Relativistic Thomas-Fermi treatment of compressed atoms and compressed nuclear matter cores of stellar dimensions

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The Feynman-Metropolis-Teller treatment of compressed atoms is extended to the relativistic regimes. Each atomic configuration is confined by a Wigner-Seitz cell and is characterized by a positive electron Fermi energy. The nonrelativistic treatment assumes a pointlike nucleus and infinite values of the electron Fermi energy can be attained. In the relativistic treatment there exists a limiting configuration, reached when the Wigner-Seitz cell radius equals the radius of the nucleus, with a maximum value of the electron Fermi energy $(E_e^F)_{\max}$, here expressed analytically in the ultrarelativistic approximation. The corrections given by the relativistic Thomas-Fermi-Dirac exchange term are also evaluated and shown to be generally small and negligible in the relativistic high-density regime. The dependence of the relativistic electron Fermi energies by compression for selected nuclei are compared and contrasted to the nonrelativistic ones and to the ones obtained in the uniform approximation. The relativistic Feynman-Metropolis-Teller approach here presented overcomes some difficulties in the Salpeter approximation generally adopted for compressed matter in physics and astrophysics. The treatment is then extrapolated to compressed nuclear matter cores of stellar dimensions with $A \simeq (m_{\rm Planck}/m_n)^3 \sim 10^{57}$ or $M_{\rm core} \sim M_\odot$. A new family of equilibrium configurations exists for selected values of the electron Fermi energy varying in the range $0 < E_e^F \leqslant (E_e^F)_{\max}$. Such configurations fulfill global but not local charge neutrality. They have electric fields on the core surface, increasing for decreasing values of the electron Fermi energy reaching values much larger than the critical value $E_c = m_e^2 c^3/(e\hbar)$ for $E_e^F = 0$. We compare and contrast our results with the ones of Thomas-Fermi model in strange stars.

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I. INTRODUCTION

In a classic article Baym, Bethe, and Pethick [1] presented the problem of matching, in a neutron star, a liquid core, composed of N_n neutrons, N_p protons, and N_e electrons, to the crust, taking into account the electrodynamical and surface tension effects. After discussing the different aspects of the problem they concluded: *The details of this picture require further elaboration; this is a situation for which the Thomas-Fermi method is useful*. This statement may at first appear surprising: The Thomas-Fermi model has been extensively applied in atomic physics (see, e.g., Gombás [2], March [3], and Lundqvist and March [4]) and has been applied extensively in atomic physics in its relativistic form (see, e.g., Ferreirinho, Ruffini, and Stella [5] and Ruffini and Stella [6]) as well as in the study of atoms with heavy nuclei in the classic works of Migdal, Popov, and Voskresenskii [7,8]. Similarly there have been considerations of the relativistic Thomas-Fermi model for quark stars pointing out the existence of critical electric fields on their surfaces (see, e.g., Alcock, Farhi, and Olinto [9] and Usov [10]). Similar results have also been obtained by Alford *et al.* [11] in the transition at very high densities, from the normal nuclear matter phase in the core to the color-flavor-locked phase of quark matter in the inner core of hybrid stars. No example exists of the application of the electromagnetic Thomas-Fermi model for neutron stars. This problem can indeed be approached with merit by studying the simplified but rigorous concept of a nuclear matter core of stellar dimensions which fulfills the relativistic Thomas-Fermi equation as discussed by Ruffini *et al.* [12], Popov *et al.* [13] and Popov [14]. As we will see this work leads to the prediction of the existence of a critical electric field at the interface between the core and the crust of a neutron star.

In Ruffini *et al.* [12] and Popov *et al.* [13] it is described a degenerate system of N_n neutrons, N_p protons, and N_e electrons constrained to a constant density distribution for the protons and it is solved the corresponding relativistic Thomas-Fermi equation and derived for the neutrons the distribution following the implementation of the β equilibrium condition. This generalizes the works of Popov [7,8,15,16] and Greiner [17,18] by eliminating the constraint $N_p \approx A/2$, clearly not valid for heavy nuclei, and enforcing the condition of β equilibrium self-consistently in a new relativistic Thomas-Fermi equation. Using then the existence of scaling laws we have extended in Popov *et al.* [13] the results from heavy nuclei to the case of nuclear matter cores of stellar dimensions. In both these treatments we had assumed the Fermi energy of the electrons $E_e^F = 0$. The aim of this article is to proceed with this dual approach and to consider first the case of compressed atoms and then, using the existence of scaling laws, the compressed nuclear matter cores of stellar dimensions with a positive value of their electron Fermi energies.

It is well known that Salpeter has been among the first to study the behavior of matter under extremely high pressures

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by considering a Wigner-Seitz cell of radius $R_{\rm WS}$ [19]. Salpeter assumed as a starting point the nucleus pointlike and a uniform distribution of electrons within a Wigner-Seitz cell and then considered corrections to the above model due to the inhomogeneity of electron distribution. The first correction corresponds to the inclusion of the lattice energy $E_C = -(9N_p^2\alpha)/(10R_{\rm WS})$, which results from the pointlike nucleus-electron Coulomb interaction and from the electron-electron Coulomb interaction inside the cell of radius $R_{\rm WS}$. The second correction is given by a series expansion of the electron Fermi energy about the average electron density n_e of the uniform approximation. The electron density is then assumed equals to $n_e[1+\epsilon(r)]$ with $\epsilon(r)$ considered as infinitesimal. The Coulomb potential energy is assumed to be the one of the pointlike nucleus with the uniform distribution of electrons of density n_e thus the correction given by $\epsilon(r)$ is neglected on the Coulomb potential. The electron distribution is then calculated at first order by expanding the relativistic electron kinetic energy about its value given by the uniform approximation, considering as infinitesimal the ratio eV/E_e^F between the Coulomb potential energy eV and the electron Fermi energy $E_e^F = \sqrt{[cP_e^F(r)]^2 + m_e^2c^4} - m_ec^2 - eV$. The inclusion of each additional Coulomb correction results in a decrease of the pressure of the cell P_S in comparison to the uniform one (see Rueda *et al.* [20] for details).

It is quite difficult to assess the self-consistency of all the recalled different approximations adopted by Salpeter. In order to validate and to see the possible limits of the Salpeter approach, we consider the relativistic generalization of the Feynman-Metropolis-Teller treatment [21] which takes into account all electromagnetic and special relativistic contributions automatically and globally. We show explicitly how this new treatment leads in the case of atoms to electron distributions that markedly differ from the ones often adopted in the literature of constant electron density distributions. At the same time it allows us to overcome some of the difficulties in current treatments. Similarly, the pointlike description of the nucleus often adopted in literature is confirmed to be unacceptable in the framework of a relativistic treatment.

In Sec. II we first recall the nonrelativistic treatment of the compressed atom by Feynman-Metropolis-Teller. In Sec. III we generalize that treatment to the relativistic regime by integrating the relativistic Thomas-Fermi equation, imposing also the condition of β equilibrium. In Sec. IV we first compare the new treatment with the one corresponding to a uniform electron distribution often used in the literature and to the Salpeter treatment. We also compare and contrast the results of the relativistic and the nonrelativistic treatment.

In Sec. V, using the same scaling laws adopted by Ruffini *et al.* [12] and Popov *et al.* [13], we turn to the case of nuclear matter cores of stellar dimensions with mass numbers $A \approx (m_{\rm Planck}/m_n)^3 \sim 10^{57}$ or $M_{\rm core} \sim M_\odot$, where m_n is the neutron mass and $m_{\rm Planck} = (\hbar c/G)^{1/2}$ is the Planck mass. Such a configuration presents global but not local charge neutrality. Analytic solutions for the ultrarelativistic limit are obtained. In particular we find: (i) explicit analytic expressions for the electrostatic field and the Coulomb potential energy; (ii) an entire range of possible Fermi energies for the electrons between zero and a maximum value $(E_e^F)_{\rm max}$, reached when $R_{\rm WS} = R_c$, which can be expressed analytically; and (iii) the explicit analytic expression of the ratio between the proton number N_p and the mass number A when $R_{\rm WS} = R_c$.

We then turn in Sec. VI to the study of the compressional energy of the nuclear matter cores of stellar dimensions for selected values of the electron Fermi energy. We show that the solution with $E_e^F = 0$ presents the largest value of the electrodynamical structure. We finally summarize the conclusions in Sec. VII.

II. THE THOMAS-FERMI MODEL FOR COMPRESSED ATOMS: THE FEYNMAN-METROPOLIS-TELLER TREATMENT

A. The classical Thomas-Fermi model

The Thomas-Fermi model assumes that the electrons of an atom constitute a fully degenerate gas of fermions confined in a spherical region by the Coulomb potential of a pointlike nucleus of charge $+eN_p$ [22,23]. Feynman, Metropolis, and Teller have shown that this model can be used to derive the equation of state of matter at high pressures by considering a Thomas-Fermi model confined in a Wigner-Seitz cell of radius $R_{\rm WS}$ [21].

We recall that the condition of equilibrium of the electrons in an atom, in the nonrelativistic limit, is expressed by

$$\frac{\left(P_e^F\right)^2}{2m_e} - eV = E_e^F, \tag{1}$$

where m_e is the electron mass, V is the electrostatic potential, and E_e^F is their constant Fermi energy.

The electrostatic potential fulfills, for $r > 0$, the Poisson equation

$$\nabla^2 V = 4\pi e n_e, \tag{2}$$

where the electron number density n_e is related to the Fermi momentum P_e^F by

$$n_e = \frac{\left(P_e^F\right)^3}{3\pi^2\hbar^3}. \tag{3}$$

For neutral atoms and ions n_e vanishes at the boundary so the electron Fermi energy is, respectively, zero or negative. In the case of compressed atoms n_e does not vanish at the boundary while the Coulomb potential energy eV does. Consequently E_e^F is positive.

Defining

$$eV(r) + E_e^F = e^2 N_p \frac{\phi(r)}{r}, \tag{4}$$

and introducing the new dimensionless radial coordinate η as

$$r = b\eta \quad \text{with} \quad b = \frac{(3\pi)^{2/3}}{2^{7/3}} \frac{1}{N_p^{1/3}} \frac{\hbar^2}{m_e e^2} = \frac{\sigma}{N_p^{1/3}} r_{\rm Bohr}, \tag{5}$$

where $\sigma = (3\pi)^{2/3}/2^{7/3} \approx 0.88$, $r_{\rm Bohr} = \hbar^2/(m_e e^2)$ is the Bohr radius, we obtain the following expression for the

electron number density

$$n_e(\eta) = \frac{N_p}{4\pi b^3}\left[\frac{\phi(\eta)}{\eta}\right]^{3/2}, \tag{6}$$

and then Eq. (2) can be written in the form

$$\frac{d^2\phi(\eta)}{d\eta^2} = \frac{\phi(\eta)^{3/2}}{\eta^{1/2}}, \tag{7}$$

which is the classic Thomas-Fermi equation [23]. A first boundary condition for this equation follows from the pointlike structure of the nucleus

$$\phi(0) = 1. \tag{8}$$

A second boundary condition comes from the conservation of the number of electrons $N_e = \int_0^{R_{\rm WS}} 4\pi n_e(r) r^2 dr$

$$1 - \frac{N_e}{N_p} = \phi(\eta_0) - \eta_0\phi'(\eta_0), \tag{9}$$

where $\eta_0 = R_{\rm WS}/b$ defines the radius $R_{\rm WS}$ of the Wigner-Seitz cell.

In the case of compressed atoms $N_e = N_p$ so the Coulomb potential energy eV vanishes at the boundary $R_{\rm WS}$. As a result, using Eqs. (1) and (3), the Fermi energy of electrons satisfies the universal relation

$$\frac{\sigma r_{\rm Bohr}}{e^2}\frac{E_e^F}{N_p^{4/3}} = \frac{\phi(\eta_0)}{\eta_0}, \tag{10}$$

while the Wigner-Seitz cell radius $R_{\rm WS}$ satisfies the universal relation

$$\frac{R_{\rm WS}}{\sigma r_{\rm Bohr} N_p^{-1/3}} = \eta_0. \tag{11}$$

Therefore in the classic treatment η_0 can approach zero and consequently the range of the possible values of the Fermi energy extends from zero to infinity. The results are summarized in Figs. 1 and 2.

B. The Thomas-Fermi-Dirac model

Dirac has introduced modifications to the original Thomas-Fermi theory to include effects of the exchange interaction [24]. In this case the condition of equilibrium of the electrons in the atom is generalized as follows:

$$\frac{\left(P_e^F\right)^2}{2m_e} - eV - \frac{e^2}{\pi\hbar}P_e^F = E_e^F. \tag{12}$$

The electron number density is now connected to the Coulomb potential energy by

$$n_e = \frac{1}{3\pi^5}\frac{1}{r_{\rm Bohr}^3}\left[1 + \sqrt{1 + 2\pi^2\frac{r_{\rm Bohr}}{e^2}\left(eV + E_e^F\right)}\right]^3. \tag{13}$$

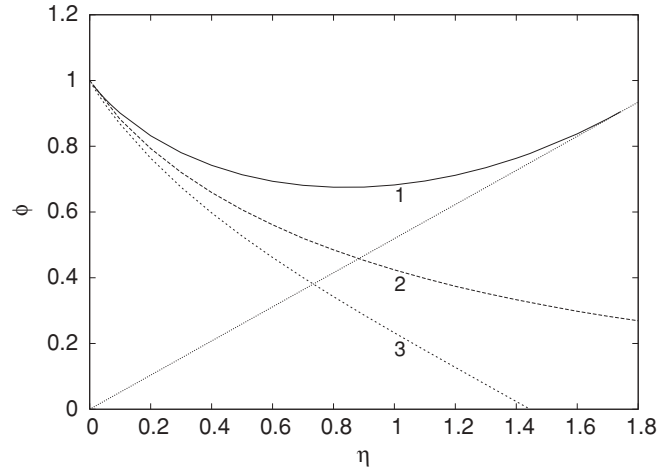

FIG. 1. Physically relevant solutions of the Thomas-Fermi equation (7) with the boundary conditions (8) and (9). Curve 1 refers to a neutral compressed atom. Curve 2 refers to a neutral free atom. Curve 3 refers to a positive ion. The dotted straight line is the tangent to curve 1 at the point $[\eta_0, \phi(\eta_0)]$ corresponding to overall charge neutrality [see Eq. (9)].

Defining

$$\frac{1}{2\pi^2}\frac{e^2}{r_{\rm Bohr}} + eV(r) + E_e^F = e^2 N_p\frac{\chi(r)}{r}, \tag{14}$$

the Eq. (2) can be written in dimensionless form as

$$\frac{d^2\phi(\eta)}{d\eta^2} = \eta\left[d + \left(\frac{\phi(\eta)}{\eta}\right)^{1/2}\right]^3, \tag{15}$$

where $d = [3/(32\pi^2)]^{1/3}(1/N_p)^{2/3}$. The boundary condition for Eq. (15) are $\phi(0) = 1$ and $\eta_0\phi'(\eta_0) = \phi(\eta_0)$.

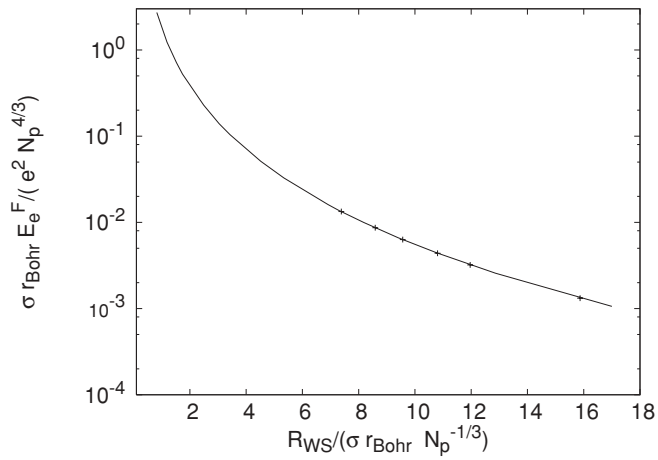

FIG. 2. The electron Fermi energy E_e^F, in units of $e^2 N_p^{4/3}/(\sigma r_{\rm Bohr})$, is plotted as a function of the Wigner-Seitz cell radius $R_{\rm WS}$, in units of $\sigma r_{\rm Bohr} N_p^{-1/3}$ [see Eqs. (10) and (11)]. Points refer to the numerical integrations of the Thomas-Fermi equation (7) performed originally by Feynman, Metropolis, and Teller in Ref. [21].

III. THE RELATIVISTIC GENERALIZATION OF THE FEYNMAN-METROPOLIS-TELLER TREATMENT

A. The relativistic Thomas-Fermi model for atoms

In the relativistic generalization of the Thomas-Fermi equation the pointlike approximation of the nucleus must be abandoned [5,6] since the relativistic equilibrium condition

$$E_e^F = \sqrt{(P_e^F c)^2 + m_e^2 c^4} - m_e c^2 - eV(r), \tag{16}$$

which generalizes the Eq. (1), would lead to a nonintegrable expression for the electron density near the origin. Consequently we adopt an extended nucleus. Traditionally the radius of an extended nucleus is given by the phenomenological relation $R_c = r_0 A^{1/3}$, where A is the number of nucleons and $r_0 = 1.2 \times 10^{-13}$ cm. Further, it is possible to show from the extremization of the semiempirical Weizsacker mass formula that the relation between A and N_p is given by (see, e.g., Segrè [25] and Ferreirinho, Ruffini, and Stella [5])

$$N_p \approx \left[\frac{2}{A} + \frac{2a_C}{a_A}\frac{1}{A^{1/3}}\right]^{-1} \approx \left[\frac{2}{A} + \frac{3}{200}\frac{1}{A^{1/3}}\right]^{-1}, \tag{17}$$

where $a_C \approx 0.71$ MeV, $a_A \approx 93.15$ MeV are the Coulomb and the asymmetry coefficients respectively. In the limit of small A Eq. (17) gives

$$N_p \approx \frac{A}{2}. \tag{18}$$

In Ref. [13] we have relaxed the condition $N_p \approx A/2$ adopted, e.g., in Refs. [8,17] as well as the condition $N_p \approx [2/A + 3/(200A^{1/3})]^{-1}$ adopted, e.g., in Refs. [5,6] by imposing explicitly the β decay equilibrium among neutron, protons, and electrons.

In particular, following the previous treatments (see, e.g., Ref. [13]), we have assumed a constant distribution of protons confined in a radius R_c defined by

$$R_c = \Delta \frac{\hbar}{m_\pi c} N_p^{1/3}, \tag{19}$$

where m_π is the pion mass and Δ is a parameter such that $\Delta \approx 1$ ($\Delta < 1$) corresponds to nuclear (supranuclear) densities when applied to ordinary nuclei. Consequently, the proton density can be written as

$$n_p(r) = \frac{N_p}{\frac{4}{3}\pi R_c^3}\theta(R_c - r) = \frac{3}{4\pi}\frac{m_\pi^3 c^3}{\hbar^3}\frac{1}{\Delta^3}\theta(R_c - r), \tag{20}$$

where $\theta(x)$ is the Heaviside function which by definition is given by

$$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{21}$$

The electron density is given by

$$n_e(r) = \frac{\left(P_e^F\right)^3}{3\pi^2\hbar^3} = \frac{1}{3\pi^2\hbar^3 c^3}[e^2V^2(r) + 2m_e c^2 eV(r)]^{3/2}, \tag{22}$$

where V is the Coulomb potential.

The overall Coulomb potential satisfies the Poisson equation

$$\nabla^2 V(r) = -4\pi e[n_p(r) - n_e(r)], \tag{23}$$

with the boundary conditions $V(\infty) = 0$ (due to global charge neutrality) and finiteness of $V(0)$.

Using Eqs. (4) and (5) and replacing the particle densities (20) and (22) into the Poisson equation (23) we obtain the relativistic Thomas-Fermi equation

$$\frac{d^2\phi(\eta)}{d\eta^2} = -\frac{3\eta}{\eta_c^3}\theta(\eta_c - \eta) + \frac{\phi^{3/2}}{\eta^{1/2}}\left[1 + \left(\frac{N_p}{N_p^{\rm crit}}\right)^{4/3}\frac{\phi}{\eta}\right]^{3/2}, \tag{24}$$

where $\phi(0) = 0$, $\phi(\infty) = 0$, and $\eta_c = R_c/b$. The critical number of protons $N_p^{\rm crit}$ is defined by

$$N_p^{\rm crit} = \sqrt{\frac{3\pi}{4}}\alpha^{-3/2}, \tag{25}$$

where, as usual, $\alpha = e^2/(\hbar c)$.

It is interesting that by introducing the new dimensionless variable

$$x = \frac{r}{\lambda_\pi} = \frac{b}{\lambda_\pi}\eta, \tag{26}$$

and the function

$$\chi = \alpha N_p \phi, \tag{27}$$

where $\lambda_\pi = \hbar/(m_\pi c)$, Eq. (24) assumes a canonical form, the master relativistic Thomas-Fermi equation (see Ruffini [26])

$$\frac{1}{3x}\frac{d^2\chi(x)}{dx^2} = -\frac{\alpha}{\Delta^3}\theta(x_c - x) + \frac{4\alpha}{9\pi}\left[\frac{\chi^2(x)}{x^2} + 2\frac{m_e}{m_\pi}\frac{\chi}{x}\right]^{3/2}, \tag{28}$$

where $x_c = R_c/\lambda_\pi$ with the boundary conditions $\chi(0) = 0$, $\chi(\infty) = 0$. The neutron density $n_n(r)$, related to the neutron Fermi momentum $P_n^F = (3\pi^2\hbar^3 n_n)^{1/3}$, is determined, as in the previous case [13], by imposing the condition of β equilibrium

$$\begin{aligned} E_n^F &= \sqrt{\left(P_n^F c\right)^2 + m_n^2 c^4} - m_n c^2 \\ &= \sqrt{\left(P_p^F c\right)^2 + m_p^2 c^4} - m_p c^2 + eV(r), \end{aligned} \tag{29}$$

which in turn is related to the proton density n_p and the electron density by Eqs. (22) and (23). Integrating numerically these equations we have obtained a new generalized relation between A and N_p for any value of A. In the limit of small A this result agrees with the phenomenological relations given by Eqs. (17) and (18), as is clearly shown in Fig. 3.

B. The relativistic Thomas-Fermi model for compressed atoms

We turn now to the case of compressed atoms in which the electron Fermi energy is positive. The relativistic generalization of the equilibrium condition (1) now reads

$$E_e^F = \sqrt{\left(P_e^F c\right)^2 + m_e^2 c^4} - m_e c^2 - eV(r) > 0\,. \tag{30}$$

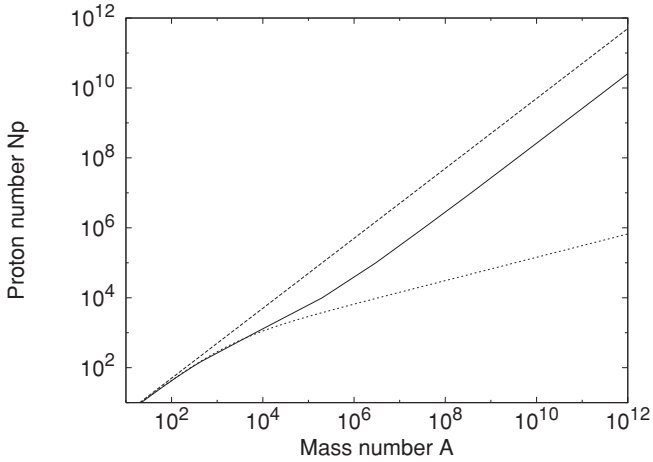

FIG. 3. The A-N_p relation at nuclear density (solid line) obtained from first principles compared with the phenomenological expressions given by $N_p \approx A/2$ (dashed line) and Eq. (17) (dotted line). The asymptotic value, for $A \to (m_{\rm Planck}/m_n)^3$, is $N_p \approx 0.0046A$.

Adopting an extended-nucleus with a radius given by Eq. (19) and a proton density given by Eq. (20) the Poisson equation (23), with the following electron density

$$n_e(r) = \frac{\left(P_e^F\right)^3}{3\pi^2\hbar^3} = \frac{1}{3\pi^2\hbar^3c^3}[e^2\hat{V}^2(r) + 2m_ec^2e\hat{V}(r)]^{3/2}, \tag{31}$$

gives again the master relativistic Thomas-Fermi equation (28) where $\chi/r = e\hat{V}(r)/(c\hbar)$ and $e\hat{V} = eV + E_e^F$.

In this case Eq. (28) has to be integrated with the boundary conditions $\chi(0) = 0$, $\chi(x_{\rm WS}) = x_{\rm WS}\chi'(x_{\rm WS})$, $x_{\rm WS} = R_{\rm WS}/\lambda_\pi$. Using Eqs. (4), (26), and (27) we obtain the electron Fermi energy in the form

$$E_e^F = m_\pi c^2 \frac{\chi(x_{\rm WS})}{x_{\rm WS}}. \tag{32}$$

The neutron density $n_n(r)$, related to the neutron Fermi momentum $P_n^F = (3\pi^2\hbar^3 n_n)^{1/3}$, is determined by imposing the condition of β equilibrium

$$\begin{aligned} E_n^F &= \sqrt{\left(P_n^Fc\right)^2 + m_n^2c^4} - m_nc^2 \\ &= \sqrt{\left(P_p^Fc\right)^2 + m_p^2c^4} - m_pc^2 + eV(r) + E_e^F. \end{aligned} \tag{33}$$

Using this approach, it is then possible to determine the β equilibrium nuclide as a function of the density of the system. In fact, electrons and protons can be converted to neutrons in inverse β decay $p + e^- \to n + \nu_e$ if the condition $E_n^F < \sqrt{(P_p^Fc)^2 + m_p^2c^4} - m_pc^2 + eV(r) + E_e^F$ holds. The condition of equilibrium (33) is crucial, for example, in the construction of a self-consistent equation of state of high-energy-density matter present in white dwarfs and neutron star crusts [20]. In the case of zero electron Fermi energy the generalized $A - N_p$ relation of Fig. 3 is obtained.

C. The relativistic Thomas-Fermi-Dirac model for compressed atoms

We now take into account the exchange corrections to the relativistic Thomas-Fermi equation (28). In this case we have (see Ref. [8], for instance)

$$E_e^F = \sqrt{\left(cP_e^F\right)^2 + m_e^2c^4} - m_ec^2 - eV - \frac{\alpha}{\pi}cP_e^F = \text{const.} \tag{34}$$

Introducing the function $\chi(r)$ as before

$$E_e^F + eV = e\hat{V} = \hbar c\frac{\chi}{r}, \tag{35}$$

we obtain the electron number density

$$\begin{aligned} n_e = \frac{1}{3\pi^2\hbar^3c^3}\Bigg\{&\gamma(m_ec^2 + e\hat{V}) + [(e\hat{V})^2 + 2m_ec^2e\hat{V}]^{1/2} \\ &\times\left[\frac{(1+\gamma^2)(m_ec^2 + e\hat{V})^2 - m_e^2c^4}{(m_ec^2 + e\hat{V})^2 - m_e^2c^4}\right]^{1/2}\Bigg\}^3, \end{aligned} \tag{36}$$

where $\gamma = (\alpha/\pi)/(1 - \alpha^2/\pi^2)$.

If we take the approximation $1 + \gamma^2 \approx 1$ the above equation becomes

$$n_e = \frac{1}{3\pi^2\hbar^3c^3}\{\gamma(m_ec^2 + e\hat{V}) + [(e\hat{V})^2 + 2m_ec^2e\hat{V}]^{1/2}\}^3. \tag{37}$$

The second term on the right-hand-side of Eq. (37) has the same form of the electron density given by the relativistic Thomas-Fermi approach without the exchange correction (31) and therefore the first term shows the explicit contribution of the exchange term to the electron density.

Using the full expression of the electron density given by Eq. (36) we obtain the relativistic Thomas-Fermi-Dirac equation

$$\begin{aligned} &\frac{1}{3x}\frac{d^2\chi(x)}{dx^2} \\ &= -\frac{\alpha}{\Delta^3}\theta(x_c - x) \\ &\quad + \frac{4\alpha}{9\pi}\Bigg\{\gamma\left(\frac{m_e}{m_\pi} + \frac{\chi}{x}\right) + \left[\left(\frac{\chi}{x}\right)^2 + 2\frac{m_e}{m_\pi}\frac{\chi}{x}\right]^{1/2} \\ &\quad \times\left[\frac{(1+\gamma^2)(m_e/m_\pi + \chi/x)^2 - (m_e/m_\pi)^2}{(m_e/m_\pi + \chi/x)^2 - (m_e/m_\pi)^2}\right]^{1/2}\Bigg\}^3, \end{aligned} \tag{38}$$

which by applying the approximation $1 + \gamma^2 \approx 1$ becomes

$$\begin{aligned} \frac{1}{3x}\frac{d^2\chi(x)}{dx^2} = -\frac{\alpha}{\Delta^3}\theta(x_c - x) + \frac{4\alpha}{9\pi}\Bigg\{&\gamma\left(\frac{m_e}{m_\pi} + \frac{\chi}{x}\right) \\ &+ \left[\left(\frac{\chi}{x}\right)^2 + 2\frac{m_e}{m_\pi}\frac{\chi}{x}\right]^{1/2}\Bigg\}^3. \end{aligned} \tag{39}$$

The boundary conditions for Eq. (38) are $\chi(0) = 0$ and $\chi(x_{\rm WS}) = x_{\rm WS}\chi'(x_{\rm WS})$. The neutron density can be obtained as before by using the β equilibrium condition (33) with the electron Fermi energy given by Eq. (34).

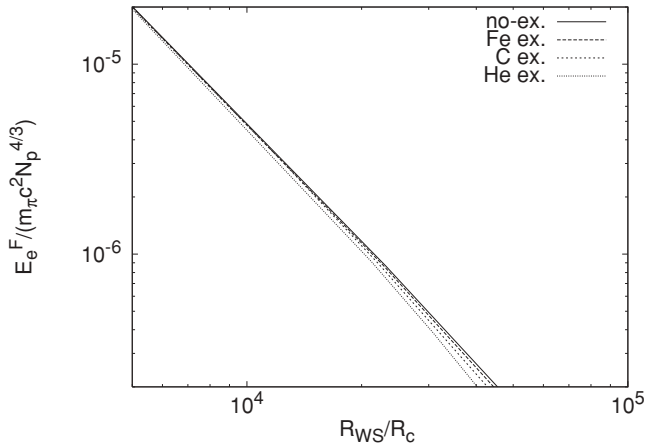

FIG. 4. The electron Fermi energy in units of $m_\pi c^2 N_p^{4/3}$ is plotted for helium, carbon, and iron, as a function of the ratio $R_{\rm WS}/R_c$ in the relativistic Feynman-Metropolis-Teller (FMT) treatment with and without the exchange effects. Here $R_{\rm WS}$ denotes the Wigner-Seitz cell radius and R_c is the nucleus radius as given by Eq. (19). It is clear that the exchange terms are appreciable only in the low-density region and are negligible as $R_{\rm WS} \to R_c$.

In Fig. 4 we show the results of the numerical integration of the relativistic Thomas-Fermi equation (28) and of the relativistic Thomas-Fermi-Dirac equation (38) for helium, carbon, and iron. In particular, we show the electron Fermi energy multiplied by $N_p^{-4/3}$ as a function of the ratio $R_{\rm WS}/R_c$ between the Wigner-Seitz cell radius $R_{\rm WS}$ and the nucleus radius R_c given by Eq. (19).

The effects of the exchange term are appreciable only in the low-density (low compression) region, i.e., when $R_{\rm WS} \gg R_c$ (see Fig. 4). We can then conclude in total generality that the correction given by the Thomas-Fermi-Dirac exchange term is small in the nonrelativistic low-compression (low-density) regime and negligible in the relativistic high-compression (high-density) regime.

IV. COMPARISON AND CONTRAST WITH APPROXIMATE TREATMENTS

There exists in the literature a variety of semiqualitative approximations adopted in order to describe the electron component of a compressed atom (see, e.g., Bürvenich, Mishustin, and Greiner [27] for applications of the uniform approximation and, e.g., Chabrier and Potekhin [28], Potekhin, Chabrier, and Rogers [29], Douchin and Haensel [30], and Haensel and Zdunik [31–33] for applications of the Salpeter approximate treatment).

We shall see how the relativistic treatment of the Thomas-Fermi equation affects the current analysis of compressed atoms in the literature by introducing qualitative and quantitative differences which deserve attention.

A. Relativistic FMT treatment versus relativistic uniform approximation

One of the most used approximations in the treatment of the electron distribution in compressed atoms is the one in which, for a given nuclear charge $+eN_p$, the Wigner-Seitz cell radius $R_{\rm WS}$ is defined by

$$N_p = \frac{4\pi}{3} R_{\rm WS}^3 n_e, \tag{40}$$

where $n_e = (P_e^F)^3/(3\pi^2\hbar^3)$. The Eq. (40) ensures the global neutrality of the Wigner-Seitz cell of radius $R_{\rm WS}$ assuming a uniform distribution of electrons inside the cell.

We shall first compare the Feynman-Metropolis-Teller treatment, previously introduced, with the uniform approximation for the electron distribution. In view of the results of the preceding section, hereafter we shall consider the nonrelativistic and the relativistic formulation of the Feynman-Metropolis-Teller treatment with no Thomas-Fermi-Dirac exchange correction.

In Fig. 5 we have plotted the electron number density obtained from Eq. (31) where the Coulomb potential is related to the function χ, which is obtained from numerical integration of the relativistic Thomas-Fermi equation (28) for different compressions for helium and iron. We have normalized the electron density to the average electron number density $n_0 = 3N_e/(4\pi R_{\rm WS}^3) = 3N_p/(4\pi R_{\rm WS}^3)$ as given by Eq. (40).

We can see in Fig. 5 how our treatment, based on the numerical integration of the relativistic Thomas-Fermi equation (28) and imposing the condition of β equilibrium (33), leads to electron density distributions that differ markedly from the constant electron density approximation.

From Eqs. (30) and (40) and taking into account the global neutrality condition of the Wigner-Seitz cell $eV(R_{\rm WS}) = 0$, the electron Fermi energy in the uniform approximation can be written as

$$E_e^F \simeq \left[-\frac{m_e}{m_\pi} + \sqrt{\left(\frac{m_e}{m_\pi}\right)^2 + \left(\frac{9\pi}{4}\right)^{2/3} \frac{N_p^{2/3}}{x_{\rm WS}^2}} \right] m_\pi c^2. \tag{41}$$

We show in Fig. 6 the electron Fermi energy as a function of the average electron density $n_0 = 3N_e/(4\pi R_{\rm WS}^3) = 3N_p/(4\pi R_{\rm WS}^3)$ in units of the nuclear density $n_{\rm nuc} = 3A/(4\pi\Delta^3 N_p \lambda_\pi^3)$. For selected compositions we show the results for the relativistic Feynman-Metropolis-Teller treatment, based on the numerical integration of the relativistic Thomas-Fermi equation (28) and for the relativistic uniform approximation.

As clearly shown in Fig. 5 and summarized in Fig. 6 the relativistic treatment leads to results strongly dependent at low compression from the nuclear composition. The corresponding value of the electron Fermi energy derived from a uniform approximation overevaluates the true electron Fermi energy (see Fig. 6). In the limit of high compression the relativistic curves asymptotically approach the uniform one (see also Fig. 5).

The uniform approximation becomes exact in the limit when the electron Fermi energy acquires its maximum value as given by

$$(E_e^F)_{\rm max} \simeq \left[-\frac{m_e}{m_\pi} + \sqrt{\left(\frac{m_e}{m_\pi}\right)^2 + \left(\frac{3\pi^2}{2}\right)^{2/3} \left(\frac{N_p}{A}\right)^{2/3}} \right] m_\pi c^2, \tag{42}$$

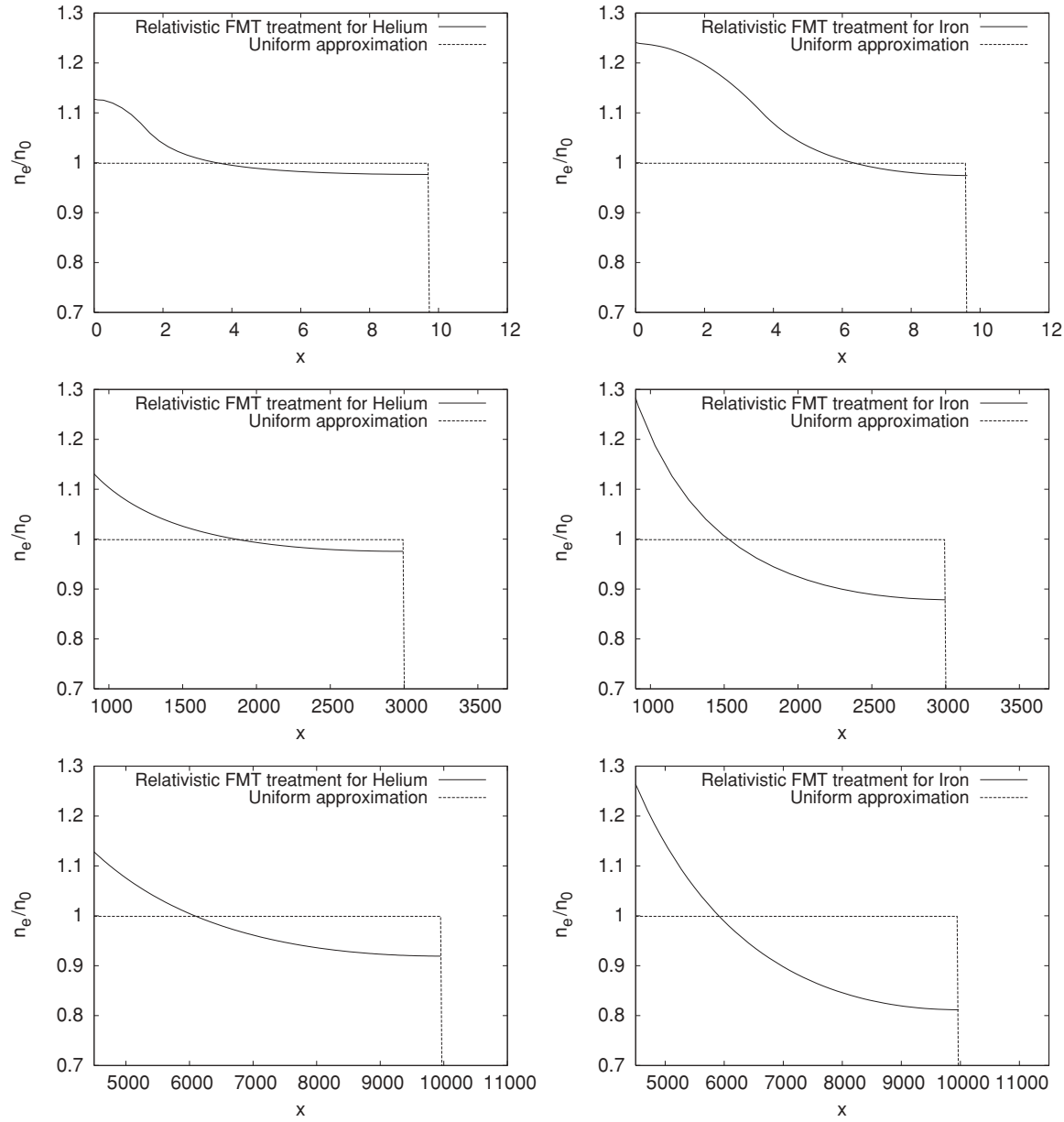

FIG. 5. The electron number density n_e in units of the average electron number density $n_0 = 3N_e/(4\pi R_{\rm WS}^3)$ is plotted as a function of the dimensionless radial coordinate $x = r/\lambda_\pi$ for the selected compressions $x_{\rm WS} = 9.7$ (upper panels), $x_{\rm WS} = 3 \times 10^3$ (middle panels) and $x_{\rm WS} = 10^4$ (bottom panels), in both the relativistic Feynman-Metropolis-Teller approach and the uniform approximation respectively for helium (panels on the left) and iron (panels on the right).

which is attained when $R_{\rm WS}$ coincides with the nuclear radius R_c. Here, the maximum electron Fermi energy (42) is obtained by replacing in Eq. (41) the value of the normalized Wigner-Seitz cell radius $x_{\rm WS} = x_c = R_c/\lambda_\pi \approx [(3/2)\pi]^{1/3} A^{1/3}$.

B. Relativistic FMT treatment vs. Salpeter approximate treatment

Corrections to the uniform distribution were also studied by Salpeter [19] and his approximations are largely applied in physics (see, e.g., Chabrier and Potekhin [28] and Potekhin, Chabrier, and Rogers [29]) and astrophysics (see, e.g., Douchin and Haensel [30] and Haensel and Zdunik [31–33]).

Keeping the pointlike nucleus assumption, Salpeter [19] studied the corrections to the above models due to the inhomogeneity of the electron distribution inside the Wigner-Seitz cell. He expressed an analytic formula for the total energy of a Wigner-Seitz cell based on Coulomb corrections to the uniform distribution of electrons. The first correction corresponds to the inclusion of the lattice energy $E_C = -(9N_p^2\alpha)/(10R_{\rm WS})$, which results from the pointlike nucleus-electron Coulomb interaction and from the electron-electron Coulomb interaction inside the cell of radius $R_{\rm ws}$. The second correction is given

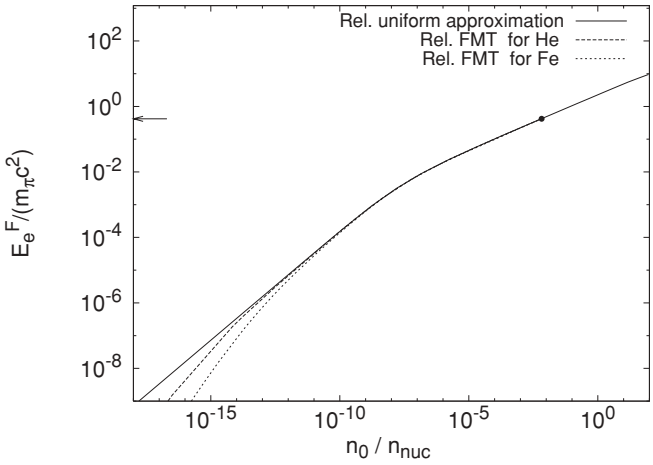

FIG. 6. The electron Fermi energy E_e^F in units of the pion rest energy is plotted as a function of the average electron density $n_0 = 3N_e/(4\pi R_{\rm WS}^3)$ in units of the nuclear density $n_{\rm nuc} = 3A/(4\pi\Delta^3 N_p \lambda_\pi^3)$ for a uniform approximation (solid line) compared and contrasted to the ones obtained considering the relativistic Feynman-Metropolis-Teller approach. The arrow and the dot indicate the value of the maximum electron Fermi energy as given by Eq. (42), consistent with the finite size of the nucleus.

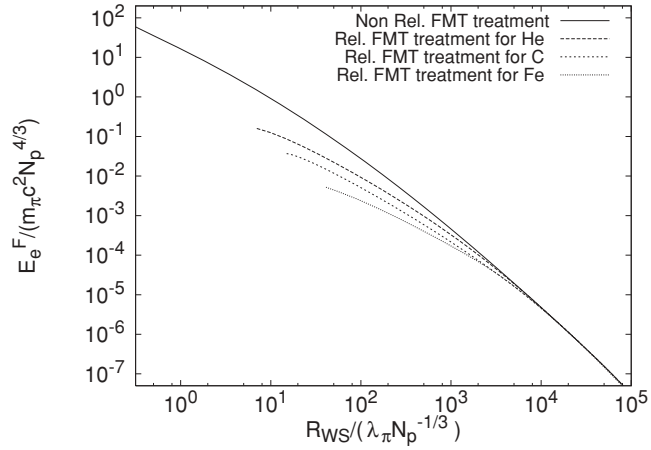

FIG. 7. The electron Fermi energies in units of $m_\pi c^2 N_p^{4/3}$ for helium, carbon, and iron are plotted as a function of the ratio $R_{\rm WS}/(\lambda_\pi N_p^{-1/3})$ respectively in the nonrelativistic and in the relativistic Feynman-Metropolis-Teller (FMT) treatment. The dimensionless quantities have been chosen in order to obtain a universal curve in the nonrelativistic treatment following Eqs. (10) and (11). The relativistic treatment leads to results where the electron Fermi energy is dependent on the nuclear composition and systematically smaller than the nonrelativistic ones. The electron Fermi energy can attain arbitrary large values, in the nonrelativistic treatment, as the pointlike nucleus is approached.

by a series expansion of the electron Fermi energy about the average electron density n_e given by Eq. (40) the uniform approximation $n_e = 3N_p/(4\pi R_{\rm WS}^3)$. The electron density is then assumed to be equal to $n_e[1+\epsilon(r)]$ with $\epsilon(r)$ considered as infinitesimal. The Coulomb potential energy is assumed to be the one of the pointlike nuclei with the uniform distribution of electrons of density n_e, thus the correction given by $\epsilon(r)$ is neglected on the Coulomb potential. The electron distribution is then calculated at first order by expanding the relativistic electron kinetic energy

$$\epsilon_k = \sqrt{\left[cP_e^F(r)\right]^2 + m_e^2c^4} - m_ec^2 = \sqrt{(3\pi^2 n_e)^{2/3}[1+\epsilon(r)]^{2/3} + m_e^2c^4} - m_ec^2, \quad (43)$$

about its value given by the uniform approximation

$$\epsilon_k^{\rm unif} = \sqrt{(3\pi^2 n_e)^{2/3} + m_e^2c^4} - m_ec^2, \quad (44)$$

considering as infinitesimal the ratio eV/E_e^F between the Coulomb potential energy eV and the electron Fermi energy $E_e^F = \sqrt{[cP_e^F(r)]^2 + m_e^2c^4} - m_ec^2 - eV$. The effect of the Dirac electron-exchange correction [24] on the equation of state was also considered by Salpeter [19]. However, adopting the general approach of Migdal *et al.* [8], these effects are negligible in the relativistic regime (see Sec. III C).

The inclusion of each additional Coulomb correction results in a decreasing of the pressure of the cell P_S (see Ref. [20] for details). However, despite the fact that it is very interesting in identifying piecewise contributions to the total pressure, the validity of the Salpeter approach needs verification through a more general treatment. For instance, the failure of the Salpeter formulas can be seen at densities of the order of $\sim 10^2$–10^3 g cm^{-3} for nuclei with large N_p, as in the case of iron, where the pressure becomes negative (see Table I). Therefore, the problem of solving the relativistic Thomas-Fermi equation within the Feynman-Metropolis-Teller approach becomes a necessity, since this approach gives all the possible Coulomb and relativistic contributions automatically and correctly.

C. Relativistic FMT treatment versus nonrelativistic FMT treatment

We now compare and contrast the Fermi energy, given by Eq. (32), of a compressed atom in the nonrelativistic and the relativistic limit (see Fig. 7). There are the following major differences:

(i) The electron Fermi energy in the relativistic treatment is strongly dependent on the nuclear composition, while the nonrelativistic treatment presents a universal behavior in the units of Fig. 7. In the limit of low densities the relativistic curves approach the universal nonrelativistic curve. In the nonrelativistic treatment the ratio $E_e^F/(m_\pi c^2 N_p^{4/3})$ does not depend on the number of protons N_p if the Wigner-Seitz cell radius $R_{\rm WS}$ is multiplied by $N_p^{1/3}$ [see Eqs. (10) and (11)]. This universality is lost in the relativistic treatment since there is no way to eliminate the dependence of the electron Fermi energy on the nuclear composition (see Eq. (28) and [5]).

(ii) The relativistic treatment leads to values of the electron Fermi energy consistently smaller than the ones of the nonrelativistic treatment.

(iii) While in the nonrelativistic treatment the electron Fermi energy can reach, by compression, infinite values as $R_{\rm WS} \to 0$, in the relativistic treatment it reaches a perfectly finite value given by Eq. (42) attained when $R_{\rm WS}$ coincides with the nuclear radius R_c.

It is clear then, from above considerations, the relativistic treatment of the Thomas-Fermi equation introduces significant differences from the current approximations in the literature: (a) the uniform electron distribution [27], (b) the approximate perturbative solutions departing from the uniform distribution [19], and (c) the nonrelativistic treatment [21]. We have recently applied these results of the relativistic Feynman-Metropolis-Teller treatment of a compressed atom to the study of white dwarfs and their consequences on the determination of their masses, radii, and critical mass [20].

V. APPLICATION TO NUCLEAR MATTER CORES OF STELLAR DIMENSIONS

We turn now to nuclear matter cores of stellar dimensions of $A \simeq (m_{\rm Planck}/m_n)^3 \sim 10^{57}$ or $M_{\rm core} \sim M_\odot$.

Following the treatment presented in Popov *et al.* [13], we use the existence of scaling laws and proceed to the ultrarelativistic limit of Eqs. (20), (28), (31), and (33). For positive values of the electron Fermi energy E_e^F, we introduce the new function $\phi = 4^{1/3}(9\pi)^{-1/3}\chi\Delta/x$ and the new variable $\hat{x} = kx$, where $k = (12/\pi)^{1/6}\sqrt{\alpha}\Delta^{-1}$, as well as the variable $\xi = \hat{x} - \hat{x}_c$ in order to describe better the region around the core radius.

Equation (28) becomes

$$\frac{d^2\hat{\phi}(\xi)}{d\xi^2} = -\theta(-\xi) + \hat{\phi}(\xi)^3, \tag{45}$$

where $\hat{\phi}(\xi) = \phi(\xi + \hat{x}_c)$ and the curvature term $2\hat{\phi}'(\xi)/(\xi + \hat{x}_c)$ has been neglected.

The Coulomb potential energy is given by

$$eV(\xi) = \left(\frac{9\pi}{4}\right)^{1/3}\frac{1}{\Delta}m_\pi c^2\hat{\phi}(\xi) - E_e^F, \tag{46}$$

corresponding to the electric field

$$E(\xi) = -\left(\frac{3^5\pi}{4}\right)^{1/6}\frac{\sqrt{\alpha}}{\Delta^2}\frac{m_\pi^2 c^3}{e\hbar}\hat{\phi}'(\xi) \tag{47}$$

and the electron number-density

$$n_e(\xi) = \frac{1}{3\pi^2\hbar^3c^3}\left(\frac{9\pi}{4}\right)\frac{1}{\Delta^3}(m_\pi c^2)^3\hat{\phi}^3(\xi). \tag{48}$$

In the core center we must have $n_e = n_p$. From Eqs. (20) and (48) we then have $\xi = -\hat{x}_c$, $\hat{\phi}(-\hat{x}_c) = 1$.

In order to consider a compressed nuclear density core of stellar dimensions, we then introduce a Wigner-Seitz cell determining the outer boundary of the electron distribution which, in the new radial coordinate ξ is characterized by $\xi^{\rm WS}$. In view of the global charge neutrality of the system the electric field goes to zero at $\xi = \xi^{\rm WS}$. This implies, from Eq. (47), that $\hat{\phi}'(\xi^{\rm WS}) = 0$.

We now turn to the determination of the Fermi energy of the electrons in this compressed core. The function $\hat{\phi}$ and its first derivative $\hat{\phi}'$ must be continuous at the surface $\xi = 0$ of the nuclear density core. This boundary-value problem can be solved analytically and indeed Eq. (45) has the first integral,

$$2[\hat{\phi}'(\xi)]^2 = \begin{cases} \hat{\phi}^4(\xi) - 4\hat{\phi}(\xi) + 3, & \xi < 0, \\ \hat{\phi}^4(\xi) - \phi^4(\xi^{\rm WS}), & \xi > 0, \end{cases} \tag{49}$$

with boundary conditions at $\xi = 0$:

$$\hat{\phi}(0) = \frac{\hat{\phi}^4(\xi^{\rm WS}) + 3}{4}, \quad \hat{\phi}'(0) = -\sqrt{\frac{\hat{\phi}^4(0) - \hat{\phi}^4(\xi^{\rm WS})}{2}}. \tag{50}$$

Having fulfilled the continuity condition we integrate Eq. (49), obtaining for $\xi \leqslant 0$

$$\hat{\phi}(\xi) = 1 - 3[1 + 2^{-1/2}\sinh(a - \sqrt{3}\xi)]^{-1}, \tag{51}$$

where the integration constant a has the value

$$\sinh(a) = \sqrt{2}\left[\frac{11 + \hat{\phi}^4(\xi^{\rm WS})}{1 - \hat{\phi}^4(\xi^{\rm WS})}\right]. \tag{52}$$

In the interval $0 \leqslant \xi \leqslant \xi^{\rm WS}$, the field $\hat{\phi}(\xi)$ is implicitly given by

$$F\left[\arccos\frac{\hat{\phi}(\xi^{\rm WS})}{\hat{\phi}(\xi)}, \frac{1}{\sqrt{2}}\right] = \hat{\phi}(\xi^{\rm WS})(\xi - \xi^{\rm WS}), \tag{53}$$

where $F(\varphi, k)$ is the elliptic function of the first kind and $F(0, k) \equiv 0$. For $F(\varphi, k) = u$, the inverse function $\varphi = F^{-1}(u, k) = {\rm am}(u, k)$ is the well-known Jacobi amplitude. In terms of it, we can express the solution (53) for $\xi > 0$ as,

$$\hat{\phi}(\xi) = \hat{\phi}(\xi^{\rm WS})\left(\cos\left\{{\rm am}\left[\hat{\phi}(\xi^{\rm WS})(\xi - \xi^{\rm WS}), \frac{1}{\sqrt{2}}\right]\right\}\right)^{-1}. \tag{54}$$

In the present case of $E_e^F > 0$ the ultrarelativistic approximation is indeed always valid up to $\xi = \xi^{\rm WS}$ for high compression factors, i.e., for $R_{\rm WS} \simeq R_c$. In the case of $E_e^F = 0$, $\xi^{\rm WS} \to \infty$, there is a breakdown of the ultrarelativistic approximation when $\xi \to \xi^{\rm WS}$. Details are given in Figs. 8, 9, and 10.

We can now estimate two crucial quantities of the solutions: the Coulomb potential at the center of the configuration and the electric field at the surface of the core:

$$eV(0) \simeq \left(\frac{9\pi}{4}\right)^{1/3}\frac{1}{\Delta}m_\pi c^2 - E_e^F, \tag{55}$$

$$E_{\rm max} \simeq 2.4\frac{\sqrt{\alpha}}{\Delta^2}\left(\frac{m_\pi}{m_e}\right)^2 E_c|\hat{\phi}'(0)|, \tag{56}$$

where $E_c = m_e^2c^3/(e\hbar)$ is the critical electric field for vacuum polarization. These functions depend on the value $\hat{\phi}(\xi^{\rm WS})$ via Eqs. (49)–(53). At the boundary $\xi = \xi^{\rm WS}$, due to the global charge neutrality, both the electric field $E(\xi^{\rm WS})$ and the Coulomb potential $eV(\xi^{\rm WS})$ vanish. From Eq. (46), we determine the value of $\hat{\phi}(\xi)$ at $\xi = \xi^{\rm WS}$

$$\hat{\phi}(\xi^{\rm WS}) = \Delta\left(\frac{4}{9\pi}\right)^{1/3}\frac{E_e^F}{m_\pi c^2}, \tag{57}$$

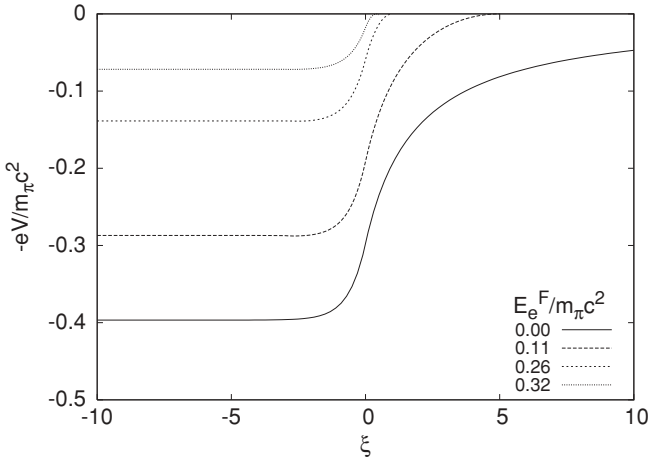

FIG. 8. The electron Coulomb potential energies, in units of the pion rest energy in a nuclear matter core of stellar dimensions with $A \simeq 10^{57}$ or $M_{\rm core} \sim M_\odot$ and $R_c \approx 10^6$ cm, are plotted as a function of the dimensionless variable ξ for different values of the electron Fermi energy, also in units of the pion rest energy. The solid line corresponds to null electron Fermi energy. By increasing the value of the electron Fermi energy the electron Coulomb potential energy depth is reduced.

as a function of the electron Fermi energies E_e^F. From the above Eq. (57), one can see that there exists a solution, characterized by the value of electron Fermi energy

$$\frac{(E_e^F)_{\rm max}}{m_\pi c^2} = \frac{1}{\Delta}\left(\frac{9\pi}{4}\right)^{1/3}, \tag{58}$$

such that $\hat{\phi}(\xi^{\rm WS}) = 1$. From Eq. (53) and $\xi = 0$, we also have

$$\xi^{\rm WS}[\hat{\phi}(\xi^{\rm WS})] = \left\{\frac{1}{\hat{\phi}(0)} F\left[\arccos\left(4 - \frac{3}{\hat{\phi}(0)}\right), \frac{1}{\sqrt{2}}\right]\right\}. \tag{59}$$

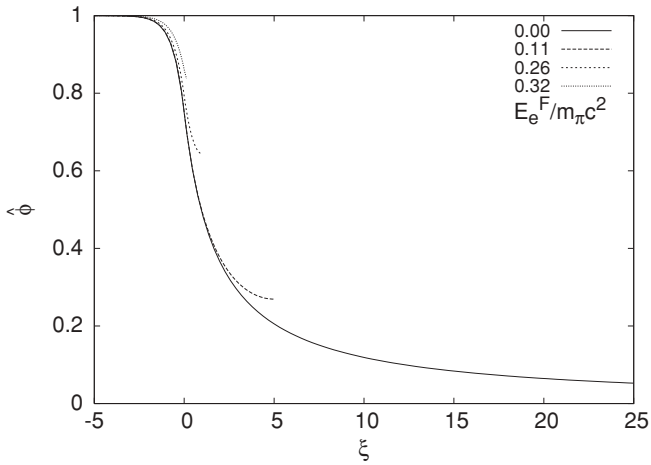

FIG. 9. Solutions of the ultrarelativistic Thomas-Fermi equation (45) for different values of the Wigner-Seitz cell radius $R_{\rm WS}$ and, correspondingly, of the electron Fermi energy in units of the pion rest energy as in Fig. 8, near the core surface. The solid line corresponds to null electron Fermi energy.

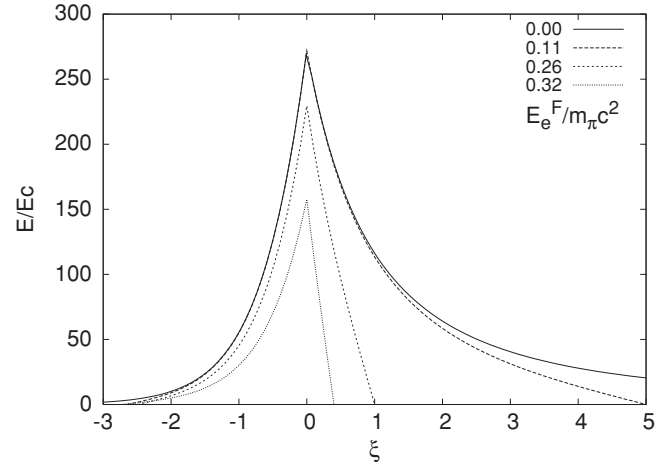

FIG. 10. The electric field in units of the critical field for vacuum polarization $E_c = m_e^2 c^3/(e\hbar)$ is plotted as a function of the coordinate ξ for different values of the electron Fermi energy in units of the pion rest energy. The solid line corresponds to the case of null electron Fermi energy. When the value of the electron Fermi energy increases there is a reduction of the peak of the electric field.

For $\hat{\phi}(\xi^{\rm WS}) = 1$, from Eq. (50) follows $\hat{\phi}(0) = 1$ hence Eq. (59) becomes

$$\xi^{\rm WS}[\hat{\phi}(0)] = F\left[0, \frac{1}{\sqrt{2}}\right]. \tag{60}$$

It is well known that if the inverse Jacobi amplitude $F[0, 1/\sqrt{2}]$ is zero, then

$$\xi^{\rm WS}[\hat{\phi}(\xi^{\rm WS}] = \hat{\phi}(0) = 1) = 0. \tag{61}$$

Indeed from $\hat{\phi}(\xi^{\rm WS}) = 1$ follows $\hat{\phi}(0) = 1$ and $\xi^{\rm WS} = 0$. When $\xi^{\rm WS} = 0$ from Eq. (50) follows $\hat{\phi}'(0) = 0$ and, using Eq. (56), $E_{\rm max} = 0$. In other words, for the value of E_e^F fulfilling Eq. (57), no electric field exists on the boundary of the core and from Eq. (48) and Eqs. (19) and (20) it follows that indeed this is the solution fulfilling both global $N_e = N_p$ and local $n_e = n_p$ charge neutrality. In this special case, starting from Eq. (33) and $A = N_p + N_n$, we obtain

$$\left(E_e^F\right)_{\rm max}^{3/2} = \frac{\frac{9\pi}{4}(\hbar c)^3 \frac{A}{R_c^3} - \left(E_e^F\right)_{\rm max}^3}{2^{3/2}\left[\left(\frac{9\pi}{4}(\hbar c)^3 \frac{A}{R_c^3} - \left(E_e^F\right)_{\rm max}^3\right)^{2/3} + m_n^2 c^4\right]^{3/4}}. \tag{62}$$

In the ultrarelativistic approximation $(E_e^F)_{\rm max}^3 / \frac{9\pi}{4}(\hbar c)^3 \frac{A}{R_c^3} \ll 1$ so Eq. (62) can be approximated to

$$\left(E_e^F\right)_{\rm max} = 2^{1/3}\frac{m_n}{m_\pi}\gamma\left[-1 + \sqrt{1 + \frac{\beta}{2\gamma^3}}\right]^{2/3} m_\pi c^2, \tag{63}$$

where

$$\beta = \frac{9\pi}{4}\left(\frac{\hbar}{m_n c}\right)^3 \frac{A}{R_c^3}, \qquad \gamma = \sqrt{1 + \beta^{2/3}}. \tag{64}$$

The corresponding limiting value to the N_p/A ratio is obtained as follows:

$$\frac{N_p}{A} = \frac{2\gamma^3}{\beta}\left[-1+\sqrt{1+\frac{\beta}{2\gamma^3}}\right]^2. \tag{65}$$

Inserting Eqs. (63) and (64) into Eq. (65) one obtains the ultrarelativistic limit of Eq. (42), since the electron Fermi energy, in view of the scaling laws introduced in Ref. [13], is independent of the value of A and depends only on the density of the core.

The N_p independence in the limiting case of maximum electron Fermi energy attained when $R_{\rm WS} = R_c$, in which the ultrarelativistic treatment approaches the uniform one, and the N_p dependence for smaller compressions $R_{\rm WS} > R_c$ can be understood as follows. Let see the solution to the ultrarelativistic equation (45) for small $\xi > 0$. Analogously to the Feynman-Metropolis-Teller approach to the nonrelativistic Thomas-Fermi equation, we solve the ultrarelativistic equation (45) for small ξ. Expanding $\hat{\phi}(\xi)$ about $\xi = 0$ in a semiconvergent power series

$$\frac{\hat{\phi}(\xi)}{\hat{\phi}(0)} = 1 + \sum_{n=2}^{\infty} a_n \xi^{n/2} \tag{66}$$

and substituting it into the ultrarelativistic equation (45), we have

$$\sum_{k=3}^{\infty} a_k \frac{k(k-2)}{4}\xi^{(k-4)/2} = \phi^2(0)\exp\left[3\ln\left(1+\sum_{n=2}^{\infty} a_n \xi^{n/2}\right)\right]. \tag{67}$$

This leads to a recursive determination of the coefficients:

$$\begin{aligned} &a_3 = 0, \quad a_4 = \phi^2(0)/2, \quad a_5 = 0, \\ &a_6 = \phi^2(0)a_2/2, \quad a_7 = 0, \quad a_8 = \phi^2(0)\big(1-a_2^2\big)/8, \ldots, \end{aligned} \tag{68}$$

with $a_2 = \hat{\phi}'(0)/\hat{\phi}(0)$ determined by the initial slope, namely the boundary condition $\hat{\phi}'(0)$ and $\hat{\phi}(0)$ in Eq. (50):

$$\hat{\phi}(0) = \frac{\hat{\phi}^4(\xi^{\rm WS})+3}{4}, \quad \hat{\phi}'(0) = -\sqrt{\frac{\hat{\phi}^4(0)-\hat{\phi}^4(\xi^{\rm WS})}{2}}. \tag{69}$$

Thus the series solution (66) is uniquely determined by the boundary value $\hat{\phi}(\xi^{\rm WS})$ at the Wigner-Seitz cell radius.

Now we consider the solution up to the leading orders

$$\begin{aligned} \hat{\phi}(\xi) = \hat{\phi}(0) + \hat{\phi}'(0)\xi + \tfrac{1}{2}\hat{\phi}^3(0)\xi^2 + \tfrac{1}{2}\hat{\phi}^3(0)a_2\xi^3 \\ + \tfrac{1}{8}\hat{\phi}^3(0)\big(1-a_2^2\big)\xi^4 + \cdots. \end{aligned} \tag{70}$$

Using Eq. (70), the electron Fermi energy (57) becomes

$$\begin{aligned} E_e^F = \big(E_e^F\big)_{\rm max}\big[1 + a_2\xi^{\rm WS} + \tfrac{1}{2}\hat{\phi}^2(0)(\xi^{\rm WS})^2 + \tfrac{1}{2}\hat{\phi}^2(0)a_2(\xi^{\rm WS})^3 \\ + \tfrac{1}{8}\hat{\phi}^2(0)\big(1-a_2^2\big)(\xi^{\rm WS})^4 + \cdots\big]\hat{\phi}(0), \end{aligned} \tag{71}$$

where $(E_e^F)_{\rm max} = (9\pi/4)^{1/3}\Delta^{-1}$ is the maximum Fermi energy which is attained when the Wigner-Seitz cell radius equals the nucleus radius R_c [see Eq. (58)]. For $\hat{\phi}(\xi^{\rm WS}) < 1$, we approximately have $\hat{\phi}(0) = 3/4$, $\hat{\phi}'(0) = -(3/4)^2/\sqrt{2}$ and the initial slope $a_2 = \hat{\phi}'(0)/\hat{\phi}(0) = -(3/4)/\sqrt{2}$. Therefore Eq. (71) becomes

$$\begin{aligned} E_e^F \approx \big(E_e^F\big)_{\rm max}\Bigg[1 - \frac{3}{4\sqrt{2}}\xi^{\rm WS} + \frac{1}{2}\left(\frac{3}{4}\right)^2(\xi^{\rm WS})^2 \\ - \frac{1}{2^{3/2}}\left(\frac{3}{4}\right)^3(\xi^{\rm WS})^3 + \frac{1}{8}\left(\frac{3}{4}\right)^2\left(\frac{41}{32}\right)(\xi^{\rm WS})^4 + \cdots\Bigg]. \end{aligned} \tag{72}$$

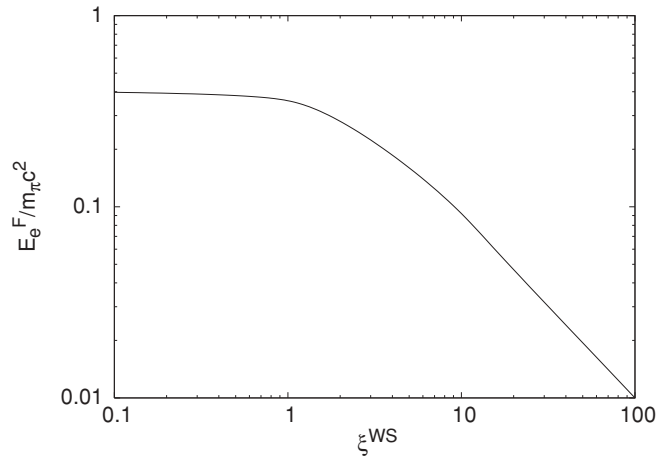

FIG. 11. The Fermi energy of electrons in units of the pion rest energy is plotted for different Wigner-Seitz cell dimensions (i.e., for different compressions) $\xi^{\rm WS}$ in the ultrarelativistic approximation. In the limit $\xi^{\rm WS} \to 0$ the electron Fermi energy approaches asymptotically the value $(E_e^F)_{\rm max}$ given by Eq. (63).

By the definition of the coordinate ξ, we know all terms except the first term in the square bracket depend on the values of N_p. In the limit of maximum compression when the electron Fermi energy acquires its maximum value, namely when $\xi^{\rm WS} = 0$, the electron Fermi energy (72) is the same as the one obtained from the uniform approximation which is independent of N_p. For smaller compressions, namely for $\xi^{\rm WS} > 0$, the electron Fermi energy deviates from the one given by the uniform approximation, becoming N_p dependent.

In Fig. 11 we plot the Fermi energy of electrons, in units of the pion rest energy as a function of the dimensionless parameter $\xi^{\rm WS}$ and, as $\xi^{\rm WS} \to 0$, the limiting value given by Eq. (63) is clearly displayed.

In Alcock, Farhi, and Olinto [9], in order to study the electrodynamical properties of strange stars, the ultrarelativistic Thomas-Fermi equation was numerically solved in the case of bare strange stars as well as in the case of strange stars with a crust (see, e.g., curves (a) and (b) in Fig. 6 of Ref. [9]). Figure 6 of Ref. [9] plots what they called the Coulomb potential energy, which we will denote as $V_{\rm Alcock}$. The potential $V_{\rm Alcock}$ was plotted for different values of the electron Fermi momentum at the edge of the crust. Actually, such a potential $V_{\rm Alcock}$ is not the Coulomb potential eV but it coincides with our function $e\hat{V} = eV + E_e^F$; that is, the potential $V_{\rm Alcock}$ corresponds to the Coulomb potential shifted by the the Fermi energy of the

electrons. We then have from Eq. (46)

$$e\hat{V}(\xi) = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{\Delta} m_\pi c^2 \hat{\phi}(\xi) = V_{\rm Alcock}. \quad (73)$$

This explains why in Ref. [9], for different values of the Fermi momentum at the crust, the depth of the potential $V_{\rm Alcock}$ remains unchanged. Instead, the correct behavior of the Coulomb potential differs markedly and, indeed, its depth decreases with increasing of compression, as can be seen in Fig. 8.

VI. COMPRESSIONAL ENERGY OF NUCLEAR MATTER CORES OF STELLAR DIMENSIONS

We turn now to the compressional energy of these family of compressed nuclear matter cores of stellar dimensions, each characterized by a different Fermi energy of the electrons. The kinematic energy spectra of complete degenerate electrons, protons, and neutrons are

$$\epsilon^i(p) = \sqrt{(pc)^2 + m_i^2 c^4}, \quad p \leqslant P_i^F, \quad i = e, p, n. \quad (74)$$

So the compressional energy of the system is given by

$$\mathcal{E} = \mathcal{E}_B + \mathcal{E}_e + \mathcal{E}_{\rm em}, \qquad \mathcal{E}_B = \mathcal{E}_p + \mathcal{E}_n, \quad (75)$$

$$\mathcal{E}_{\rm i} = 2\int_i \frac{d^3r d^3p}{(2\pi\hbar)^3}\epsilon^i(p), \quad i = e, p, n, \quad \mathcal{E}_{\rm em} = \int \frac{E^2}{8\pi} d^3r. \quad (76)$$

Using the analytic solution (54) we calculate the energy difference between two systems, I and II,

$$\Delta\mathcal{E} = \mathcal{E}\left[E_e^F(II)\right] - \mathcal{E}\left[E_e^F(I)\right], \quad (77)$$

with $E_e^F(II) > E_e^F(I) \geqslant 0$ at fixed A and R_c.

We first consider the infinitesimal variation of the total energy $\delta\mathcal{E}_{\rm tot}$ with respect to the infinitesimal variation of the electron Fermi energy δE_e^F

$$\delta\mathcal{E} = \left[\frac{\partial\mathcal{E}}{\partial N_p}\right]_{V^{\rm WS}} \left[\frac{\partial N_p}{\partial E_e^F}\right] \delta E_e^F + \left[\frac{\partial\mathcal{E}}{\partial V^{\rm WS}}\right]_{N_p} \left[\frac{\partial V^{\rm WS}}{\partial E_e^F}\right] \delta E_e^F. \quad (78)$$

For the first term of this relation we have

$$\left[\frac{\partial\mathcal{E}}{\partial N_p}\right]_{V^{\rm WS}} = \left[\frac{\partial\mathcal{E}_p}{\partial N_p} + \frac{\partial\mathcal{E}_n}{\partial N_p} + \frac{\partial\mathcal{E}_e}{\partial N_p} + \frac{\partial\mathcal{E}_{\rm em}}{\partial N_p}\right]_{V^{\rm WS}} \simeq \left[E_p^F - E_n^F + E_e^F + \frac{\partial\mathcal{E}_{\rm em}}{\partial N_p}\right]_{V^{\rm WS}}, \quad (79)$$

where the general definition of chemical potential $\partial\epsilon_i/\partial n_i = \partial\mathcal{E}_i/\partial N_i$ is used ($i = e, p, n$) neglecting the mass defect $m_n - m_p - m_e$. Further using the condition of the β equilibrium (33) we have

$$\left[\frac{\partial\mathcal{E}}{\partial N_p}\right]_{V^{\rm WS}} = \left[\frac{\partial\mathcal{E}_{\rm em}}{\partial N_p}\right]_{V^{\rm WS}}. \quad (80)$$

For the second term of the Eq. (78) we have

$$\left[\frac{\partial\mathcal{E}}{\partial V^{\rm WS}}\right]_{N_p} = \left[\frac{\partial\mathcal{E}_p}{\partial V^{\rm WS}} + \frac{\partial\mathcal{E}_n}{\partial V^{\rm WS}} + \frac{\partial\mathcal{E}_e}{\partial V^{\rm WS}} + \frac{\partial\mathcal{E}_{\rm em}}{\partial V^{\rm WS}}\right]_{N_p} = \left[\frac{\partial\mathcal{E}_e}{\partial V^{\rm WS}}\right]_{N_p} + \left[\frac{\partial\mathcal{E}_{\rm em}}{\partial V^{\rm WS}}\right]_{N_p}, \quad (81)$$

since in the process of increasing the electron Fermi energy, namely by decreasing the radius of the Wigner-Seitz cell, the system by definition maintains the same number of baryons A and the same core radius R_c.

Now $\delta\mathcal{E}$ reads

$$\delta\mathcal{E} = \left\{\left[\frac{\partial\mathcal{E}_e}{\partial V^{\rm WS}}\right]_{N_p} \frac{\partial V^{\rm WS}}{\partial E_e^F} + \left[\frac{\partial\mathcal{E}_{\rm em}}{\partial V^{\rm WS}}\right]_{N_p} \frac{\partial V^{\rm WS}}{\partial E_e^F} + \left[\frac{\partial\mathcal{E}_{\rm em}}{\partial N_p}\right]_{V^{\rm WS}} \frac{\partial N_p}{\partial E_e^F}\right\} \delta E_e^F, \quad (82)$$

so only the electromagnetic energy and the electron energy give non-null contributions.

From this equation it follows that

$$\Delta\mathcal{E} = \Delta\mathcal{E}_{\rm em} + \Delta\mathcal{E}_{\rm e}, \quad (83)$$

where $\Delta\mathcal{E}_{\rm em} = \mathcal{E}_{\rm em}[E_e^F(II)] - \mathcal{E}_{\rm em}[E_e^F(I)]$ and $\Delta\mathcal{E}_{\rm e} = \mathcal{E}_e[E_e^F(II)] - \mathcal{E}_e[E_e^F(I)]$.

In the particular case in which $E_e^F(II) = (E_e^F)_{\rm max}$ and $E_e^F(I) = 0$ we obtain

$$\Delta\mathcal{E} \simeq 0.75\frac{3^{5/3}}{2}\left(\frac{\pi}{4}\right)^{1/3} \frac{1}{\Delta\sqrt{\alpha}} \left(\frac{\pi}{12}\right)^{1/6} N_p^{2/3} m_\pi c^2, \quad (84)$$

which is positive.

The compressional energy of a nuclear matter core of stellar dimensions increases with its electron Fermi energy as expected.

VII. CONCLUSIONS

The results presented in this article are in the realm of theoretical physics of nuclear physics and of atomic physics and give special attention to relativistic effects. They generalize to the relativistic regimes classical results obtained by Feynman, Metropolis, and Teller [21] and, by the introduction of scaling laws, they generalize the classical results obtained by Popov and collaborators [7,8,13] in heavy nuclei to massive cores of $\sim M_\odot$. As such they find their justification. They acquire also special meaning in astrophysics: The considerations contained in Secs. I– IV lead to a consistent treatment of white dwarfs and the ones in Secs. V and VI lead to a deeper understanding of neutron star physics.

We have generalized to the relativistic regime the classic work of Feynman, Metropolis, and Teller by solving the relativistic Thomas-Fermi equation in a Wigner-Seitz cell corresponding to a compressed atom. The relativistic Thomas-Fermi equation has also been called the Vallarta-Rosen equation [6,34]. The integration of this equation does not admit regular solutions for a pointlike nucleus and both the nuclear radius and the nuclear composition have necessarily

to be taken into account [5,6]. This introduces a fundamental difference from the nonrelativistic Thomas-Fermi model where a pointlike nucleus is traditionally adopted.

As in previous works by Ferreirinho, Ruffini, and Stella [5], Ruffini and Stella [6], and Ruffini *et al.* [12], the protons in the nuclei have been assumed to be at constant density and the electron distribution has been derived by the relativistic Thomas-Fermi equation and the neutron component by the β equilibrium among neutrons, protons, and electrons.

We have examined, for completeness, the relativistic generalization of the Thomas-Fermi-Dirac equation by taking into account the exchange terms [24], adopting the general approach of Migdal, Popov, and Voskresenskii [8], and we have shown that these effects, generally small, can be neglected in the relativistic treatment.

There are marked differences between the relativistic and the nonrelativistic treatments. The first is that the existence of a finite-size nucleus introduces a limit to the compressibility: The dimension of the Wigner-Seitz cell can never be smaller then the nuclear size. Consequently, the electron Fermi energy which in the nonrelativistic approach can reach arbitrarily large values reaches in the present case a perfectly finite value: An expression for this finite value of the electron Fermi energy has been given in analytic form. There are in the literature many papers adopting a relativistic treatment for the electrons, with a pointlike approximation for the nucleus, which are clearly inconsistent (see, e.g., Chabrier and Potekhin [28] and Potekhin, Chabrier, and Rogers [29]).

The second is the clear difference of the electron distribution as a function of the radius and of the nuclear composition as contrasted to the uniform approximation (see Fig. 5 in Sec. IV), often adopted in the literature (see, e.g., Bürvenich, Mishustin, and Greiner [27]). Therefore the validity of inferences based on a uniform approximation should be duly verified both in the relativistic and in the nonrelativistic regime.

The third is that the relativistic Feynman-Metropolis-Teller treatment allows us to treat precisely the electrodynamical interaction within a compressed atom with all the relativistic corrections. This allows us to validate and in some cases overcome the difficulties of treatments describing the electrodynamical effect by a sequence of successive approximations. This is the case of validation of the Salpeter approach at high densities and the overcome of negative pressures at low densities. The new treatment evidences a softening of the dependence of the electron Fermi energy on the compression factor, as well as a gradual decrease of the exchange terms in proceeding from the nonrelativistic to the fully relativistic regimes. It is then possible to derive, as shown in Table I in Sec. IV, a consistent equation of state for compressed matter.

The equation of state obtained in Table I has been recently applied to the study of the general relativistic white-dwarf equilibrium configurations by Rueda *et al.* [20]. The contribution of quantum statistics and weak and electromagnetic interactions here considered have been further generalized by considering the contribution of the general relativistic equilibrium of white-dwarf matter. This is expressed by the simple formula $\sqrt{g_{00}}\mu_{\rm ws} = \text{const.}$, which links the chemical potential of the Wigner-Seitz cell $\mu_{\rm ws}$ with the general relativistic gravitational potential g_{00} at each point of the configuration. The configuration outside each Wigner-Seitz cell is strictly neutral and therefore no global electric field is necessary to warranty the equilibrium of the white dwarf. These equations modify the ones used by Chandrasekhar by taking into account the Coulomb interaction between the nuclei and the electrons as well as inverse β decay. They also generalize the work of Salpeter by considering a unified self-consistent approach to the Coulomb interaction in each Wigner-Seitz cell. The consequences on the numerical value of the Chandrasekhar-Landau mass limit have been then presented as well as on the mass-radius relation of white dwarfs [20]. This leads to the possibility of a direct confrontation of these results with observations, in view of the current great interest for the cosmological implications of the type Ia supernovae [35–38] and in the case of low-mass white-dwarf companions of the Pulsar PSRJ1141-6545 [39] as well as the role of white dwarfs in novae [40].

In Secs. V and VI we have then extrapolated these results to the case of nuclear matter cores of stellar dimensions for $A \approx (m_{\rm Planck}/m_n)^3 \sim 10^{57}$ or $M_{\rm core} \sim M_\odot$. The aim here is to explore the possibility of obtaining for these systems a self-consistent solution presenting global and not local charge neutrality. The results generalize the considerations presented in the previous article by Popov *et al.* [13] corresponding to a nuclear matter core of stellar dimensions with null Fermi energy of the electrons. The ultrarelativistic approximation allows us to obtain analytic expressions for the fields in the case of positive electron Fermi energies. An entire family of configurations exist with values of the Fermi energy of the electrons ranging from zero to a maximum value $(E_e^F)_{\rm max}$ which is reached when the Wigner- Seitz cell coincides with the core radius. The configuration with $E_e^F = (E_e^F)_{\rm max}$ corresponds to the configuration with $N_p = N_e$ and $n_p = n_e$: For this limiting value of the Fermi energy the system fulfills both the global and the local charge neutrality and, correspondingly, no electrodynamical structure is present in the core. The other configurations present generally overcritical electric fields close to their surface. The configuration with $E_e^F = 0$ has the maximum value of the electric field at the core surface, well above the critical value E_c (see Figs. 8, 9, and 10). All these cores with overcritical electric fields are stable against the vacuum polarization process due to the Pauli blocking by the degenerate electrons (see, e.g., Ruffini, Vereshchagin, and Xue [41]). We have also compared and contrasted our treatment of the relativistic Thomas-Fermi solutions to the corresponding one addressed in the framework of strange stars by Alcock, Farhi, and Olinto [9], pointing out in these treatments some inconsistency in the definition of the Coulomb potential. We have finally compared the compressional energy of configurations with selected values of the electron Fermi energy.

The above problem is theoretically well defined and represents a necessary step in order to approach the more complex problem of a neutron star core and its interface with the neutron star crust.

Neutron stars are composed of two sharply different components: the liquid core at nuclear and/or supranuclear density consisting of neutrons, protons, and electrons and

TABLE I. Pressure for iron as a function of the density ρ in the uniform approximation (P), in the Salpeter approximation (P_S) and in the relativistic Feynman-Metropolis-Teller approach ($P_{\rm FMTrel}$). Here $x_S = P^F_{e,S}/(m_e c)$ and $x_{\rm FMTrel} = P^F_e/(m_e c)$ are respectively the normalized Salpeter Fermi momentum and the relativistic Feynmann-Metropolis-Teller Fermi momentum.

ρ (g/cm^3)	x_S	$x_{\rm FMTrel}$	P (bar)	P_S (bar)	$P_{\rm FMTrel}$ (bar)
2.63×10^2	0.05	0.0400	2.9907×10^{10}	-1.8800×10^8	9.9100×10^9
2.10×10^3	0.10	0.0857	9.5458×10^{11}	4.4590×10^{11}	5.4840×10^{11}
1.68×10^4	0.20	0.1893	3.0227×10^{13}	2.2090×10^{13}	2.2971×10^{13}
5.66×10^4	0.30	0.2888	2.2568×10^{14}	1.8456×10^{14}	1.8710×10^{14}
1.35×10^5	0.40	0.3887	9.2964×10^{14}	8.0010×10^{14}	8.0790×10^{14}
2.63×10^5	0.50	0.4876	2.7598×10^{15}	2.4400×10^{15}	2.4400×10^{15}
4.53×10^5	0.60	0.5921	6.6536×10^{15}	6.0040×10^{15}	6.0678×10^{15}
7.19×10^5	0.70	0.6820	1.3890×10^{16}	1.2693×10^{16}	1.2810×10^{16}
1.08×10^6	0.80	0.7888	2.6097×10^{16}	2.4060×10^{16}	2.4442×10^{16}
2.10×10^6	1.00	0.9853	7.3639×10^{16}	6.8647×10^{16}	6.8786×10^{16}
3.63×10^6	1.20	1.1833	1.6902×10^{17}	1.5900×10^{17}	1.5900×10^{17}
5.77×10^6	1.40	1.3827	3.3708×10^{17}	3.1844×10^{17}	3.1898×10^{17}
8.62×10^6	1.6	1.5810	6.0754×10^{17}	5.7588×10^{17}	5.7620×10^{17}
1.23×10^7	1.80	1.7790	1.0148×10^{18}	9.6522×10^{17}	9.6592×10^{17}
1.68×10^7	2.0	1.9770	1.5981×10^{18}	1.5213×10^{18}	1.5182×10^{18}
3.27×10^7	2.50	2.4670	4.1247×10^{18}	3.9375×10^{18}	3.9101×10^{18}
5.66×10^7	3.00	2.965	8.8468×10^{18}	8.4593×10^{18}	8.4262×10^{18}
1.35×10^8	4.00	3.956	2.9013×10^{19}	2.7829×10^{19}	2.7764×10^{19}
2.63×10^8	5.00	4.939	7.2160×10^{19}	6.9166×10^{19}	6.9062×10^{19}
8.85×10^8	7.50	7.423	3.7254×10^{20}	3.5700×10^{20}	3.5700×10^{20}

a crust of degenerate electrons in a lattice of nuclei (see, e.g., Baym, Bethe, and Pethick [1]) and Harrison *et al.* [42]) and possibly of free neutrons due to neutron drip when this process occurs (see, e.g., Ref. [1]). Consequently, the boundary conditions for the electrons at the surface of the neutron star core will have generally a positive value of the electron Fermi energy in order to take into account the compressional effects of the neutron star crust on the core [43]. The case of zero electron Fermi energy corresponds to the limiting case of absence of the crust.

In a set of interesting papers [44–49], Glendenning and collaborators relaxed the local charge neutrality condition for the description of the mixed phases in hybrid stars. In such configurations the global charge neutrality condition, as opposed to the local one, is applied to the limited regions where mixed phases occur while in the pure phases the local charge neutrality condition still holds. We have generalized Glendenning's considerations by looking to a violation of the local charge neutrality condition on the entire configuration, still keeping its overall charge neutrality. This effect cannot occur locally and requires a global description of the equilibrium configuration. To exemplify this novel approach we have considered in Rueda *et al.* [50] the simplest, nontrivial, self-gravitating system of degenerate neutrons, protons, and electrons in β equilibrium in the framework of relativistic quantum statistics and the Einstein-Maxwell equations. The impossibility of imposing the condition of local charge neutrality on such systems is proved in complete generality. The crucial role of the constancy of the generalized electron Fermi energy is emphasized and consequently the coupled system of the general relativistic Thomas-Fermi equations and the Einstein-Maxwell equations is solved. We then give an explicit solution corresponding to a violation of the local charge neutrality condition on the entire star, still fulfilling the global charge neutrality when electromagnetic, weak, and general relativistic effects are taken into account.

The results presented in the second part of this article on nuclear matter cores of stellar dimensions evidence the possibility of having the existence of critical electromagnetic fields at the core surface. We have further extended this analysis by considering the case of a neutron star. At nuclear and supranuclear densities we have considered a core described by a self-gravitating system of degenerate neutrons, protons, and electrons within the framework of relativistic quantum statistics and Einstein-Maxwell equations [43]. At densities lower than the nuclear density such a core is surrounded by a crust. A globally neutral neutron star configuration is therefore examined in contrast with the traditional ones constructed by imposing local charge neutrality. To illustrate the application of this approach we have adopted the Baym, Bethe, and Pethick [1] strong-interaction model of the baryonic matter in the core and in the white dwarflike material of the crust. The existence of an overcritical electric field at the boundary of the core predicted in the present paper is there confirmed. We find an electric field extending over a thin shell of thickness $\sim\hbar/(m_e c)$ between the core and the crust which largely overcritical in the limit of decreasing values of the crust mass [43].

We are further extending these treatments by describing the strong interaction between nucleons through σ-ω-ρ meson exchange in the context of the extended Walecka

model [51], all duly expressed in general relativity. We demonstrate that, as in the noninteracting case, the thermodynamic equilibrium condition given by the constancy of the Fermi energy of each particle species can be properly generalized to include the contribution of all fields [52].

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