for the α-particle (triton) binding energies with the addition of the MT 3-I and MT 3-II 3*N*Fs, with the averaged version of MT I-III used as the *NN* potential, are −35.7 (−8.68) and −34.5 (−8.45) MeV, respectively. Unfortunately, we could not compare these results for binding energies with other calculations, but we have listed our recent results with different combinations of MT V *NN* potential and previously mentioned 3*N* potential models (i.e., MT 3-I and MT 3-II) [14]. As one can see from the comparison of our results with and without 3*N*Fs (while MT I-III^{ave} is used as the *NN* potential model) with the previously calculated binding energies (while MT V is used as the *NN* potential model) the MT I-III^{av} *NN* potential model provide more reasonable results in comparison to MT-V for triton and α-particle binding energies.

All these numbers are not meant to provide insight into the physics of three and four interacting nucleons, but have the purpose of demonstrating the high accuracy of numerical results that can be obtained by considering the present non-PW approach, in comparison with other methods. The advantage of the method is its simplified and straightforward formalism, which is appropriate to treat typical nuclear forces consisting of attractive and repulsive (short-range) parts. The results presented indicate that the 3D approach leads not only to numerical results with the same accuracy of PW-based methods, but also to integral equations with much less analytical

TABLE V. Triton binding energy for the $P_{5,5}GL$ potential in MeV. The numbers in parentheses are α-particle binding energies.

Method	E_t
SKFR [29]	−29.10
FY(PW) [3]	−28.87(−8.04)
FY(3D)	−28.9(−8.04)
Expt.	−28.30(−8.48)

TABLE VI. Triton and α-particle binding energies with and without 3*N*Fs (in MeV).

Potential	E_t	E_α
MT I-III^{av}	−7.55	−28.8
MT I-III^{av}+MT 3-I	−8.68	−35.7
MT I-III^{av}+MT 3-II	−8.45	−34.5
MT V [13]	−7.74	−31.3
MT V+MT 3-I [14]	−8.92	−38.8
MT V+MT 3-II [14]	−8.70	−37.5
Expt.	−8.48	−28.30

and algebraic complexity in comparison to corresponding equations formulated in PW-based methods. In a 3D case, there are only a finite number of coupled three-dimensional integral equations to be solved, whereas, in the PW case, after truncation, there are a finite number of coupled equations with kernels containing relatively complicated geometrical expressions.

IV. SUMMARY AND OUTLOOK

In the present paper, we solve the FY three-dimensional integral equations for spin-dependent and spin-averaged *NN* potential models (i.e., S3, MT I-III, YS type, and $P_{5.5}GL$) and the scalar two-meson exchange three-body interaction. These potentials provide reasonable results for binding energies in comparison to the potential models that have been used in previous works. Our results for these potential models are in good agreement with the corresponding previous values when considering VAR, HHE, SKFR, SIDE, and DFY techniques. In particular, they are matched with PW calculations in the FY scheme.

This non-PW approach, by working directly with momentum-vector variables, is shown to be a good, efficient alternative to other methods to treat three- and four-nucleon bound-state calculations. Recently, following this approach, the coupled FY equations have been formulated, with and without 3*N*Fs, as a function of vector Jacobi momenta, where the formalism is given in terms of the magnitudes of the momenta and the angles between them. It has been demonstrated that the three-dimensional FY integral equations can be handled in a straightforward and numerically reliable fashion. In comparison to commonly used angular-momentum decompositions, this direct approach leads to a finite number of coupled equations with kernels containing greatly simplified expressions.

It should be clear that this approach is more efficient for scattering problems, especially in the energy regions where the PW-based calculations have slow convergence. The formulation of 3*N* scattering and ^{3}H photodisintegration in a realistic 3D approach has been done successfully [37,38] and the calculation is underway. Molecular, atomic, and nuclear or subnuclear physics are but a few examples of various fields of physics where quantum-mechanical few-body problems play an important role. Since the 3D approach is general, it can be applied to any system, from molecules to elementary particles. Another valuable application of this non-PW approach is in few-body atomic bound states with realistic potentials.

We should also mention a renormalization group approach that we have considered when solving integral equations for the nucleon-nucleon interaction [39]. In leading order, by using the one-pion exchange potential plus a Dirac-δ function, this is considered a nonperturbative renormalization procedure, relying on a subtracted kernel where a scaling parameter is introduced. The role of the scaling parameter is similar to the cut-off momentum parameter, but with a big advantage in view of its flexibility. Since the approach is renormalization-group invariant, one can arbitrarily move the reference scale without affecting the relevant physical results. An extension of this approach is being investigated [40], where a recursive subtraction procedure is applied to the scattering matrix solution with next-leading-order and next-to-next-leading-order two-pion exchange interactions. Also, we are considering the application of the present 3D approach for the *NN* interaction in the renormalization group scheme that was used in Ref. [39].

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