Bulk viscosity of a pion gas

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We compute the bulk viscosity of a gas of pions at temperatures below the QCD crossover temperature, for the physical value of m_π, to lowest order in chiral perturbation theory. Bulk viscosity is controlled by number-changing processes which become exponentially slow at low temperatures when the pions become exponentially dilute, leading to an exponentially large bulk viscosity $\zeta \sim (F_0^8/m_\pi^5) \exp(2m_\pi/T)$, where $F_0 \simeq 93$ MeV is the pion decay constant.

I. INTRODUCTION

One of the most prominent discoveries of the heavy ion program at the BNL Relativistic Heavy Ion Collider (RHIC) has been the success of hydrodynamics [1] with a zero [2] or very small [3] viscosity. Though the exact value of the viscosity cannot yet be extracted due to uncertainties in the initial state and other effects, it is a robust result that the viscosity near the QCD crossover temperature is small, $\eta/s < 0.5$ [3]. On the other hand, perturbative calculations show that the viscosity to entropy ratio η/s at high temperatures $T \gg 1$ GeV, where perturbation theory should work, is significantly higher [4]. Both theoretical [5] and data-driven [6] analyses of the pion gas indicate that η/s also rises at low temperatures, suggesting that the relative viscosity bottoms out near the crossover [7], similar to the behavior in conventional fluids [8].

The bulk viscosity is also expected to be important in the hydrodynamics of heavy ion collisions [9]. Bulk viscosity vanishes for a conformal system, a good approximation to QCD at high temperatures; therefore the bulk viscosity to entropy ratio ζ/s is small at high temperatures [10]. Near the crossover temperature, QCD is very far from conformal, as indicated by the peak in $(\epsilon - 3P)/T^4$ [11,12], and it is expected that ζ/s may display a peak at this scale [13,14]. At lower temperatures, QCD is well described by a pion gas. Existing studies of pion gases indicate that the bulk viscosity falls away at low temperatures [5,6]. This suggests that the ratio ζ/s shows the opposite behavior of η/s, peaking near the transition and falling off to either side [15].

However, previous analyses of the bulk viscosity of a pion gas have been very incomplete. In particular, neither standard reference [5,6] considers number-changing processes. But such processes are essential to the relaxation of particle number to equilibrium and frequently control the bulk viscosity, as emphasized by Jeon [16]. Therefore we believe that what the calculations in the literature were computing was not really the bulk viscosity of a pion gas, but rather the constant for a relaxation process which treated kinetic but not chemical equilibration. To make a fair comparison with the calculations of ζ/s at higher temperatures, one should compute the true bulk viscosity of a pion gas at low temperatures. When the bulk viscosity calculated in this way becomes large, it indicates that the pion gas will lose chemical equilibrium, a physically interesting property.

In this paper, we will provide a calculation of the bulk viscosity of a pion gas, including the relaxation via number-changing reactions to chemical equilibrium. We will work to lowest nontrivial order in chiral perturbation theory, the effective theory of low energy pions. That is, we will make an expansion to lowest order in $m_\pi/4\pi F_0$ and $T/4\pi F_0$, treating T/m_π as a free parameter of order 1. (Here F_0 is the pion decay constant, m_π is the pion mass treating the π_0 and $\pi^\pm$ as degenerate, and T is the temperature as usual.) Our treatment is therefore only valid at temperature scales low enough that there are almost no resonances (such as ρ mesons) and few kaons relative to pions; we will not try to extrapolate close to the crossover temperature.

In the next section of the paper, we will review the physics of bulk viscosity in a gas of relativistic, massive, weakly coupled bosons, emphasizing the role played by number-changing processes. We show that the bulk viscosity is controlled by m_π/T and by the rate of number-changing processes. In Sec. III we present the calculation of the number-changing rate within chiral perturbation theory. Our numerical results and conclusions are presented in Sec. IV, but can be summarized here. We find that, as temperature falls, number-changing processes become less efficient and the bulk viscosity actually grows, scaling as $\zeta/s \sim F_0^8 T^{-8}$ for $T \sim m_\pi$ and $\zeta/s \sim F_0^8 T^{-1/2} m_\pi^{-15/2} \exp(3m_\pi/T)$ for $m_\pi \gg T$. Therefore the behavior of bulk viscosity is not the opposite of the behavior of shear viscosity, and in particular both the bulk viscosity to entropy ratio and the bulk viscosity itself diverge exponentially in the low temperature limit.

II. KINETIC DESCRIPTION OF BULK VISCOSITY

By definition, bulk viscosity ζ is a reduction of the pressure in an expanding system, and an increase in pressure in a contracting system, proportional to the rate of volume change,[1]

$$P = P_{\text{eq}} - \zeta \nabla \cdot \boldsymbol{v} = P_{\text{eq}} - \zeta \frac{dV/dt}{V}. \qquad (2.1)$$

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[1]When we write noncovariantly, we implicitly work in the instantaneous local rest frame. We use boldface $\boldsymbol{p}$, $\boldsymbol{v}$ for vectors and normal letters p, v for their magnitudes; P is always the pressure.

This arises because the volume change induces a departure from equilibrium, which in turn modifies the pressure. To see how this occurs for a pion gas, we need to describe the system in terms of a calculable approximation scheme. Since physical QCD is near the chiral limit, the pion is a pseudo-Goldstone boson of the (spontaneously but also explicitly broken) chiral symmetry, and pions are therefore weakly coupled at low momenta and well described by chiral perturbation theory (see, for instance, Refs. [17,18]). Weak coupling means that thermal pions will have well-defined quasiparticles which will be well described by Boltzmann equations. Defining the species sum and integration

$$\int_{a\boldsymbol{p}} \equiv \sum_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \,, \tag{2.2}$$

the pressure is related to the occupancy of species a at momentum $\boldsymbol{p}$, $f_a(\boldsymbol{p})$, via

$$P = \frac{1}{3}\int_{a\boldsymbol{p}} 2p^2 f_a(\boldsymbol{p}) \,. \tag{2.3}$$

$f_a(\boldsymbol{p})$ in turn evolves according to the Boltzmann equation

$$2E_{\boldsymbol{p}}\frac{\partial f_a(\boldsymbol{p},t)}{\partial t} + 2\boldsymbol{p}\cdot\frac{\partial f_a(\boldsymbol{p},t)}{\partial \boldsymbol{x}}$$
$$= -\mathcal{C}[f] = -\mathcal{C}_{\text{elastic}}[f] - \mathcal{C}_{\text{inel}}[f] \,, \tag{2.4}$$

with $\mathcal{C}[f]$ the collision operator, which we discuss in more detail below.

The left-hand side of the Boltzmann equation drives the system from equilibrium. Since the bulk viscosity involves one space-time gradient, we can find it by expanding the Boltzmann equation to first order in gradients; since the left-hand side is explicitly first order in gradients, we may substitute $f_a(\boldsymbol{p},t)$ with its equilibrium form

$$f_0 = \left(\exp\left[\frac{E_{\boldsymbol{p}} - \boldsymbol{v}\cdot\boldsymbol{p}}{T}\right] - 1\right)^{-1} \,. \tag{2.5}$$

We take the energy to be $E = \sqrt{p^2 + m_\pi^2}$, meaning that we will neglect interaction self-energy corrections in comparison to the explicit pion mass. Clearly this treatment does not allow us to consider QCD in the strict chiral symmetry limit, where interaction effects are the only thing which lead to modified dispersion. It would be interesting to return to this case in the future, but we expect it to be rather subtle; for instance, the lowest order interaction effect actually does not change the dispersion relation [19,20], and the next order only shifts the speed of propagation away from the speed of light [19,21], which we believe also does not lead to nonvanishing bulk viscosity. Therefore interaction effects appear to arise at a rather high order in $(T/4\pi F_0)^2$. Therefore interaction effects can be neglected for $T \lesssim m_\pi$, which is what we are considering. In this case, explicitly evaluating the left-hand side of the Boltzmann equation yields

$$2E_{\boldsymbol{p}}\frac{\partial f_a(\boldsymbol{p},t)}{\partial t} + 2\boldsymbol{p}\cdot\frac{\partial f_a(\boldsymbol{p},t)}{\partial \boldsymbol{p}}$$
$$= 2f_0(1+f_0)\left(\frac{E^2}{T^2}\frac{dT}{dt} + \frac{p_i p_j}{T}\partial_i v_j\right) \,. \tag{2.6}$$

We are interested in the case $\partial_i v_j = \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}$. The temperature changes with time because expansion causes cooling; at first order in gradients, the time dependence of the temperature has its usual equilibrium relation to $\boldsymbol{\nabla}\cdot\boldsymbol{v}$, $dT/dt = -c_s^2 T\boldsymbol{\nabla}\cdot\boldsymbol{v}$ with $c_s^2 \equiv dP/d\epsilon$ the squared speed of sound [10]. Therefore the left-hand side of the Boltzmann equation is

$$2f_0(1+f_0)\frac{p^2 - 3c_s^2 E^2}{3T}\boldsymbol{\nabla}\cdot\boldsymbol{v} \,. \tag{2.7}$$

This "source" for departure from equilibrium has no net energy content. To see this, note that

$$P = \int_{a\boldsymbol{p}} \frac{2p^2}{3} f_0(\boldsymbol{p}) \,, \qquad \epsilon = \int_{a\boldsymbol{p}} 2E_{\boldsymbol{p}}^2 f_0(\boldsymbol{p}) \,, \tag{2.8}$$

$$c_s^2 = \frac{dP}{d\epsilon} = \frac{dP/dT}{d\epsilon/dT} = \frac{\int_{a\boldsymbol{p}} \frac{2p^2}{3}\frac{E}{T^2}f_0(1+f_0)}{\int_{a\boldsymbol{p}} 2E^2\frac{E}{T^2}f_0(1+f_0)} \,, \tag{2.9}$$

and therefore

$$\int_{a\boldsymbol{p}} E\, f_0(1+f_0)\frac{2p^2}{3T} = c_s^2 \int_{a\boldsymbol{p}} E\, f_0(1+f_0)\frac{2E^2}{T} \,, \tag{2.10}$$

which shows that there is no energy content for the source for departure from equilibrium. That this occurs is just a check that we have correctly identified the time dependence of the temperature. However, the source *does* carry a net particle number, namely,

$$\frac{dn}{dt}\bigg|_{\text{LHS}} = \int_{a\boldsymbol{p}} f_0(1+f_0)2\frac{p^2 - 3c_s^2 E^2}{3T}\boldsymbol{\nabla}\cdot\boldsymbol{v} \neq 0. \tag{2.11}$$

This means that expansion leaves excess pions, relative to the equilibrium number at the given energy density. The relaxation of this excess particle number controls equilibration and bulk viscosity.

Next we turn to the collision term. At lowest (fourth) order in $\frac{T,m_\pi}{4\pi F_0}$, the collision term contains only elastic $\pi\pi \leftrightarrow \pi\pi$ scattering. Such terms drive $f_a(\boldsymbol{p})$ toward its equilibrium form *except* that they cannot change total particle number. That is, there is no solution to the linearized Boltzmann equation with Eq. (2.7) on the left-hand side and only $\pi\pi \leftrightarrow \pi\pi$ collision processes on the right-hand side, since the left-hand side includes a change to the net particle number, while the right-hand side cannot change particle number. Therefore a calculation involving only number-conserving processes is incomplete and inconsistent, as emphasized by Jeon [16] in the context of scalar $\lambda\phi^4$ theory.[2] Therefore we must include as well the lowest order number-changing process. Since QCD is parity symmetric but the pion is a parity-odd

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[2]Nevertheless the two previous references on bulk viscosity in a pion gas treated only elastic processes. Ref. [6] got a finite answer by using the methodology developed in Ref. [22], which assumes particle number is conserved and therefore allows a nonzero chemical potential in the equilibrium distribution function. Ref. [5] got a finite answer by doing a one-loop diagrammatic evaluation without "ladder" graphs, which amounts to neglecting the departure from equilibrium in all f's other than $f(\boldsymbol{p})$ in the Boltzmann equation.

scalar, all interaction terms are even in the pion field, and the lowest order kinematically allowed number-changing process is $\pi\pi \leftrightarrow \pi\pi\pi\pi$.

At this point there is a simplification. As in the case of scalar $\lambda\phi^4$ theory [16] (but unlike the case of weakly coupled QCD [10]), number-changing processes are much less efficient than number-conserving processes in a pion gas. Number-conserving processes drive the nonequilibrium distribution function $f(\boldsymbol{p}) = f_0 + \delta f$ toward an almost-equilibrium form,

but with a chemical potential for particle number,

$$f_\mu^a(\boldsymbol{p}) \equiv \left(\exp \frac{E - \mu_a - \boldsymbol{p} \cdot \boldsymbol{v}}{T + \delta T} - 1 \right)^{-1} . \tag{2.12}$$

Here δT is determined by the condition that the energy content of f_μ is the same as the energy content of f_0. But number-conserving processes cannot lead to the relaxation of μ toward zero, because the elastic collision term vanishes if $f(\boldsymbol{p}) = f_\mu(\boldsymbol{p})$:

$$
\begin{aligned}
0 = -\mathcal{C}_{\text{elastic}}[f_\mu] = \frac{1}{2!} \int_{b\boldsymbol{p}',ck dk'} |\mathcal{M}^{ab,cd}_{\boldsymbol{p}\boldsymbol{p}'\rightarrow kk'}|^2 (2\pi)^4 \delta^4(p^\mu + p'^\mu - k^\mu - k'^\mu) \\
\times \left(f_\mu^a(\boldsymbol{p}) f_\mu^b(\boldsymbol{p}')[1 + f_\mu^c(\boldsymbol{k})][1 + f_\mu^d(\boldsymbol{k}')] - [f \leftrightarrow (1+f)] \right),
\end{aligned}
\tag{2.13}
$$

because the gain term $\propto f(\boldsymbol{p})$ and the loss term $\propto [1 + f(\boldsymbol{p})]$ cancel. Therefore $f(\boldsymbol{p})$ will equal $f_\mu(\boldsymbol{p})$ plus a small correction. The value of μ will dominate the pressure shift.

We cannot make the substitution $f(\boldsymbol{p}) = f_\mu(\boldsymbol{p})$ in the elastic part of the collision operator. But if we consider the integral $\int_{a\boldsymbol{p}}$ of Eq. (2.4), then the integral over $\mathcal{C}_{\text{elastic}}$ exactly vanishes, independent of the form of $f(\boldsymbol{p})$. We *can* approximate $f(\boldsymbol{p}) = f_\mu(\boldsymbol{p})$ in the smaller inelastic part of $\mathcal{C}$, yielding

$$\nabla \cdot \boldsymbol{v} \int_{a\boldsymbol{p}} f_0(1+f_0) \frac{p^2 - 3c_s^2 E_{\boldsymbol{p}}^2}{3T} = \int_{a\boldsymbol{p}} (-\mathcal{C}_{\text{inel}}[f^a]) \equiv -C_{\text{inel}} . \tag{2.14}$$

There are two contributions to this collision term. One contribution arises when $\boldsymbol{p} = \boldsymbol{p}_1$ is one of the four pions,

$$
\begin{aligned}
C^{4\rightarrow 2}_{\text{inel}} = \frac{1}{3!2!} \int_{a\boldsymbol{p}_1 b\boldsymbol{p}_2 c\boldsymbol{p}_3 d\boldsymbol{p}_4, e k_1 f k_2} \left| \mathcal{M}^{abcd,ef}_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_4\rightarrow k_1 k_2} \right|^2 (2\pi)^4 \delta^4 \left(\sum_{i=1..4} p_i^\mu - \sum_{i=1,2} k_i^\mu \right) \\
\times \left(f_\mu^a(\boldsymbol{p}_1) f_\mu^b(\boldsymbol{p}_2) f_\mu^c(\boldsymbol{p}_3) f_\mu^d(\boldsymbol{p}_4)[1 + f_\mu^e(\boldsymbol{k}_1)][1 + f_\mu^f(\boldsymbol{k}_2)] - [f \leftrightarrow (1+f)] \right).
\end{aligned}
\tag{2.15}
$$

The other contribution, $C^{2\rightarrow 4}_{\text{inel}}$, arises when $p = k_1$ is one of the two pions. It is the same except $\frac{1}{3!2!}$ is replaced with $-\frac{1}{4!1!}$, so it cancels half of the above contribution. (These prefactors are symmetry factors to eliminate overcounting; for instance, if b, c, d are identical, then only $1/3!$ of the phase space should be integrated over; and if b, c, d are all distinct, then the sum $\sum_{bcd}$ over counts the possibilities by a factor of $3!$. The sign difference arises from the relative sign between gain and loss terms.)

Next we expand $f_\mu^a(\boldsymbol{p})$ to first order in $\mu, \delta T$:

$$f_\mu^a(\boldsymbol{p}) \simeq f_0(\boldsymbol{p}) + f_0(\boldsymbol{p})[1 + f_0(\boldsymbol{p})] \left(\frac{\mu}{T} + \frac{E\delta T}{T^2} \right) . \tag{2.16}$$

Inserting in Eq. (2.15) and expanding to first order in $\mu, \delta T$, the distribution functions become

$$f_0(\boldsymbol{p}_1) f_0(\boldsymbol{p}_2) f_0(\boldsymbol{p}_3) f_0(\boldsymbol{p}_4)[1 + f_0(\boldsymbol{k}_1)][1 + f_0(\boldsymbol{k}_2)] \left[(4-2)\frac{\mu}{T} + \left(\sum E_{\boldsymbol{p}} - \sum E_k \right) \frac{\delta T}{T^2} \right] . \tag{2.17}$$

The sum of energies cancels by energy conservation, leaving

$$
\begin{aligned}
C_{\text{inel}}[f_\mu] = \frac{2(2\mu)}{T} \frac{1}{4!2!} \int_{a\boldsymbol{p}_1 b\boldsymbol{p}_2 c\boldsymbol{p}_3 d\boldsymbol{p}_4, e k_1 f k_2} \left| \mathcal{M}^{abcd,ef}_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_4\rightarrow k_1 k_2} \right|^2 (2\pi)^4 \delta^4 \left(\sum_{l=1..4} p_l^\mu - \sum_{l=1,2} k_l^\mu \right) \\
\times \left(f_0(\boldsymbol{p}_1) f_0(\boldsymbol{p}_2) f_0(\boldsymbol{p}_3) f_0(\boldsymbol{p}_4)[1 + f_0(\boldsymbol{k}_1)][1 + f_0(\boldsymbol{k}_2)] \right),
\end{aligned}
\tag{2.18}
$$

which determines μ. The value of μ in turn determines the pressure correction,

$$P - P_{\text{eq}} = \int_{a\boldsymbol{p}} \frac{2p^2}{3} f_0(\boldsymbol{p})[1 + f_0(\boldsymbol{p})] \left(\frac{\mu}{T} + \frac{E\delta T}{T^2} \right) . \tag{2.19}$$

Recall that δT is set by the condition that the perturbation carry no net energy, which using Eq. (2.16) is

$$\delta T \int_{a\boldsymbol{p}} f_0(1+f_0) \frac{2E^3}{T^2} = -\mu \int_{a\boldsymbol{p}} f_0(1+f_0) \frac{2E^2}{T} . \tag{2.20}$$

Combining this with Eq. (2.10), we find

$$P - P_{\text{eq}} = \mu \int_{a\mathbf{p}} f_0 (1+f_0) 2 \frac{p^2 - 3c_s^2 E^2}{3T} . \quad (2.21)$$

Putting everything together with the definition in Eq. (2.1), we find

$$\zeta = \frac{T \left(\int_{a\mathbf{p}} f_0 (1+f_0) 2 \frac{p^2 - 3c_s^2 E^2}{3T} \right)^2}{4\hat{C}_{\text{inel}}} , \quad (2.22)$$

$$\hat{C}_{\text{inel}} = \frac{1}{4!2!} \int_{a\mathbf{p}_1 b\mathbf{p}_2 c\mathbf{p}_3 d\mathbf{p}_4, e\mathbf{k}_1 f\mathbf{k}_2} \left| \mathcal{M}^{abcd,ef}_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \to \mathbf{k}_1\mathbf{k}_2} \right|^2$$

$$\times (2\pi)^4 \delta^4 \left(\sum_{i=1..4} p_i^\mu - \sum_{i=1,2} k_i^\mu \right)$$

$$\times (f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) f_0(\mathbf{p}_3) f_0(\mathbf{p}_4) [1+f_0(\mathbf{k}_1)][1+f_0(\mathbf{k}_2)]) .$$

$$(2.23)$$

The integration in the numerator is elementary, so evaluating the denominator will be our main challenge.

Using the technique developed in Refs. [4,10,23], we would arrive at the same result by using the single parameter *Ansatz* for the departure from equilibrium shown in Eq. (2.16). In the notation used there, each term in the numerator is $\tilde{S}$ and the denominator is $\tilde{C}$. The factor of 4 is essentially $(\mu + \mu + \mu + \mu - \mu - \mu)^2/\mu^2$ and can be understood as follows: each number-changing collision changes particle number by 2 (one factor of 2), and a chemical potential makes the forward process faster than the backward process by $2\mu/T$ (the other factor of 2).

III. CHIRAL PERTURBATION THEORY

Quantum chromodynamics is considered as the fundamental theory for describing strong interactions between quarks and gluons. However, at energies below the breaking scale of chiral symmetry, quarks and gluons are confined within the asymptotic hadron states, such as pions, kaons, and η mesons. In this energy regime, the QCD coupling constant becomes so large that the theory is highly nonperturbative and we still lack an analytical method to solve it. However, the situation gets better if we write an effective field theory describing the meson states. It is an experimental fact that, at sufficiently low energies, the light mesons interact weakly with each other, with the strength of interactions controlled by a derivative expansion which is described by chiral perturbation theory [18,24], an effective theory for the interactions of light pseudoscalar mesons.

In the chiral limit, the QCD Lagrangian possesses an $SU(N)_L \times SU(N)_R \times U(1)_V$ global symmetry. Here N denotes the number of flavors. The axial symmetry $U(1)_A$ of the QCD Lagrangian, present at the classical level, is broken due to a quantum anomaly. Experimental facts, such as the hadron spectrum and quark condensate, indicate that the $SU(N)_L \times SU(N)_R \times U(1)_V$ spontaneously breaks down into $SU(N)_V \times U(1)_V$. According to Goldstone's theorem, in this process, massless Goldstone bosons, which are identified with the pseudoscalar mesons, are generated. Since we are dealing

with a pure pion gas, we only focus on the case that $N = 2$, that is, only up and down quarks are of concern in our discussion.

In this specific case, the three kinds of pions are considered as the Goldstone bosons, and they transform as a triplet under the subgroup $SU_V(2)$. Moreover, pion fields, the three-component vector $\vec{\Phi} = (\phi_1, \phi_2, \phi_3)$, are isomorphic to the quotient group $SU(2)_L \times SU(2)_R/SU(2)_V$.

In the chiral limit, one can, in terms of pion fields $\vec{\Phi} = (\phi_1, \phi_2, \phi_3)$, construct the general Lagrangian invariant under $SU(2)_L \times SU(2)_R \times U(1)_V$, with the ground state invariant only under the subgroup $SU(2)_V \times U(1)_V$. But in fact, instead of being massless, pions have small but finite masses around 135 MeV. This is because chiral symmetry is not an exact one. It is broken by a small amount due to the nonvanishing masses of up and down quarks. In order to give masses to pions, one also needs to add an explicit symmetry-breaking term into the Lagrangian, which is treated as a small perturbation.

The general effective Lagrangian can be organized by chiral order,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots,$$

where the subscripts indicate the chiral order. $\mathcal{L}_2$ with the smallest chiral order contains the minimum number of derivatives and quark mass terms. It reads [18]

$$\mathcal{L}_2 = -\frac{F_0^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F_0^2 m_\pi^2}{4} \text{Tr}(U + U^\dagger). \quad (3.1)$$

Here

$$U = \exp\left(i \frac{\vec{\tau} \cdot \vec{\Phi}}{F_0} \right) = \exp\left[i \frac{\phi(x)}{F_0} \right], \quad (3.2)$$

$$\phi(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \quad (3.3)$$

where $F_0 \approx 93$ MeV is the pion decay constant and $\vec{\tau}$ are the three Pauli matrices.

The matrix element for elastic scattering is well known in chiral perturbation theory [18] and does not concern us, since Eq. (2.22) shows that the bulk viscosity is controlled by number-changing processes. We need the matrix element for $4\pi \to 2\pi$ processes. Three classes of diagrams can arise, as depicted in Fig. 1. For each class, we must sum over the distinct permutations of the external lines.

Expanding $\mathcal{L}_2$, one can find the corresponding matrix elements. For the representative permutations shown in Fig. 1, the matrix elements read (here for simplicity of writing down the matrix elements, all the four-momenta are viewed as incoming)

$$\mathcal{M}_1 = \sum_{g=1,2,3} \mathcal{V}(a, b, e, g) \frac{-i}{p_g^2 + m_\pi^2} \mathcal{V}(c, d, f, g), \quad (3.4)$$

$$\mathcal{M}_2 = \sum_{g=1,2,3} \mathcal{V}(a, b, c, g) \frac{-i}{p_g^2 + m_\pi^2} \mathcal{V}(d, e, f, g), \quad (3.5)$$

$$\mathcal{M}_3 = \frac{i}{9F_0^4} \delta^{ab} \delta^{cd} \delta^{ef} \left[4(p_a \cdot p_b + p_c \cdot p_d + p_e \cdot p_f) - 3m_\pi^2 \right]$$

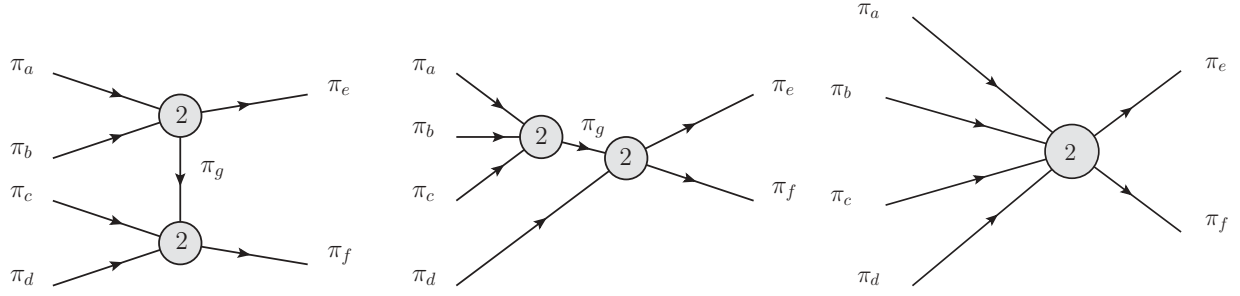$$+ \text{all distinct pairings of the set } \{a, b, c, d, e, f\},$$

$$(3.6)$$

FIG. 1. Three classes of diagrams needed to evaluate the inelastic scattering rate to lowest order in chiral perturbation theory. Here, subscripts $a-f$ are the Cartesian isospin indices, and the number "2" in a circle denotes the chiral order of the vertex.

where $\sum\limits_{g=1,2,3}$ is a sum over the species type in the propagator, p_g is the four-momentum of the propagator, and

$$
\begin{aligned}
\mathcal{V}(\alpha, \beta, \gamma, g) = \left(i/3F_0^2\right) \big[& \delta^{\alpha g}\delta^{\beta\gamma}\left(2p_\alpha \cdot p_\beta + 2p_\alpha \cdot p_\gamma \right. \\
& \left. - 4p_\beta \cdot p_\gamma + m_\pi^2\right) + \delta^{\beta g}\delta^{\alpha\gamma}\left(2p_\alpha \cdot p_\beta \right. \\
& \left. + 2p_\beta \cdot p_\gamma - 4p_\alpha \cdot p_\gamma + m_\pi^2\right) \\
& + \delta^{\gamma g}\delta^{\alpha\beta}\left(2p_\alpha \cdot p_\gamma + 2p_\beta \cdot p_\gamma \right. \\
& \left. - 4p_\alpha \cdot p_\beta + m_\pi^2\right) \big].
\end{aligned}
\tag{3.7}
$$

Therefore, the transition amplitude of the lowest order in question is

$$
|\mathcal{M}|^2 = \left| \sum_{\text{perm}} \mathcal{M}_1 + \sum_{\text{perm}} \mathcal{M}_2 + \mathcal{M}_3 \right|^2,
\tag{3.8}
$$

where $\sum\limits_{\text{perm}}$ means a sum is taken over all distinct permutations of the external lines.

This transition amplitude has a very complicated form, so we cannot finish the integral $\hat{C}_{\text{inel}}$ analytically. Therefore, we resort to numerical methods. For the numerical calculation, we work in the local plasma rest frame; that is, the rest-frame four-velocity is $u^\mu = (1, 0, 0, 0)$. The distribution function in this frame is just $f_0(\vec{p}) = [\exp(E_p) - 1]^{-1}$. The main challenge is to perform the phase-space integration over six external states. We consider the process as $4\pi \to 2\pi$, that is four incoming particles and two outgoing particles. We perform unconstrained integrations over the four incoming particle momenta in spherical coordinates with $\boldsymbol{p}_a$ as the z axis and $\boldsymbol{p}_b$ lying in the xz plane,

$$
\begin{aligned}
&\int \frac{d^3\boldsymbol{p}_a\, d^3\boldsymbol{p}_b\, d^3\boldsymbol{p}_d\, d^3\boldsymbol{p}_e}{(2\pi)^{12}\, 16 E_a E_b E_c E_d} \\
&= \frac{1}{8\,(2\pi)^{10}} \int \frac{p_a^2\, dp_a}{E_a} \frac{p_b^2\, dp_b\, d\theta_b}{E_b} \frac{p_b^2\, dp_b\, d\Omega_c}{E_c} \frac{p_b^2\, dp_b\, d\Omega_d}{E_d},
\end{aligned}
\tag{3.9}
$$

and then apply the energy-momentum conserving delta function to simplify the two-particle final phase-space integration in the manner shown in Refs. [25,26]. The final state phase space can be rewritten as

$$
\int \frac{d^3\boldsymbol{p}_e\, d^3\boldsymbol{p}_f}{(2\pi)^6\, 4 E_e E_f} \delta^4(p_a + p_b + p_c + p_d - p_e - p_f)
$$

$$
= \frac{\sqrt{1 - 4m^2/s}}{2^9 \pi^6} \int d\Omega^*,
\tag{3.10}
$$

where Ω^* is defined in the center-of-mass frame of the total incoming momentum $k^\mu = p_a^\mu + p_b^\mu + p_c^\mu + p_d^\mu$, and $s = -k^2$ is the Mandelstam variable. In the c.m. frame, it is most convenient to work in spherical coordinates, with the z axis chosen along the boost axis to the plasma rest frame. All dot products between incoming momenta are easily expressed in terms of the plasma frame variables, as is the Mandelstam variable s. For final state particle energies and dot products between an incoming and an outgoing momentum, we need to apply the boost between c.m. and plasma frame variables. An alternative approach is to consider the process $2\pi \to 4\pi$ and apply the energy-momentum conserving delta function on the four-particle final state phase space as shown in Refs. [25,26]; but this approach is a little more involved. The resulting 11-dimensional integrations are performed by Monte Carlo integration using CUBA [27].

We determine the pressure, speed of sound, and numerator of Eq. (2.22) by performing the integrals in Eqs. (2.8), (2.9), and (2.22) numerically. It has become customary to compare viscosities with the entropy density $s = \partial P/\partial T$, which has the same units as ζ. Differentiating Eq. (2.8),

$$
s = \int_{a\boldsymbol{p}} \frac{2E p^2}{3T^2} f_0(1 + f_0),
\tag{3.11}
$$

which we also handle numerically.

IV. RESULTS AND DISCUSSION

The results of numerical calculation of the bulk viscosity are shown in the Table I and Fig. 2. The most obvious feature of the bulk viscosity is that ζ and ζ/s both rise as the temperature is lowered. This is the same behavior as the shear viscosity, in contrast to the high temperature regime, $T \gg \Lambda_{\text{QCD}}$, where η/s rises but ζ/s falls with rising temperature.

We can understand the rising behavior of ζ/s with lower temperature, for $T \sim m_\pi$, as follows. First, the strength of conformal symmetry breaking depends on m_π/T, which gets larger as T gets smaller. Second, as the temperature gets lower, the typical momentum scale for pions gets lower. Since pions are pseudo-Goldstone bosons, they interact mostly through high-derivative interactions, which get weaker as

TABLE I. Values of ζ and s at certain temperatures.

T (MeV)	10	20	30	40	50	60	70
ζ (GeV^3)	3.6×10^{11}	2.1×10^5	9.3×10^2	3.9×10^1	4.1×10^0	7.0×10^{-1}	1.6×10^{-1}
s (GeV^3)	2.4×10^{-10}	3.9×10^{-7}	5.8×10^{-6}	2.7×10^{-5}	7.4×10^{-5}	1.6×10^{-4}	2.9×10^{-4}

T (MeV)	80	90	100	110	120	130	140
ζ (GeV^3)	4.7×10^{-2}	1.6×10^{-2}	5.9×10^{-3}	2.4×10^{-3}	1.1×10^{-3}	5.2×10^{-4}	2.6×10^{-4}
s (GeV^3)	4.7×10^{-4}	7.1×10^{-4}	1.0×10^{-3}	1.4×10^{-3}	1.9×10^{-3}	2.5×10^{-3}	3.2×10^{-3}

the energy scale is lowered. Therefore the system remains out of equilibrium longer, leading to higher viscosities. This last effect becomes very important when $T \ll m_\pi$. In this case the density of pions falls exponentially, $n \sim (m_\pi T)^{3/2} \exp(-m_\pi/T)$. The probability to have *four* pions in one place at one time, to participate in a number-changing collision, is therefore *exponentially* small,[3] so the rate of number-changing processes is exponentially suppressed and the bulk viscosity becomes exponentially large. This behavior was pointed out in the context of scalar field theory by Jeon [16].

In the low temperature limit $T \ll m_\pi$, the behavior of the bulk viscosity can be calculated analytically. In this regime the distribution function for incoming pions is well approximated by the nonrelativistic form $f_0(p) \simeq e^{-m/T} e^{-p^2/2m_\pi T}$. The typical value of the momentum p is $p \sim \sqrt{m_\pi T} \ll m_\pi$, which greatly simplifies both the initial particle phase space and the matrix element. For the purposes of evaluating the matrix element $\mathcal{M}_{4\to2}$, at leading order we can make the approximation that

$$p_a = p_b = p_c = p_d = (m, \vec{0}), \quad p_e = (2m, \sqrt{3}\vec{m}),$$
$$p_f = (2m, -\sqrt{3}\vec{m}). \quad (4.1)$$

Under this approximation the summation of the matrix element over species can be found in closed form: $\sum_{a,b,c,d,e,f} |\mathcal{M}|^2 \sim 2025 m_\pi^4/2F_0^8$. Factoring it out of the integral, and approximating $s \sim 16m^2$, the remaining angular integrations can be performed easily. Then the phase-space integral in $\hat{C}_{\text{inel}}$

---

[3] Or for the inverse process, the probability to have two pions with enough energy to generate four pions is exponentially small.

reduces to

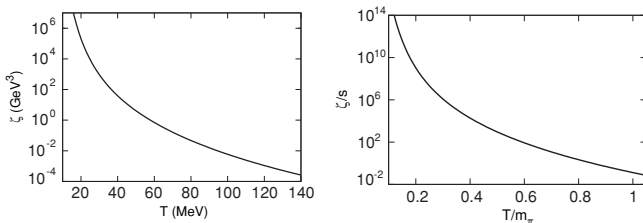$$\int \frac{d^3\boldsymbol{p}_a\, d^3\boldsymbol{p}_b\, d^3\boldsymbol{p}_c\, d^3\boldsymbol{p}_d\, d^3\boldsymbol{p}_e\, d^3\boldsymbol{p}_f}{(2\pi)^{18} 64 E_a E_b E_c E_d E_e E_f}$$
$$\times (2\pi)^4 \delta^4(p_a + p_b + p_c + p_d - p_e - p_f)$$
$$\times f_0(p_a) f_0(p_b) f_0(p_c) f_0(p_d)[1+f_0(p_e)][1+f_0(p_f)] \quad (4.2)$$
$$\simeq \frac{\sqrt{3}}{4096\pi^9 m_\pi^4} \int p_a^2\, dp_a\, p_b^2\, dp_b\, p_c^2\, dp_c\, p_d^2\, dp_d$$
$$\times e^{-4m_\pi/T} e^{-(p_a^2+p_b^2+p_c^2+p_d^2)/2m_\pi T}$$
$$= \frac{\sqrt{3} m_\pi^2 T^6 e^{-4m_\pi/T}}{16384\pi^7}.$$

We also need to carry out the integral in Eq. (2.22), which includes determining the speed of sound from Eq. (2.10). Here there is a subtlety: if we compute c_s^2 to lowest order and put it in Eq. (2.22), again computing in the nonrelativistic approximation, we get zero. Both equations must be expanded to second order in T/m_π, yielding

$$c_s^2 = \frac{T}{m_\pi} - \frac{T^2}{2m_\pi^2} + O\left(T^3/m_\pi^3\right), \quad (4.3)$$

$$\int_{ap} f_0(1+f_0) 2 \frac{p^2 - 3c_s^2 E^2}{3T} = \exp(-m_\pi/T)$$
$$\times \left(-3\frac{m_\pi^{1/2} T^{5/2}}{(2\pi)^{3/2}} + O(T^{7/2} m_\pi^{-1/2})\right), \quad (4.4)$$

where the factor of 3 counts the number of pion species. Combining these results, the low temperature limit of the bulk viscosity is

$$\zeta(T \ll m) \simeq \frac{16384\sqrt{3}\pi^4}{225} \frac{F_0^8}{m_\pi^5} \exp\left(\frac{2m_\pi}{T}\right), \quad (4.5)$$

$$\frac{\zeta}{s}(T \ll m) \simeq \frac{32768\sqrt{6}\pi^{\frac{11}{2}} F_0^8}{675 m_\pi^{\frac{15}{2}} T^{\frac{1}{2}}} \exp\left(\frac{3m_\pi}{T}\right),$$

where we used the leading-order behavior of Eq. (3.11), $s \simeq [3m_\pi^{5/2} T^{1/2}/(2\pi)^{3/2}] \exp(-m_\pi/T)$. These low temperature asymptotics are consistent with our numerical results.

We should emphasize that at temperatures such that the bulk viscosity is very large, $\zeta \nabla \cdot \boldsymbol{u} \gtrsim P$, the near-equilibrium expansion implicit in defining and using ζ has broken down. When this occurs, the system in question has fallen out of

FIG. 2. Numerical calculation of bulk viscosity ζ and the bulk viscosity to entropy ratio ζ/s.

chemical equilibrium; in fact, $\zeta \nabla \cdot \boldsymbol{u} > P$ can be taken as a criterion for the breakdown of chemical equilibrium and the freezing out of number-changing processes. And when ζ becomes *exponentially* large, the approximation that we treat QCD without including electromagnetic interactions ceases to be valid. At low temperatures the dominant number-changing process would actually be $\pi^0 \to 2\gamma$ (and its crossings). We will not consider this extension here.

In conclusion, we have computed the bulk viscosity of a pion gas, the natural low-temperature limit of QCD. We find that the bulk viscosity rises at low temperatures, growing exponentially as $\zeta \sim \exp(2m_\pi/T)$ in the $T \ll m_\pi$ limit. This growth implies that kinetic theory will generally break down at low temperatures, explaining chemical freeze-out.

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