Bohr Hamiltonian with a deformation-dependent mass term for the Davidson potential

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Analytical expressions for spectra and wave functions are derived for a Bohr Hamiltonian, describing the collective motion of deformed nuclei, in which the mass is allowed to depend on the nuclear deformation. Solutions are obtained for separable potentials consisting of a Davidson potential in the β variable, in the cases of γ-unstable nuclei, axially symmetric prolate deformed nuclei, and triaxial nuclei, implementing the usual approximations in each case. The solution, called the deformation-dependent mass (DDM) Davidson model, is achieved by using techniques of supersymmetric quantum mechanics (SUSYQM), involving a deformed shape invariance condition. Spectra and $B(E2)$ transition rates are compared to experimental data. The dependence of the mass on the deformation, dictated by SUSYQM for the potential used, reduces the rate of increase of the moment of inertia with deformation, removing a main drawback of the model.

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I. INTRODUCTION

The Bohr Hamiltonian [1] and its extensions, the geometrical collective model [2,3], have provided for several decades a sound framework for understanding the collective behavior of atomic nuclei. It was customary to consider in the Bohr Hamiltonian the mass to be a constant. However, evidence was accumulating that this approximation might be inadequate. In particular:

(i) The moments of inertia are predicted to increase proportionally to β^2, where β is the collective variable corresponding to nuclear deformation, whereas the experimentally determined (from the spectra) moment of inertia shows a much more moderate increase as a function of the experimentally determined [from the $B(E2)$ transition rates] deformation, especially for well-deformed nuclei [4]. This discrepancy has led to arguments that the use of the Bohr Hamiltonian is justified for vibrational and transitional nuclei, but its applicability to deformed nuclei needs further clarification.

(ii) Detailed comparisons to experimental data have recently pointed out [5,6] that the mass tensor of the collective Hamiltonian cannot be considered as a constant and should be taken as a function of the collective coordinates, with quadrupole and hexadecapole terms present in addition to the monopole one.

(iii) In the framework of the interacting boson model (IBM) [7], which offers an algebraic description of atomic nuclei complementary to that of the Bohr Hamiltonian, it is known that in its geometrical limit [7], obtained through the use of coherent states [7], terms of the form $\beta^2\pi^2$ and/or more complicated terms appear [8], in addition to the usual term of the kinetic energy, π^2. Thus it might be appropriate to search for a modified form of the Bohr Hamiltonian, in which the kinetic energy term will be modified by terms containing β^2 and/or more complicated terms.

Based on this evidence, a Bohr Hamiltonian with a mass depending on the collective variable β can be considered. Position-dependent effective masses have been studied recently in a general framework [9], while several Hamiltonians known to be soluble through techniques of supersymmetric quantum mechanics (SUSYQM) [10,11], have been appropriately generalized [12] to include position-dependent effective masses, the three-dimensional harmonic oscillator being among them [12].

In the present work we are going to show that a Bohr Hamiltonian with a Davidson potential [13] in β (a harmonic oscillator potential with a term proportional to $1/\beta^2$ added to it) can be generalized to include a mass depending on β, $B = B_0/(1 + a\beta^2)^2$, where B_0 and a are constants. We shall call this approach the deformation-dependent mass (DDM) Davidson model. Three cases of potentials, for which exact separation of variables can be achieved, will be considered:

(i) Potentials independent [14] of the collective variable γ (an angle measuring departure from axial symmetry), called γ-unstable potentials, appropriate for describing vibrational and near-vibrational nuclei.

(ii) Potentials of the form [14–18] $v(\beta, \gamma) = u(\beta) + w(\gamma)/\beta^2$, with $u(\beta)$ being the Davidson potential [13], and with $w(\gamma)$ having a deep minimum at $\gamma = 0$, corresponding to axially symmetric prolate deformed nuclei.

(iii) Potentials of the form $v(\beta, \gamma) = u(\beta) + w(\gamma)/\beta^2$, with $u(\beta)$ being the Davidson potential [13], and with $w(\gamma)$ having a deep minimum at $\gamma = \pi/6$, corresponding to triaxial nuclei [19,20].

Analytical results for spectra and $B(E2)$ transition rates will be provided for all three cases, implementing the usual

approximations in each limit [21–23], whereas comparison to experimental results will be undertaken in the first two, for which able bulk of experimental data exists. A special solution regarding γ-unstable nuclei was given earlier in Ref. [24].

The analytical spectra and wave functions of the Bohr Hamiltonians considered are obtained by using techniques of supersymmetric quantum mechanics [10,11], equivalent [11] to the factorization method of Infeld and Hull [25]. The integrability of the Hamiltonian is achieved by imposing a deformed shape invariance condition [12]. These tools are described in more detail in Sec. VI.

It should be noticed that the concept of a nonconstant mass in the framework of the Bohr Hamiltonian was used long ago in numerical solutions of a generalized Bohr Hamiltonian [26], as well as in relevant mean-field calculations [27]. The main difference of the present work from these earlier approaches is that analytical solutions are obtained here. In addition, in the present case the number of free parameters remains small (two or three), while the functional dependence of the mass on the deformation for the potential used is dictated by SUSYQM. The relation of the present work to these earlier approaches will be discussed in Sec. XII.

The structure of the present work is as follows. In Sec. II the formalism of position-dependent effective masses, which we use to allow the mass to depend on the deformation β, is briefly reviewed, and applied to the Bohr Hamiltonian in Sec. III. The three exactly separable cases described above are considered in Sec. IV, in which the common overall form of the radial equation in all three cases is pointed out, while in Sec. V we focus on the use of the Davidson potential in the radial equation. The solvability of the Hamiltonian is achieved in Sec. VI by imposing a deformed shape invariance condition, leading to the energy spectrum given in Sec. VII and the wave functions given in Sec. VIII. Normalization coefficients are given in Sec. IX, while a detail on their numerical calculation is included as Appendix. $B(E2)$ transition probabilities are considered in Sec. X, while in Sec. XI comparisons of spectra and $B(E2)$s to experimental data are carried out. Finally, connections to earlier work are discussed in Sec. XII, while Sec. XIII contains discussion of the present results and plans for further work.

II. FORMALISM OF POSITION-DEPENDENT EFFECTIVE MASSES

For reasons of completeness, we briefly review the basics of the formalism needed in handling effective masses depending on the position. The main problem encountered is the generalization of the kinetic energy term. We show how this can be solved in an unambiguous way.

When the mass $m(\mathbf{x})$ is position dependent [9], it does not commute with the momentum $\mathbf{p} = -i\hbar\nabla$. Therefore, there are many ways to generalize the usual form of the kinetic energy, $\mathbf{p}^2/(2m_0)$, where m_0 is a constant mass, to obtain a Hermitian operator. To avoid any specific choices, one can use the general two-parameter form proposed by von Roos [28], with a Hamiltonian,

$$H = -\frac{\hbar^2}{4}[m^{\delta'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\lambda'}(\mathbf{x}) + m^{\lambda'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\delta'}(\mathbf{x})] + V(\mathbf{x}), \quad (1)$$

where V is the relevant potential and the parameters δ', κ', λ' are constrained by the condition $\delta' + \kappa' + \lambda' = -1$. Assuming a position-dependent mass of the form,

$$m(\mathbf{x}) = m_0 M(\mathbf{x}), \quad M(\mathbf{x}) = \frac{1}{(f(\mathbf{x}))^2}, \quad f(\mathbf{x}) = 1 + g(\mathbf{x}), \quad (2)$$

where m_0 is a constant mass and $M(\mathbf{x})$ is a dimensionless position-dependent mass, the Hamiltonian becomes

$$H = -\frac{\hbar^2}{4m_0}[f^{\delta}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\lambda}(\mathbf{x}) + f^{\lambda}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\delta}(\mathbf{x})] + V(\mathbf{x}), \quad (3)$$

with $\delta + \kappa + \lambda = 2$. It is known [9] that this Hamiltonian can be put into the form,

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})} + V_{\rm eff}(\mathbf{x}), \quad (4)$$

with

$$V_{\rm eff}(\mathbf{x}) = V(\mathbf{x}) + \frac{\hbar^2}{2m_0}\left[\frac{1}{2}(1-\delta-\lambda)f(\mathbf{x})\nabla^2 f(\mathbf{x}) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f(\mathbf{x}))^2\right], \quad (5)$$

where δ and λ are free parameters.

In the final part of the paper, in which comparison to experiment will be carried out by fitting the theoretical predictions to the experimental data, it will be seen that the predictions for the theoretical spectra turn out to be independent of the choice made for δ and λ.

III. BOHR HAMILTONIAN WITH DEFORMATION-DEPENDENT EFFECTIVE MASS

A. Deformation-dependent effective mass formalism

The original Bohr Hamiltonian [1] is

$$H_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \quad (6)$$

where β and γ are the usual collective coordinates (β being a deformation coordinate measuring departure from spherical shape, and γ being an angle measuring departure from axial symmetry), while Q_k ($k = 1, 2, 3$) are the components of angular momentum in the intrinsic frame, and B is the mass parameter, which is usually considered constant.

We wish to construct a Bohr equation with a mass depending on the deformation coordinate β, in accordance

with the formalism described above,

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{7}$$

where B_0 is a constant. We then need the usual Pauli–Podolsky prescription [29]:

$$(\nabla\Phi)^i = g^{ij}\frac{\partial\Phi}{\partial x^j}, \quad \nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j\Phi, \tag{8}$$

to construct a Schrödinger equation corresponding to the Hamiltonian of Eq. (4) in a five-dimensional space equipped with the Bohr-Wheeler coordinates β, γ. Because the deformation function f depends only on the radial coordinate β, only the β part of the resulting equation will be affected, the final result reading

$$H\Psi = \left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)} + v_{\text{eff}}\right]\Psi = \epsilon\Psi, \tag{9}$$

where reduced energies $\epsilon = B_0E/\hbar^2$ and reduced potentials $v = B_0V/\hbar^2$ have been used, with

$$v_{\text{eff}} = v(\beta,\gamma) + \tfrac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \tfrac{1}{2}\left(\tfrac{1}{2}-\delta\right)\left(\tfrac{1}{2}-\lambda\right)(\nabla f)^2. \tag{10}$$

B. Connection to curved space

In Ref. [9] it was proved that the position-dependent effective mass formalism can be equivalently expressed in a curved space. We shall prove here that this connection is possible also in the case of the Bohr Hamiltonian, paving the way for connecting in Sec. XII the present results to earlier related work.

Ordering the coordinates as

$$q_1 = \Phi, \quad q_2 = \Theta, \quad q_3 = \psi, \quad q_4 = \beta, \quad q_5 = \gamma, \tag{11}$$

the kinetic energy in the standard Bohr Hamiltonian [1] can be represented as

$$T = \frac{B}{2}\left(\frac{ds}{dt}\right)^2, \tag{12}$$

where

$$ds^2 = g_{ij}dq_idq_j, \tag{13}$$

the symmetric matrix g_{ij} having the form,

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix}, \tag{14}$$

with [30]

$$g_{11} = \frac{\mathcal{J}_1}{B}\sin^2\Theta\cos^2\psi + \frac{\mathcal{J}_2}{B}\sin^2\Theta\sin^2\psi + \frac{\mathcal{J}_3}{B}\cos^2\Theta,$$
$$g_{12} = \frac{1}{B}(\mathcal{J}_2-\mathcal{J}_1)\sin\Theta\sin\psi\cos\psi,$$
$$g_{13} = \frac{\mathcal{J}_3}{B}\cos\Theta, \quad g_{22} = \frac{\mathcal{J}_1}{B}\sin^2\psi + \frac{\mathcal{J}_2}{B}\cos^2\psi, \tag{15}$$
$$g_{33} = \frac{\mathcal{J}_3}{B}, \quad g_{44} = 1, \quad g_{55} = \beta^2,$$

where the moments of inertia are

$$\mathcal{J}_k = 4B\beta^2\sin^2\left(\gamma - k\frac{2\pi}{3}\right). \tag{16}$$

The determinant of the matrix is

$$g = \frac{\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3}{B^3}\beta^2\sin^2\Theta = 4\beta^8\sin^2 3\gamma\sin^2\Theta. \tag{17}$$

The relevant volume element is then

$$dV = 2\beta^4\sin 3\gamma\sin\Theta d\Phi d\Theta d\psi d\beta d\gamma. \tag{18}$$

The inverse matrix is found to be

$$\left(g_{ij}^{-1}\right) = \begin{pmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} & 0 & 0 \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} & 0 & 0 \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_{44}^{-1} & 0 \\ 0 & 0 & 0 & 0 & g_{55}^{-1} \end{pmatrix}, \tag{19}$$

with

$$g_{11}^{-1} = \frac{B}{\sin^2\Theta}\left(\frac{\cos^2\psi}{\mathcal{J}_1} + \frac{\sin^2\psi}{\mathcal{J}_2}\right),$$
$$g_{12}^{-1} = -B\left(\frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_2}\right)\frac{\sin\psi\cos\psi}{\sin\Theta},$$
$$g_{13}^{-1} = -B\left(\frac{\cos^2\psi}{\mathcal{J}_1} + \frac{\sin^2\psi}{\mathcal{J}_2}\right)\frac{\cot\Theta}{\sin\Theta},$$
$$g_{22}^{-1} = B\left(\frac{\sin^2\psi}{\mathcal{J}_1} + \frac{\cos^2\psi}{\mathcal{J}_2}\right), \tag{20}$$
$$g_{23}^{-1} = B\left(\frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_2}\right)\cot\Theta\sin\psi\cos\psi,$$
$$g_{33}^{-1} = B\left(\frac{\cos^2\psi}{\mathcal{J}_1} + \frac{\sin^2\psi}{\mathcal{J}_2}\right)\cot^2\Theta + \frac{B}{\mathcal{J}_3},$$
$$g_{44}^{-1} = 1, \quad g_{55}^{-1} = \frac{1}{\beta^2}.$$

Using these matrix elements and the value of the determinant from Eq. (17) in Eq. (8) we obtain

$$T = -\frac{\hbar^2}{2B}\nabla^2 = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\times\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right], \tag{21}$$

where Q_k are the components of the angular momentum in the intrinsic frame,

$$Q_x = -i\left(-\frac{\cos\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \sin\psi\frac{\partial}{\partial\Theta} + \cot\Theta\cos\psi\frac{\partial}{\partial\psi}\right),$$
$$Q_y = -i\left(-\frac{\sin\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \cos\psi\frac{\partial}{\partial\Theta} - \cot\Theta\sin\psi\frac{\partial}{\partial\psi}\right),$$
$$Q_z = -i\frac{\partial}{\partial\psi}. \tag{22}$$

The connection between the position-dependent effective mass and curved spaces was considered in Ref. [9]. According to the findings of Ref. [9], one expects in the present case all elements of the matrix (14) to be divided by f^2,

$$g'_{ij} = \frac{g_{ij}}{f^2}. \tag{23}$$

As a result, the determinant of the matrix will be

$$g' = \frac{g}{f^{10}}, \tag{24}$$

and the volume element will be

$$dV' = \frac{dV}{f^5}. \tag{25}$$

The elements of the inverse matrix will be

$$g'^{-1}_{ij} = f^2 g^{-1}_{ij}. \tag{26}$$

According to Ref. [9], to obtain the Schrödinger equation in the form of Eq. (9), one has to start with the equation,

$$H_g\tilde{\Psi} = \left[-\frac{1}{2}\nabla^2 + u_g\right]\tilde{\Psi} = \left[-\frac{1}{2}\frac{1}{\sqrt{g'}}\partial_i\sqrt{g'}g'^{-1}_{ij}\partial_j + u_g\right]\tilde{\Psi}, \tag{27}$$

where

$$\tilde{\Psi} = f^{5/2}\Psi, \tag{28}$$

while reduced energies and reduced potentials are used, as in Eq. (9). The exponent in the last equation is related to the dimensionality of the space.

Substituting the g' matrix elements and determinant in Eq. (27), and performing the relevant calculation (which closely resembles the pure Bohr case, except for the 44-term), we see that Eqs. (27) and (9) do coincide with

$$u_g = u_{\rm eff} + f\ddot{f} - 2(\dot{f})^2 + 4\frac{f\dot{f}}{\beta}, \quad \dot{f} = \frac{df}{d\beta}, \quad \ddot{f} = \frac{d^2f}{d\beta^2}. \tag{29}$$

This result has several important consequences.

(i) It becomes clear that solving the Schrödinger equation (9) with deformation-dependent mass is equivalent to solving a modified Bohr equation (27) with different metric matrix g' and another effective potential, u_g. Between the two equivalent schemes, one chooses to solve Eq. (9) instead of Eq. (27), just because the former can be solved analytically through the use of SUSYQM techniques.

(ii) The wave functions $\tilde{\Psi} = f^{5/2}\Psi$ are accompanied by the volume element $dV' = dV/f^5$. As a result,

$$\int\tilde{\Psi}^*\tilde{\Psi}dV' = \int(f^{5/2}\Psi^*)(f^{5/2}\Psi)\frac{dV}{f^5} = \int\Psi^*\Psi dV, \tag{30}$$

that is, the wave functions Ψ of the deformation-dependent mass problem correspond to the usual Bohr volume element dV.

(iii) The simple relation between $\tilde{\Psi}$ and Ψ also shows that the wave functions Ψ satisfy the well-known 24 symmetries of Bohr wave functions [1], which the wave functions $\tilde{\Psi}$ satisfy by construction. If these symmetries were not satisfied, the solutions could not have been used for the description of nuclei.

Further consequences, regarding the connection of the present approach to earlier work, will be discussed in Sec. XII.

IV. EXACTLY SEPARABLE SPECIAL FORMS OF THE BOHR HAMILTONIAN

The solution of the above Bohr-like equation can be reached for certain classes of potentials using techniques developed in the context of SUSYQM [10–12]. At this point exact separation of variables can be achieved in three cases, described in the following three subsections.

A. γ-unstable nuclei

To achieve separation of variables we assume that the potential $v(\beta,\gamma)$ depends only on the variable β, that is, $v(\beta) = u(\beta)$ [14]. Potentials of this kind are called γ-unstable potentials because they are appropriate for the description of nuclei which can depart from axial symmetry without any energy cost.

One then seeks wave functions of the form [14,31],

$$\Psi(\beta,\gamma,\theta_i) = \xi(\beta)\Phi(\gamma,\theta_i), \tag{31}$$

where θ_i $(i = 1, 2, 3)$ are the Euler angles. Separation of variables gives

$$\left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} + \frac{f^2}{2\beta^2}\Lambda + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2 + u(\beta)\right]\xi(\beta) = \epsilon\xi(\beta), \tag{32}$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_k\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] \times\Phi(\gamma,\theta_i) = \Lambda\Phi(\gamma,\theta_i). \tag{33}$$

Equation (33) was solved by Bès [32]. $\Lambda = \tau(\tau+3)$ represents the eigenvalues of the second-order Casimir operator of SO(5), while τ is the seniority quantum number, characterizing the irreducible representations of SO(5). The values of angular

momentum L occurring for each τ are provided by a well-known algorithm and are listed in Refs. [7,14]. Within the ground-state band (gsb) one has $L = 2\tau$. The $L = 2$ member of the quasi-γ_1 band is degenerate with the $L = 4$ member of the gsb, the $L = 3$, 4 members of the quasi-γ_1 band are degenerate to the $L = 6$ member of the gsb, the $L = 5$, 6 members of the quasi-γ_1 band are degenerate to the $L = 8$ member of the gsb, and so on.

B. Axially symmetric prolate deformed nuclei

To achieve exact separation of variables, we assume a potential of the form [14–18],

$$v(\beta,\gamma) = u(\beta) + \frac{f^2}{\beta^2} w(\gamma), \tag{34}$$

with $w(\gamma)$ having a deep minimum at $\gamma = 0$. Then the angular momentum term can be written as [21]

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} \approx \frac{4}{3}\left(Q_1^2 + Q_2^2 + Q_3^2\right) + Q_3^2\left(\frac{1}{\sin^2\gamma} - \frac{4}{3}\right). \tag{35}$$

One then seeks wave functions of the form [21],

$$\Psi(\beta,\gamma,\theta_i) = \phi_K^L(\beta,\gamma)\mathcal{D}^L_{M,K}(\theta_i), \tag{36}$$

where $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles, L is the angular momentum quantum number, while M and K are the quantum numbers of the projections of angular momentum on the laboratory-fixed z axis and the body-fixed z' axis, respectively. Then separation of variables leads to

$$\left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} + \frac{f^2}{2\beta^2}\tilde{\Lambda} + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2 + u(\beta)\right]\xi_L(\beta) = \epsilon\xi_L(\beta), \tag{37}$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{K^2}{4}\left(\frac{1}{\sin^2\gamma} - \frac{4}{3}\right) + 2w(\gamma)\right]\eta_K(\gamma) = \Lambda\eta_K(\gamma), \tag{38}$$

where

$$\tilde{\Lambda} = \Lambda + \frac{L(L+1)}{3}, \tag{39}$$

and $\phi_K^L(\beta,\gamma) = \xi_L(\beta)\eta_K(\gamma)$. We remark that Eq. (37) has the same form as Eq. (32), obtained in the case of γ-unstable nuclei, when $\tilde{\Lambda}$ in the former is replaced by Λ in the latter. However, the results are different as far as the physics described is concerned. The angular momentum dependence, contained in $\tilde{\Lambda}$ and Λ, respectively, is different. Furthermore, the angular equation is different in each case, because of the different treatment of the γ variable, the potential being confined to $\gamma \approx 0$ in the former case, while being independent of γ in the latter.

Equation (38) was solved for a harmonic oscillator potential:

$$w(\gamma) = \tfrac{1}{2}(3c)^2\gamma^2, \tag{40}$$

in the case of $\gamma \approx 0$ [18,21], resulting in

$$\Lambda = \epsilon_\gamma - \frac{K^2}{3}, \quad \epsilon_\gamma = (6c)(n_\gamma + 1), \quad n_\gamma = 0, 1, 2, \ldots, \tag{41}$$

where n_γ is the quantum number related to γ oscillations. The allowed bands are characterized by

$$\begin{aligned} &n_\gamma = 0, \quad K = 0; \quad n_\gamma = 1, \quad K = \pm 2; \\ &n_\gamma = 2, \quad K = 0, \pm 4; \quad \ldots \end{aligned} \tag{42}$$

As a result,

$$\tilde{\Lambda} = \frac{L(L+1) - K^2}{3} + \epsilon_\gamma = \frac{L(L+1) - K^2}{3} + (6c)(n_\gamma + 1). \tag{43}$$

C. Triaxial nuclei with $\gamma = \pi/6$

In this case we assume again a potential of the form of Eq. (34), but with $w(\gamma)$ having a deep minimum at $\gamma = \pi/6$. In this case K, the angular momentum projection on the body-fixed z' axis, is not a good quantum number any more, but α, the angular momentum projection on the body-fixed x' axis, is a good quantum number, as found [22] in the study of the triaxial rotator [19,20]. Then the angular momentum term can be written as [22,23]

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} \approx 4\left(Q_1^2 + Q_2^2 + Q_3^2\right) - 3Q_1^2. \tag{44}$$

One then seeks wave functions of the form [23],

$$\Psi(\beta,\gamma,\theta_i) = \phi_\alpha^L(\beta,\gamma)\mathcal{D}^L_{M,\alpha}(\theta_i), \tag{45}$$

where $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles, L is the angular momentum quantum number, while M and α are the quantum numbers of the projections of angular momentum on the laboratory-fixed z axis and the body-fixed x' axis, respectively. Then separation of variables leads to

$$\left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} + \frac{f^2}{2\beta^2}\bar{\Lambda} + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2 + u(\beta)\right]\xi_{L,\alpha}(\beta) = \epsilon\xi_{L,\alpha}(\beta), \tag{46}$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + 2w(\gamma)\right]\eta(\gamma) = \Lambda'\eta(\gamma), \tag{47}$$

with $\phi_\alpha^L(\beta,\gamma) = \xi_{L,\alpha}(\beta)\eta(\gamma)$, and

$$\bar{\Lambda} = \frac{4L(L+1) - 3\alpha^2}{4} + \Lambda'. \tag{48}$$

Equation (47) was solved for a harmonic oscillator potential,

$$w(\gamma) = \frac{1}{4}c\left(\gamma - \frac{\pi}{6}\right)^2, \tag{49}$$

in the case of $\gamma \approx \pi/6$ [23], resulting in

$$\Lambda' = \epsilon_\gamma = \sqrt{2c}\left(n_\gamma + \tfrac{1}{2}\right), \tag{50}$$

where n_γ is the quantum number related to γ oscillations. As a result,

$$\bar{\Lambda} = \frac{4L(L+1) - 3\alpha^2}{4} + \sqrt{2c}\left(n_\gamma + \frac{1}{2}\right). \tag{51}$$

We remark that Eqs. (37) and (46) have the same form, with $\tilde{\Lambda}$ in the former replaced by $\bar{\Lambda}$ in the latter.

In the literature on triaxial nuclei it is customary, instead of the projection α of the angular momentum on the x' axis, to introduce the wobbling quantum number [2,22] $n_w = L - \alpha$. Inserting $\alpha = L - n_w$ in Eq. (51) one obtains

$$\bar{\Lambda} = \frac{L(L+4) + 3n_w(2L - n_w)}{4} + \sqrt{2c}\left(n_\gamma + \frac{1}{2}\right). \tag{52}$$

D. Common form of the radial equation

We remark that Eqs. (32), (37), and (46) have the same form, the only difference being that Λ in the first equation is replaced by $\tilde{\Lambda}$ in the second, and by $\bar{\Lambda}$ in the third one. In what follows we are going to use the symbol Λ, understanding that (i) for γ-unstable nuclei it is given by $\Lambda = \tau(\tau+3)$; (ii) for axially symmetric prolate deformed nuclei it should be replaced by $\tilde{\Lambda}$, given in Eq. (43); and (iii) for triaxial nuclei it should be replaced by $\bar{\Lambda}$, given in Eq. (52).

Equation (32) can be simplified by performing the derivations,

$$\frac{1}{2}f^2\xi'' + \left(ff' + \frac{2f^2}{\beta}\right)\xi' + \left(\frac{(f')^2}{8} + \frac{ff''}{4} + \frac{ff'}{\beta}\right)\xi - \frac{f^2}{2\beta^2}\Lambda\xi + \epsilon\xi - v_{\rm eff}\xi = 0, \tag{53}$$

with

$$v_{\rm eff} = u + \frac{1}{4}(1-\delta-\lambda)f\left(\frac{4f'}{\beta} + f''\right) + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(f')^2. \tag{54}$$

The difference in the numerical coefficient of f' observed in comparison to Eq. (2.27) of Ref. [9] is because of the different dimensionality of the space used in each case.

Setting

$$\xi(\beta) = \frac{R(\beta)}{\beta^2}, \tag{55}$$

Eq. (53) is put into the form,

$$HR = -\left(\sqrt{f}\frac{d}{d\beta}\sqrt{f}\right)^2 R + 2u_{\rm eff}R = 2\epsilon R, \tag{56}$$

where

$$u_{\rm eff} = v_{\rm eff} + \frac{f^2 + \beta f f'}{\beta^2} + \frac{f^2}{2\beta^2}\Lambda. \tag{57}$$

V. THE DAVIDSON POTENTIAL

Up to now no assumption about the specific form of the potential $u(\beta)$ and the deformation function $f(\beta)$ was made. We are now going to consider the special case of the Davidson potential [13]:

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{58}$$

where the parameter β_0 indicates the position of the minimum of the potential. The special case of $\beta_0 = 0$ corresponds to the simple harmonic oscillator.

Based on the results for the three-dimensional harmonic oscillator reported in Ref. [12], we are also going to consider for the deformation function the special form,

$$f(\beta) = 1 + a\beta^2, \quad a \ll 1. \tag{59}$$

This choice is made to lead to an exact solution. Its physical implications will be discussed in Sec. XI.

Using these forms for the potential and the deformation function in Eq. (57) one obtains

$$2u_{\rm eff} = k_1\beta^2 + k_0 + \frac{k_{-1}}{\beta^2}, \tag{60}$$

where

$$\begin{aligned} k_1 &= 2 + a^2[5(1-\delta-\lambda) + (1-2\delta)(1-2\lambda) + 6 + \Lambda], \\ k_0 &= a[5(1-\delta-\lambda) + 8 + 2\Lambda], \\ k_{-1} &= 2 + \Lambda + 2\beta_0^4. \end{aligned} \tag{61}$$

VI. DEFORMED SHAPE INVARIANCE

Our task now is to find the eigenvalues and eigenfunctions of the Hamiltonian of Eq. (56). This can be achieved by imposing shape invariance [12], which is an integrability condition guaranteeing that exact solutions of the Hamiltonian of Eq. (56) can be found. The use of shape invariance in the framework of SUSYQM is equivalent [11] to the well-known factorization method of the Schrödinger equation, introduced 60 years ago by Infeld and Hull [25]. In other words, we are now going to use a mathematical technique allowing us to find the solutions of Eq. (56).

In its simplest form in a one-dimensional space, shape invariance can be described as follows [33]. Two potentials V_1 and V_2, which are supersymmetric partners, are in general different functions of x. They are called shape invariant if they satisfy the condition,

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \tag{62}$$

where a_1, a_2 are sets of parameters independent of x, with a_2 being a function of a_1, and the remainder $R(a_1)$ is also independent of x. In other words, the two potentials have the same functional dependence on x, the difference being in the

values of the parameters appearing in each of them, and in their relative displacement by the remainder $R(a_1)$. Furthermore, it is known that the shape invariance condition of Eq. (62) can be written in the operator form,

$$A(a_1)A^\dagger(a_1) = A^\dagger(a_2)A(a_2) + R(a_1), \tag{63}$$

where A and $A^\dagger$ are the operators corresponding to the supersymmetric partners $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$. Solving the Schrödinger equation for H_1 by this method, one obtains as a "bonus" the solution of H_2 as well.

In the present case, the concept of shape invariance has to be generalized, as described in detail in Ref. [12], because the mass depends on the deformation, resulting in a deformed shape invariance condition. Instead of two Hamiltonians, one has a series of many Hamiltonians. We are interested in solving the Schrödinger equation for the first of them, which will be Eq. (56).

H in Eq. (56) may be considered as the first member $H_0 = H$ of a hierarchy of Hamiltonians,

$$H_i = A_i^+ A_i^- + \sum_{j=0}^{i} \varepsilon_j, \quad i = 0, 1, 2, \ldots, \tag{64}$$

where the first-order operators [12],

$$A_i^\pm = A^\pm(\mu_i, \nu_i) = \mp\sqrt{f}\frac{d}{d\beta}\sqrt{f} + W(\mu_i, \nu_i; \beta), \tag{65}$$

satisfy a deformed shape invariance condition,

$$A_i^- A_i^+ = A_{i+1}^+ A_{i+1}^- + \varepsilon_{i+1}, \quad i = 0, 1, 2, \ldots, \tag{66}$$

with ε_i, $i = 0, 1, 2, \ldots$, denoting some constants. (Note that the parameters λ and μ of [12] have been changed into μ and ν, respectively.)

In other words, the superpotential $W(\mu, \nu; \beta)$ fulfills the two conditions:

$$W^2(\mu, \nu; \beta) - f(\beta)W'(\mu, \nu; \beta) + \varepsilon_0 = 2u_{\text{eff}}(\beta), \tag{67}$$

and

$$\begin{aligned} &W^2(\mu_i, \nu_i; \beta) + f(\beta)W'(\mu_i, \nu_i; \beta) \\ &\quad = W^2(\mu_{i+1}, \nu_{i+1}; \beta) - f(\beta)W'(\mu_{i+1}, \nu_{i+1}; \beta) + \varepsilon_{i+1}, \\ &\quad i = 0, 1, 2, \ldots, \end{aligned} \tag{68}$$

where $\mu_0 = \mu$, $\nu_0 = \nu$, and a prime denotes derivative with respect to β.

In the case of the effective potential given in Eq. (60), $W(\mu, \nu; \beta)$ is a class 2 superpotential,

$$W(\mu, \nu; \beta) = \mu\phi(\beta) + \frac{\nu}{\phi(\beta)}, \tag{69}$$

$$\phi(\beta) = \frac{1}{\beta}, \tag{70}$$

which means that Eqs. (3.9) and (3.10) of [12] read

$$\phi'(\beta) = -\frac{1}{\beta^2} = \frac{A}{\beta^2} + B, \quad a\beta^2 = \frac{(A'/\beta^2) + B'}{(-1/\beta^2)}, \tag{71}$$

with $A = -1$, $B = 0$, $A' = 0$, and $B' = -a$.

Inserting Eqs. (69) and (70) in (67), we obtain

$$\begin{aligned} &\left(\frac{\mu}{\beta} + \nu\beta\right)^2 - (1 + a\beta^2)\left(-\frac{\mu}{\beta^2} + \nu\right) + \varepsilon_0 \\ &\quad = k_1\beta^2 + k_0 + \frac{k_{-1}}{\beta^2}, \end{aligned} \tag{72}$$

which is equivalent to the three equations,

$$\mu(\mu{+}1) = k_{-1}, \quad \nu(\nu{-}a) = k_1, \quad 2\mu\nu + \mu a - \nu + \varepsilon_0 = k_0. \tag{73}$$

Their solutions read

$$\begin{aligned} &\mu = \frac{1}{2}(-1 \pm \Delta_1), \quad \nu = \frac{a}{2}(1 \pm \Delta_2), \\ &\varepsilon_0 = k_0 - 2\mu\nu - \mu a + \nu, \\ &\Delta_1 \equiv \sqrt{1 + 4k_{-1}}, \quad \Delta_2 \equiv \sqrt{1 + 4\frac{k_1}{a^2}}, \end{aligned} \tag{74}$$

provided $1 + 4k_1/a^2 \geqslant 0$ (note that $1 + 4k_{-1}$ is always positive). As we shall show in Sec. VIII A, the conditions ensuring that the ground-state wave function is physically acceptable select the lower sign for μ and the upper one for ν:

$$\mu = -\frac{1}{2}(1 + \Delta_1), \quad \nu = \frac{a}{2}(1 + \Delta_2). \tag{75}$$

Inserting next Eqs. (69) and (70) in Eq. (68), we get

$$\begin{aligned} &\left(\frac{\mu_i}{\beta} + \nu_i\beta\right)^2 + (1 + a\beta^2)\left(-\frac{\mu_i}{\beta^2} + \nu_i\right) \\ &\quad = \left(\frac{\mu_{i+1}}{\beta} + \nu_{i+1}\beta\right)^2 - (1 + a\beta^2)\left(-\frac{\mu_{i+1}}{\beta^2} + \nu_{i+1}\right) + \varepsilon_{i+1}, \end{aligned} \tag{76}$$

leading to the three conditions,

$$\begin{aligned} \mu_i(\mu_i - 1) &= \mu_{i+1}(\mu_{i+1} + 1), \\ \nu_i(\nu_i + a) &= \nu_{i+1}(\nu_{i+1} - a), \\ 2\mu_i\nu_i - \mu_i a + \nu_i &= 2\mu_{i+1}\nu_{i+1} + \mu_{i+1}a - \nu_{i+1} + \varepsilon_{i+1}. \end{aligned} \tag{77}$$

Their solutions are

$$\mu_{i+1} = \mu_i - 1, \quad \nu_{i+1} = \nu_i + a, \tag{78}$$

and

$$\varepsilon_{i+1} = 2(\mu_i\nu_i - \mu_{i+1}\nu_{i+1}) - (\mu_i + \mu_{i+1})a + \nu_i + \nu_{i+1}. \tag{79}$$

Note that there are other solutions for μ_{i+1} and ν_{i+1}, namely $\mu_{i+1} = -\mu_i$ and $\nu_{i+1} = -\nu_i$, but the alternating signs would not be compatible with physically acceptable excited-state wave functions. Finally, the iteration of (78) leads to

$$\mu_i = \mu - i, \quad \nu_i = \nu + ia. \tag{80}$$

VII. ENERGY SPECTRUM

The energy spectrum of Eq. (56) is therefore given by

$$\begin{aligned}\epsilon_n &= \frac{1}{2}\sum_{i=0}^{n}\varepsilon_i \\ &= \frac{1}{2}\left[k_0-2\mu_n\nu_n-a\left(2\sum_{i=0}^{n-1}\mu_i+\mu_n\right)+2\sum_{i=0}^{n-1}\nu_i+\nu_n\right] \\ &= \frac{1}{2}[k_0-2\mu\nu-a\mu+\nu-4(a\mu-\nu)n+4an^2]. \qquad (81)\end{aligned}$$

On taking (75) into account, this can be rewritten as

$$\begin{aligned}\epsilon_n &= \tfrac{1}{2}\big[k_0+\tfrac{1}{2}a(3+2\Delta_1+2\Delta_2+\Delta_1\Delta_2) \\ &\quad +2a(2+\Delta_1+\Delta_2)n+4an^2\big], \qquad (82) \\ n &= 0,1,2,\ldots.\end{aligned}$$

Equation (82) only provides a formal solution to the bound-state energy spectrum. The range of n values is actually determined by the existence of corresponding physically acceptable wave functions. The relevant conditions will be considered in the next section.

We quote here the final results for the spectra, which will be used for comparison to experiment. One has

$$\begin{aligned}\epsilon_0 &= \frac{19}{4}a+\frac{5}{2}(1-\delta-\lambda)a+\frac{1}{2}\sqrt{a^2+4k_1} \\ &\quad +\frac{a}{2}\sqrt{1+4k_{-1}}+\frac{1}{4}\sqrt{(a^2+4k_1)(1+4k_{-1})}+a\Lambda, \qquad (83)\end{aligned}$$

$$\epsilon_1 = \epsilon_0+4a+\sqrt{a^2+4k_1}+a\sqrt{1+4k_{-1}}, \qquad (84)$$

$$\epsilon_2 = \epsilon_0+12a+2\sqrt{a^2+4k_1}+2a\sqrt{1+4k_{-1}}, \qquad (85)$$

where k_1, k_{-1} are given by Eq. (61), in which Λ has the form explained in Sec. IV D.

The ground-state band is obtained from Eq. (83), while the quasi-β_1 band is obtained from Eq. (84), and the quasi-β_2 band is obtained from Eq. (85).

In the special case of $a = 0$ (no dependence of the mass on the deformation) one easily obtains

$$\epsilon_1 = \epsilon_0+2\sqrt{2}, \quad \epsilon_2 = \epsilon_0+4\sqrt{2}, \qquad (86)$$

that is, the β bandheads become equidistant.

VIII. WAVE FUNCTIONS

To be physically acceptable, the bound-state wave functions should satisfy two conditions [12]:

(i) As in conventional (constant-mass) quantum mechanics, they should be square integrable on the interval of definition of $u_{\rm eff}$, that is,

$$\int_0^\infty d\beta\,|R_n(\beta)|^2<\infty. \qquad (87)$$

(ii) Furthermore, they should ensure the Hermiticity of H. For such a purpose, it is enough to impose that the operator $\sqrt{f}(d/d\beta)\sqrt{f}$ be Hermitian, which amounts to the restriction,

$$|R_n(\beta)|^2 f(\beta)\to 0 \quad \text{for} \quad \beta\to 0 \quad \text{and} \quad \beta\to\infty, \qquad (88)$$

or, equivalently,

$$\begin{aligned}|R_n(\beta)|^2 &\to 0 \quad \text{for} \quad \beta\to 0 \quad \text{and} \\ |R_n(\beta)|^2\beta^2 &\to 0 \quad \text{for} \quad \beta\to\infty. \qquad (89)\end{aligned}$$

As condition (89) is more stringent than condition (87), we should only be concerned with the former.

A. Ground-state wave function

The ground-state wave function, which is annihilated by A^-, is given by Eq. (2.29) of [12] as

$$\begin{aligned}R_0(\beta) &= R_0(\mu,\nu;\beta) \\ &= \frac{N_0}{\sqrt{f(\beta)}}\exp\left(-\int^\beta\frac{W(\mu,\nu;\tilde\beta)}{f(\tilde\beta)}d\tilde\beta\right), \qquad (90)\end{aligned}$$

where N_0 is some normalization coefficient. Here

$$\begin{aligned}\int^\beta\frac{W(\mu,\nu;\tilde\beta)}{f(\tilde\beta)}d\tilde\beta &= \int^\beta\left(\frac{\mu}{\tilde\beta}+\frac{(\nu-\mu a)\tilde\beta}{1+a\tilde\beta^2}\right)d\tilde\beta \\ &= \mu\ln\beta+\frac{1}{2a}(\nu-\mu a)\ln(1+a\beta^2). \qquad (91)\end{aligned}$$

Hence

$$R_0(\beta) = N_0\beta^{-\mu}f^{-(\nu-\mu a+a)/(2a)}. \qquad (92)$$

For $\beta\to 0$, the function $|R_0(\beta)|^2$ behaves as $\beta^{-2\mu}$. Condition (89) imposes that $-2\mu>0$ or $\mu<0$. Because k_{-1}, defined in Eq. (61), is greater than 2, it follows that Δ_1, defined in (74), is greater than 3, so that the upper sign choice for μ in (74) would lead to $\mu>1$. As this is not acceptable, we have to take the lower sign for which $\mu<-2$.

For $\beta\to\infty$, $|R_0(\beta)|^2\beta^2$ behaves as $\beta^{-2\nu/a}$. Condition (89) therefore imposes that $\nu>0$. This restriction is surely satisfied by the upper sign choice for ν in (74). For the lower one, it is not fulfilled if we restrict ourselves to small enough values of a because then k_1 in (61) will be positive and Δ_2 in (74) will be greater than 1. For sufficiently large values of a, however, both sign choices might be acceptable. Because among two acceptable wave functions, it is customary in quantum mechanics to choose the most regular one (see, e.g., [34] and references quoted therein), we assume the upper sign for ν, thus getting Eq. (75).

B. Excited-state wave functions

According to Eqs. (2.30), (3.20), and (3.21) of [12], the excited-state wave functions are given by

$$\begin{aligned}R_n(\beta) &= R_n(\mu,\nu;\beta)\propto\beta^{-n}R_0(\mu_n,\nu_n;\beta)P_n(\mu,\nu;y), \\ y &= \beta^2, \qquad (93)\end{aligned}$$

where $P_n(\mu, \nu; y)$ is an nth-degree polynomial in y, satisfying the equation,

$$P_{n+1}(\mu, \nu; y) = -2y(1+ay)\frac{d}{dy}P_n(\mu_1, \nu_1; y) + [\mu_{n+1} + \mu + n + (\nu_{n+1} + \nu + na)y]P_n(\mu_1, \nu_1; y), \tag{94}$$

with the starting value $P_0(\mu, \nu; y) = 1$.

From Eqs. (80) and (92), it follows that

$$R_0(\mu_n, \nu_n; \beta) \propto \beta^{-\mu_n} f^{-(\nu_n - \mu_n a + a)/(2a)} \propto R_0(\mu, \nu; \beta)\beta^n f^{-n}, \tag{95}$$

so that Eq. (93) becomes

$$R_n(\beta) \propto R_0(\beta) f^{-n} P_n(\mu, \nu; y). \tag{96}$$

It is then clear that $R_n(\beta)$ satisfies condition (89) for any $n = 1, 2, \ldots$, because $R_0(\beta)$ does.

It now remains to solve Eq. (94). For such a purpose, let us make the changes of variable and of function,

$$t = 1 - \frac{2}{f} = \frac{-1+ay}{1+ay}, \qquad P_n(\mu, \nu; y) = C_n f^n Q_n(\mu, \nu; t), \tag{97}$$

where C_n is some constant. From definition (97), it follows that $Q_n(\mu, \nu; t)$ an nth-degree polynomial in t. We successively get

$$y = \frac{1+t}{a(1-t)}, \quad 1+ay = \frac{2}{1-t}, \quad \frac{d}{dy} = \frac{a}{2}(1-t)^2\frac{d}{dt}. \tag{98}$$

It is then straightforward to show that Eq. (94) becomes

$$\frac{C_{n+1}}{C_n}Q_{n+1}(\mu, \nu; t) = \left\{ -(1-t^2)\frac{d}{dt} + \left[\mu + \frac{\nu}{a} + \left(\frac{\nu}{a} - \mu + 1\right)t\right]\right\} Q_n(\mu - 1, \nu + a; t). \tag{99}$$

On taking into account that the Jacobi polynomials satisfy the backward shift operator relation [see Eq. (1.8.7) of [35]],

$$2(n+1)P_{n+1}^{(\alpha,\beta)}(x) = \left\{ -(1-x^2)\frac{d}{dx} + [\alpha - \beta + (\alpha + \beta + 2)x]\right\} P_n^{(\alpha+1,\beta+1)}(x), \tag{100}$$

we see that $Q_n(\mu, \nu; t)$ is actually some Jacobi polynomial,

$$Q_n(\mu, \nu; t) = P_n^{\left(\frac{\nu}{a} - \frac{1}{2}, -\mu - \frac{1}{2}\right)}(t) = P_n^{\left(\frac{\Delta_2}{2}, \frac{\Delta_1}{2}\right)}(t), \tag{101}$$

provided we choose

$$\frac{C_{n+1}}{C_n} = 2(n+1), \quad C_0 = 1, \tag{102}$$

or, in other words, $C_n = 2^n n!$.

We therefore conclude that the wave functions are given by

$$R_n(\beta) = \frac{N_n}{N_0}R_0(\beta)P_n^{\left(\frac{\nu}{a} - \frac{1}{2}, -\mu - \frac{1}{2}\right)}(t) = N_n\beta^{-\mu} f^{-(\nu - \mu a + a)/(2a)} P_n^{\left(\frac{\nu}{a} - \frac{1}{2}, -\mu - \frac{1}{2}\right)}(t), \tag{103}$$

or

$$R_n(\beta) = N_n\beta^{(1+\Delta_1)/2} f^{-1-(\Delta_1+\Delta_2)/4} P_n^{(\Delta_2/2, \Delta_1/2)}(t), \qquad t = \frac{-1+a\beta^2}{1+a\beta^2}, \tag{104}$$

where N_n is some normalization coefficient.

The Jacobi polynomials appearing in the wave functions of the ground-state band ($n = 0$), the quasi-β_1 band ($n = 1$), and the quasi-β_2 band ($n = 2$), needed for the calculation of the relevant $B(E2)$ transitions, read

$$P_0^{(\alpha,\beta)}(x) = 1, \tag{105}$$

$$P_1^{(\alpha,\beta)}(x) = \tfrac{1}{2}[2(\alpha+1) + (\alpha+\beta+2)(x-1)], \tag{106}$$

$$P_2^{(\alpha,\beta)}(x) = \tfrac{1}{8}[4(\alpha+1)(\alpha+2) + 4(\alpha+\beta+3)(\alpha+2)(x-1) + (\alpha+\beta+3)(\alpha+\beta+4)(x-1)^2]. \tag{107}$$

IX. NORMALIZATION COEFFICIENT

To calculate N_n, let us first express the whole wave function R_n in terms of t:

$$R_n = N_n y^{(1+\Delta_1)/4}(1+ay)^{-1-(\Delta_1+\Delta_2)/4} P_n^{(\Delta_2/2, \Delta_1/2)}(t) = N_n 2^{-1-(\Delta_1+\Delta_2)/4} a^{-(1+\Delta_1)/4}(1+t)^{(1+\Delta_1)/4} \times (1-t)^{(3+\Delta_2)/4} P_n^{(\Delta_2/2, \Delta_1/2)}(t). \tag{108}$$

Now, on taking into account that

$$d\beta = \frac{dy}{2\sqrt{y}} = \frac{dt}{\sqrt{a}(1-t)^{3/2}(1+t)^{1/2}}, \tag{109}$$

we obtain

$$\int_0^\infty |R_n|^2 d\beta = |N_n|^2 2^{-2-(\Delta_1+\Delta_2)/2} a^{-1-\Delta_1/2} \times \int_{-1}^{+1} (1-t)^{\Delta_2/2}(1+t)^{\Delta_1/2}\left[P_n^{(\Delta_2/2, \Delta_1/2)}(t)\right]^2 dt, \tag{110}$$

in terms of the normalization integral of Jacobi polynomials [36].

Hence the normalization condition reads

$$|N_n|^2 2^{-2-(\Delta_1+\Delta_2)/2} a^{-1-\Delta_1/2} \times \frac{2^{(\Delta_1+\Delta_2)/2+1}\Gamma\left(n + \frac{\Delta_1}{2} + 1\right)\Gamma\left(n + \frac{\Delta_2}{2} + 1\right)}{\left(2n + \frac{\Delta_1+\Delta_2}{2} + 1\right) n!\,\Gamma\left(n + \frac{\Delta_1+\Delta_2}{2} + 1\right)} = 1, \tag{111}$$

and leads to

$$N_n = \left(2a^{\Delta_1/2+1}\left(2n + \frac{\Delta_1+\Delta_2}{2} + 1\right) n!\right)^{1/2} \times \left(\frac{\Gamma\left(n + \frac{\Delta_1+\Delta_2}{2} + 1\right)}{\Gamma\left(n + \frac{\Delta_1}{2} + 1\right)\Gamma\left(n + \frac{\Delta_2}{2} + 1\right)}\right)^{1/2}. \tag{112}$$

TABLE I. Comparison of theoretical predictions of the γ-unstable Bohr Hamiltonian with β-dependent mass (with $\delta = \lambda = 0$) to experimental data [40] of rare earth and actinide nuclei with $R_{4/2} \leqslant 2.6$ and known 0_2^+ and 2_γ^+ states. The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios, as well as the β and γ bandheads, normalized to the 2_1^+ state and labeled by $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ and $R_{2/2} = E(2_\gamma^+)/E(2_1^+)$, respectively, are shown. β_0 and a are free parameters, related to the Davidson potential [Eq. (58)] and to the dependence of the mass on the deformation [Eq. (59)]. The angular momenta of the highest levels of the ground state, β and γ bands included in the rms fit are labeled by L_g, L_β, and L_γ respectively, while n indicates the total number of levels involved in the fit and σ is the quality measure of Eq. (121). The theoretical predictions are obtained from the formulas mentioned below Eq. (121). The Xe and Ba isotopes have already been considered in Ref. [24]. See Sec. XI A for further discussion.

Nucleus	$R_{4/2}$ exp	$R_{4/2}$ th	$R_{0/2}$ exp	$R_{0/2}$ th	$R_{2/2}$ exp	$R_{2/2}$ th	β_0	a	L_g	L_β	L_γ	n	σ
^{98}Ru	2.14	2.14	2.0	2.4	2.2	2.1	0.99	0.020	24	0	4	15	0.277
^{100}Ru	2.27	2.24	2.1	2.7	2.5	2.2	1.19	0.048	28	0	4	17	0.315
^{102}Ru	2.33	2.20	2.0	2.4	2.3	2.2	1.05	0.059	16	0	5	12	0.364
^{104}Ru	2.48	2.34	2.8	3.0	2.5	2.3	1.40	0.083	8	2	8	12	0.429
^{102}Pd	2.29	2.24	2.9	2.3	2.8	2.2	1.08	0.081	26	4	4	18	0.326
^{104}Pd	2.38	2.21	2.4	2.6	2.4	2.2	1.15	0.034	18	2	4	13	0.397
^{106}Pd	2.40	2.16	2.2	2.2	2.2	2.2	0.91	0.062	16	4	5	14	0.409
^{108}Pd	2.42	2.26	2.4	2.3	2.1	2.3	1.09	0.103	14	4	4	12	0.318
^{110}Pd	2.46	2.31	2.5	2.0	2.2	2.3	0.99	0.195	12	10	4	14	0.354
^{112}Pd	2.53	2.29	2.6	2.5	2.1	2.3	1.21	0.086	6	0	3	5	0.485
^{114}Pd	2.56	2.31	2.6	2.8	2.1	2.3	1.30	0.076	16	0	11	18	0.722
^{116}Pd	2.58	2.36	3.3	3.4	2.2	2.4	1.52	0.062	16	0	9	16	0.609
^{106}Cd	2.36	2.25	2.8	2.9	2.7	2.3	1.28	0.028	12	0	2	7	0.268
^{108}Cd	2.38	2.14	2.7	2.2	2.5	2.1	0.91	0.041	24	0	5	16	0.528
^{110}Cd	2.35	2.08	2.2	1.9	2.2	2.1	0.00	0.061	16	6	5	15	0.415
^{112}Cd	2.29	2.05	2.0	1.9	2.1	2.0	0.00	0.033	12	8	11	20	0.523
^{114}Cd	2.30	2.06	2.0	1.9	2.2	2.1	0.00	0.041	14	4	3	11	0.418
^{116}Cd	2.38	2.16	2.5	2.7	2.4	2.2	1.14	0.000	14	2	3	10	0.387
^{118}Cd	2.39	2.19	2.6	2.9	2.6	2.2	1.21	0.002	14	0	3	9	0.429
^{120}Cd	2.38	2.20	2.7	2.9	2.6	2.2	1.22	0.006	16	0	2	9	0.412
^{118}Xe	2.40	2.32	2.5	2.6	2.8	2.3	1.27	0.103	16	4	10	19	0.319
^{120}Xe	2.47	2.36	2.8	3.4	2.7	2.4	1.51	0.063	26	4	9	23	0.524
^{122}Xe	2.50	2.40	3.5	3.3	2.5	2.4	1.57	0.096	16	0	9	16	0.638
^{124}Xe	2.48	2.36	3.6	3.5	2.4	2.4	1.55	0.051	20	2	11	21	0.554
^{126}Xe	2.42	2.33	3.4	3.1	2.3	2.3	1.42	0.064	12	4	9	16	0.584
^{128}Xe	2.33	2.27	3.6	3.5	2.2	2.3	1.42	0.000	10	2	7	12	0.431
^{130}Xe	2.25	2.21	3.3	3.1	2.1	2.2	1.27	0.000	14	0	5	11	0.347
^{132}Xe	2.16	2.00	2.8	2.0	1.9	2.0	0.00	0.000	6	0	5	7	0.467
^{134}Xe	2.04	2.00	1.9	2.0	1.9	2.0	0.00	0.000	6	0	5	7	0.685
^{130}Ba	2.52	2.42	3.3	3.2	2.5	2.4	1.60	0.118	12	0	6	11	0.352
^{132}Ba	2.43	2.29	3.2	2.8	2.2	2.3	1.29	0.059	14	0	8	14	0.619
^{134}Ba	2.32	2.16	2.9	2.7	1.9	2.2	1.12	0.000	8	0	4	7	0.332
^{136}Ba	2.28	2.00	1.9	2.0	1.9	2.0	0.00	0.000	6	0	2	4	0.250
^{142}Ba	2.32	2.38	4.3	4.3	4.0	2.4	1.72	0.028	14	0	2	8	0.609
^{134}Ce	2.56	2.34	3.7	3.9	2.4	2.3	1.59	0.019	34	2	8	25	0.527
^{136}Ce	2.38	2.11	1.9	2.1	2.0	2.1	0.82	0.034	16	0	3	10	0.457
^{138}Ce	2.32	2.00	1.9	2.0	1.9	2.0	0.00	0.000	14	0	2	8	0.314
^{140}Nd	2.33	2.05	1.8	1.9	1.9	2.1	0.00	0.037	6	0	2	4	0.192
^{148}Nd	2.49	2.36	3.0	2.8	4.1	2.4	1.38	0.110	12	8	4	13	0.764
^{140}Sm	2.35	2.29	1.9	1.9	2.7	2.3	0.92	0.196	8	0	2	5	0.207
^{142}Sm	2.33	2.06	1.9	1.9	2.2	2.1	0.33	0.044	8	0	2	5	0.147
^{142}Gd	2.35	2.21	2.7	2.8	1.9	2.2	1.20	0.020	16	0	2	9	0.231
^{144}Gd	2.35	2.33	2.5	2.5	2.5	2.3	1.26	0.112	6	0	2	4	0.124
^{152}Gd	2.19	2.13	1.8	1.8	3.2	2.1	0.00	0.104	16	10	7	19	0.635
^{154}Dy	2.23	2.15	2.0	2.0	3.1	2.1	0.75	0.083	26	10	7	24	0.530
^{156}Er	2.32	2.25	2.7	2.8	2.7	2.3	1.24	0.043	20	4	5	16	0.450
^{186}Pt	2.56	2.42	2.5	3.7	3.2	2.4	1.71	0.085	26	6	10	25	0.813
^{188}Pt	2.53	2.37	3.0	3.3	2.3	2.4	1.52	0.076	16	2	4	12	0.637
^{190}Pt	2.49	2.28	3.1	3.4	2.0	2.3	1.42	0.015	18	2	6	15	0.637

TABLE I. (*Continued*)

Nucleus	$R_{4/2}$ exp	$R_{4/2}$ th	$R_{0/2}$ exp	$R_{0/2}$ th	$R_{2/2}$ exp	$R_{2/2}$ th	β_0	a	L_g	L_β	L_γ	n	σ
^{192}Pt	2.48	2.34	3.8	3.7	1.9	2.3	1.56	0.032	10	0	8	12	0.681
^{194}Pt	2.47	2.36	3.9	3.6	1.9	2.4	1.55	0.049	10	4	5	11	0.667
^{196}Pt	2.47	2.33	3.2	2.9	1.9	2.3	1.37	0.079	10	2	6	11	0.639
^{198}Pt	2.42	2.21	2.2	2.2	1.9	2.2	0.96	0.089	6	2	4	7	0.370
^{200}Pt	2.35	2.00	2.4	2.0	1.8	2.0	0.00	0.000	4	0	4	5	0.392

A way of avoiding numerical problems when having to handle $\Gamma(x)$ functions with large x is given in Appendix.

X. $B(E2)$ TRANSITION RATES

$B(E2)$ transition rates,

$$B(E2;\varrho L \to \varrho' L') = \frac{5}{16\pi}\frac{|\langle \varrho' L'||T^{(E2)}||\varrho L\rangle|^2}{2L+1}, \quad (113)$$

where ϱ stands for quantum numbers other than the angular momentum L, can be calculated using the quadrupole operator $T^{(E2)}$ and the Wigner-Eckart theorem in the form

$$\langle \varrho' L' M'|T^{(E2)}_\mu|\varrho L M\rangle = \frac{1}{\sqrt{2L'+1}}\langle L2L'|M\mu M'\rangle\langle \varrho' L'||T^{(E2)}||\varrho L\rangle. \quad (114)$$

A. $B(E2)$s for γ-unstable nuclei

The calculation is carried out exactly as in Ref. [37], using the quadrupole operator [31],

$$T^{(E2)} = A\beta\left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i)\cos\gamma + \frac{1}{\sqrt{2}}\left(\mathcal{D}^{(2)}_{\mu,2}(\theta_i)+\mathcal{D}^{(2)}_{\mu,-2}(\theta_i)\right)\sin\gamma\right], \quad (115)$$

where A is a scale factor.

The results of Ref. [37] need not be repeated here. The only difference is that in the radial integral [see Eq. (21) of Ref. [37]] the wave functions $R_{n,\tau}(\beta)$ appear

$$I_{n',\tau+1;n,\tau} = \int_0^\infty \beta\xi_{n',\tau+1}(\beta)\xi_{n,\tau}(\beta)\beta^4 d\beta = \int_0^\infty \beta R_{n',\tau+1}(\beta)R_{n,\tau}(\beta)d\beta. \quad (116)$$

The τ dependence of the wave functions $R_n(\beta)$ of Eq. (104) is contained in Δ_1, Δ_2, known from Eq. (74) to contain k_1, k_{-1}, which in turn are known from Eq. (61) to contain $\Lambda = \tau(\tau+3)$.

B. $B(E2)$s for axially symmetric prolate deformed nuclei

The quadrupole operator is again given by Eq. (115). The calculation is carried out exactly as in Ref. [18], the results of which need not be repeated here. The only difference is that in the radial integral [see Eq. (B5) of Ref. [18]] the wave functions $R_{n,L}(\beta)$ appear

$$B_{n,L,n',L'} = \int_0^\infty \beta\xi_{n,L}(\beta)\xi_{n',L'}(\beta)\beta^4 d\beta = \int_0^\infty \beta R_{n,L}(\beta)R_{n',L'}(\beta)d\beta. \quad (117)$$

The L dependence of the wave functions $R_n(\beta)$ of Eq. (104) is contained in Δ_1, Δ_2, known from Eq. (74) to contain k_1, k_{-1}, which in turn are known from Eq. (61) to contain $\tilde{\Lambda}$ of Eq. (43).

C. $B(E2)$s for triaxial nuclei with $\gamma = \pi/6$

The calculation is carried out exactly as in Ref. [38], using the quadrupole operator,

$$T^{(E2)}_\mu = A\beta\left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i)\cos\left(\gamma-\frac{2\pi}{3}\right) + \frac{1}{\sqrt{2}}\left(\mathcal{D}^{(2)}_{\mu,2}(\theta_i) + \mathcal{D}^{(2)}_{\mu,-2}(\theta_i)\right)\sin\left(\gamma-\frac{2\pi}{3}\right)\right], \quad (118)$$

where A is a scale factor, while the quantity $\gamma - 2\pi/3$ in the trigonometric functions is obtained from $\gamma - 2\pi k/3$ for $k = 1$, because in the present case the projection α along the body-fixed $\hat{x}'$ axis is used.

The results of Ref. [38] need not be repeated here. The only difference is that in the radial integral [see Eq. (14) of Ref. [38]] the wave functions $R_{n,\alpha,L}(\beta)$ appear

$$I_\beta(n,L,\alpha;n',L',\alpha') = \int_0^\infty \beta\xi_{n,\alpha,L}(\beta)\xi_{n',L',\alpha'}(\beta)\beta^4 d\beta = \int_0^\infty \beta R_{n,\alpha,L}(\beta)R_{n',\alpha',L'}(\beta)\,d\beta. \quad (119)$$

The α, L dependence of the wave functions $R_n(\beta)$ of Eq. (104) is contained in Δ_1, Δ_2, known from Eq. (74) to contain k_1, k_{-1}, which in turn are known from Eq. (61) to contain $\bar{\Lambda}$ of Eq. (51).

XI. NUMERICAL RESULTS

From Eq. (9) it is clear that in the present case the moments of inertia are not proportional to $\beta^2\sin^2(\gamma - 2\pi k/3)$ but to

TABLE II. Same as Table I, but for axially symmetric prolate deformed rare earth and actinide nuclei with $R_{4/2} > 2.9$. β_0, a, and c are free parameters, related to the Davidson potential [Eq. (58)], to the dependence of the mass on the deformation [Eq. (59)], and to the γ potential [Eq. (40)]. The theoretical predictions are obtained from the equations mentioned in Sec. XI B, where further discussion can be found.

nucleus	$R_{4/2}$ exp	$R_{4/2}$ th	$R_{0/2}$ exp	$R_{0/2}$ th	$R_{2/2}$ exp	$R_{2/2}$ th	β_0	c	a	L_g	L_β	L_γ	n	σ
^{150}Nd	2.93	3.13	5.2	7.9	8.2	5.8	0.0	2.1	0.003	14	6	4	13	2.012
^{152}Sm	3.01	3.14	5.6	8.4	8.9	6.5	0.0	2.4	0.000	16	14	9	23	3.327
^{154}Sm	3.25	3.27	13.4	13.0	17.6	18.6	1.30	6.9	0.021	16	6	7	17	0.515
^{154}Gd	3.02	3.09	5.5	6.5	8.1	4.1	0.0	1.4	0.024	26	26	7	32	3.546
^{156}Gd	3.24	3.25	11.8	10.8	13.0	14.3	0.0	5.3	0.026	26	12	16	34	0.933
^{158}Gd	3.29	3.29	15.0	14.5	14.9	15.1	1.99	5.3	0.025	12	6	6	14	0.323
^{160}Gd	3.30	3.30	17.6	17.3	13.1	13.2	2.38	4.5	0.020	16	4	8	17	0.125
^{162}Gd	3.29	3.30	19.8	19.8	12.0	12.1	2.52	4.1	0.008	14	0	4	10	0.078
^{156}Dy	2.93	3.13	4.9	7.4	6.5	5.3	0.0	1.9	0.014	28	10	13	31	1.789
^{158}Dy	3.21	3.22	10.0	9.6	9.6	10.3	0.26	3.8	0.023	28	8	8	25	0.496
^{160}Dy	3.27	3.27	14.7	14.7	11.1	12.1	1.92	4.3	0.005	28	4	23	38	0.510
^{162}Dy	3.29	3.30	17.3	15.7	11.0	11.2	2.23	3.8	0.020	18	8	14	26	0.742
^{164}Dy	3.30	3.30	22.6	22.5	10.4	10.2	2.68	3.4	0.000	20	0	10	19	0.100
^{166}Dy	3.31	3.31	15.0	14.9	11.2	11.2	2.39	3.7	0.047	6	2	5	8	0.077
^{160}Er	3.10	3.16	7.1	8.1	6.8	6.6	0.00	2.4	0.013	26	2	5	18	0.699
^{162}Er	3.23	3.23	10.7	10.7	8.8	10.1	1.29	3.7	0.013	20	4	12	23	0.770
^{164}Er	3.28	3.27	13.6	12.2	9.4	9.6	1.83	3.3	0.026	22	10	18	33	0.918
^{166}Er	3.29	3.28	18.1	16.8	9.8	9.9	2.22	3.4	0.002	16	10	14	26	0.698
^{168}Er	3.31	3.31	15.3	14.4	10.3	10.2	2.29	3.4	0.041	18	6	8	19	0.404
^{170}Er	3.31	3.30	11.3	10.1	11.9	12.9	1.64	4.4	0.083	24	10	19	35	0.837
^{162}Yb	2.92	3.07	3.6	6.8	4.8	4.0	0.00	1.4	0.003	24	0	4	15	1.036
^{164}Yb	3.13	3.18	7.9	8.3	7.0	7.4	0.00	2.7	0.023	18	0	5	13	0.357
^{166}Yb	3.23	3.23	10.2	8.9	9.1	9.7	0.66	3.5	0.038	24	10	13	29	0.973
^{168}Yb	3.27	3.26	13.2	11.2	11.2	11.5	1.52	4.1	0.028	34	4	7	25	1.070
^{170}Yb	3.29	3.27	12.7	11.2	13.6	14.1	1.36	5.1	0.035	20	10	17	31	0.963
^{172}Yb	3.31	3.30	13.2	12.2	18.6	18.9	1.66	6.6	0.055	16	10	5	17	0.742
^{174}Yb	3.31	3.31	19.4	19.3	21.4	21.5	2.44	7.5	0.019	20	4	5	16	0.104
^{176}Yb	3.31	3.30	13.9	13.7	15.4	15.5	1.97	5.4	0.036	20	2	5	15	0.287
^{178}Yb	3.31	3.27	15.7	15.5	14.5	14.6	1.88	5.3	0.000	6	4	2	6	0.127
^{166}Hf	2.97	3.08	4.4	6.9	5.1	4.3	0.00	1.5	0.006	22	0	3	13	0.873
^{168}Hf	3.11	3.17	7.6	8.1	7.1	6.9	0.00	2.5	0.023	22	4	4	16	0.494
^{170}Hf	3.19	3.21	8.7	8.7	9.5	8.8	0.00	3.2	0.033	34	4	4	22	0.970
^{172}Hf	3.25	3.24	9.2	9.8	11.3	11.7	0.00	4.3	0.031	38	4	6	26	0.549
^{174}Hf	3.27	3.25	9.1	10.4	13.5	13.6	0.00	5.0	0.033	26	4	5	19	0.832
^{176}Hf	3.28	3.28	13.0	11.5	15.2	16.1	1.31	5.8	0.038	18	10	8	21	0.950
^{178}Hf	3.29	3.28	12.9	12.3	12.6	13.0	1.70	4.6	0.028	18	6	6	17	0.356
^{180}Hf	3.31	3.30	11.8	11.5	12.9	13.0	1.92	4.4	0.068	12	4	5	12	0.157
^{176}W	3.22	3.21	7.8	9.1	9.6	9.5	0.00	3.5	0.027	22	4	5	17	0.881
^{178}W	3.24	3.22	9.4	8.6	10.5	8.9	0.00	3.2	0.039	18	10	2	15	0.987
^{180}W	3.26	3.25	14.6	13.1	10.8	11.5	1.64	4.2	0.000	24	0	7	18	0.603
^{182}W	3.29	3.29	11.3	11.5	12.2	12.5	1.77	4.3	0.050	18	4	6	16	0.195
^{184}W	3.27	3.28	9.0	8.9	8.1	8.0	1.57	2.7	0.080	10	4	6	12	0.093
^{186}W	3.23	3.25	7.2	7.2	6.0	6.3	1.20	2.1	0.099	14	4	6	14	0.130
^{176}Os	2.93	3.10	4.5	6.9	6.4	4.6	0.00	1.6	0.016	24	6	5	19	1.747
^{178}Os	3.02	3.12	4.9	7.2	6.6	5.1	0.00	1.8	0.017	16	6	5	15	1.836
^{180}Os	3.09	3.22	5.6	7.1	6.6	6.9	0.00	2.4	0.078	10	6	7	14	1.021
^{184}Os	3.20	3.21	8.7	9.9	7.9	8.5	1.21	3.1	0.011	22	0	6	16	0.886
^{186}Os	3.17	3.19	7.7	7.0	5.6	6.0	0.00	2.1	0.063	14	10	13	24	0.702
^{188}Os	3.08	3.15	7.0	7.2	4.1	4.4	1.07	1.5	0.033	12	2	7	13	0.170
^{190}Os	2.93	3.07	4.9	5.6	3.0	3.1	0.00	1.0	0.051	10	2	6	11	0.419
^{228}Ra	3.21	3.24	11.3	11.0	13.3	13.3	0.57	5.0	0.016	22	4	3	15	0.177
^{228}Th	3.24	3.26	14.4	14.3	16.8	17.0	1.50	6.4	0.002	18	2	5	14	0.214
^{230}Th	3.27	3.27	11.9	11.6	14.7	14.7	1.44	5.3	0.034	24	4	4	17	0.243
^{232}Th	3.28	3.28	14.8	14.0	15.9	16.5	1.80	5.9	0.022	30	10	12	31	0.426
^{232}U	3.29	3.29	14.5	13.8	18.2	18.4	1.74	6.6	0.028	20	10	4	18	0.394

TABLE II. (*Continued*)

Nucleus	$R_{4/2}$ exp	$R_{4/2}$ th	$R_{0/2}$ exp	$R_{0/2}$ th	$R_{2/2}$ exp	$R_{2/2}$ th	β_0	c	a	L_g	L_β	L_γ	n	σ
^{234}U	3.30	3.30	18.6	18.3	21.3	21.8	2.19	7.8	0.011	28	8	7	24	0.244
^{236}U	3.30	3.30	20.3	20.0	21.2	21.2	2.38	7.5	0.009	30	4	5	21	0.143
^{238}U	3.30	3.31	20.6	20.6	23.6	24.7	2.38	8.8	0.009	30	4	27	43	0.665
^{238}Pu	3.31	3.31	21.4	21.4	23.3	23.3	2.61	8.1	0.016	26	2	4	17	0.067
^{240}Pu	3.31	3.31	20.1	19.9	26.6	26.6	2.40	9.4	0.018	26	4	4	18	0.117
^{242}Pu	3.31	3.31	21.5	21.4	24.7	24.7	2.52	8.7	0.012	26	2	2	15	0.107
^{248}Cm	3.31	3.31	25.0	24.8	24.2	24.3	2.72	8.5	0.004	28	4	2	17	0.159
^{250}Cf	3.32	3.31	27.0	26.9	24.2	24.2	2.88	8.4	0.003	8	2	4	8	0.053

$[\beta^2/f^2(\beta)]\sin^2(\gamma - 2\pi k/3)$. The function $\beta^2/f^2(\beta)$ is shown in Fig. 1 for different values of the parameter a. It is clear that the increase of the moment of inertia is slowed down by the function $f(\beta)$, as it is expected as nuclear deformation sets in Ref. [4].

The effect of the deformation-dependent mass on the moments of inertia can be seen in Fig. 2, where the moments of inertia [4] for the ground-state band,

$$\Theta(L) = \frac{2L-1}{E(L) - E(L-2)}, \tag{120}$$

normalized to $\Theta(2)$, are shown in the case of axially symmetric prolate deformed nuclei, for the specific values of $\beta_0 = 2$ and $c = 5$, and varying parameter a. It is clear that the rapid increase of the moments of inertia with L, seen for $a = 0$, is gradually moderated by increasing a.

A. Spectra of γ-unstable nuclei

Root mean square (rms) fits of spectra have been performed, using the quality measure,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}[E_i(\text{exp}) - E_i(\text{th})]^2}{(n-1)E(2_1^+)^2}}. \tag{121}$$

The theoretical predictions for the levels of the ground-state band are obtained from Eq. (83), while the levels of the quasi-β_1 band are obtained from Eq. (84). The levels of the quasi-γ_1 band are obtained through their degeneracies to members of the ground-state band, mentioned below Eq. (33).

The results shown in Table I have been obtained for $\delta = \lambda = 0$. (The Xe and Ba isotopes have already been considered in Ref. [24].) One can easily verify that different choices for δ and λ lead to a renormalization of the parameter values a and β_0, the predicted energy levels remaining exactly the same.

Concerning the physical content of the parameter a, it is instructive to consider in detail in Table I the Xe isotopes (already discussed in Ref. [24]), known [39] to lie in a γ-unstable region. They extend from the borders of the neutron shell (^{134}Xe$_{80}$ is just below the $N = 82$ shell closure) to the midshell (^{120}Xe$_{66}$) and even beyond, exhibiting increasing collectivity [increasing $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios] from the border to the midshell. Moving from the border of the neutron shell to the midshell, the following remarks apply.

(i) ^{134}Xe and ^{132}Xe are almost pure vibrators. Therefore no need for deformation dependence of the mass exists, the least-square fitting leading to $a = 0$. Furthermore, no

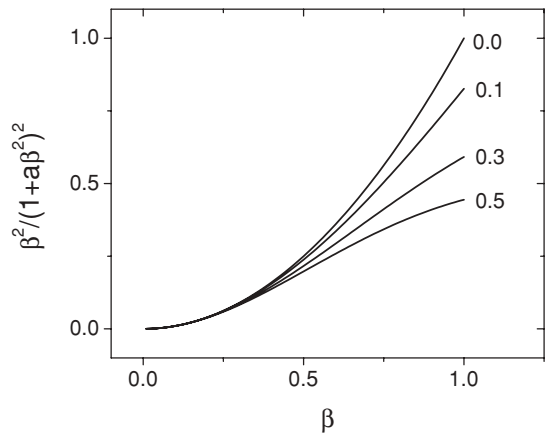

FIG. 1. The function $\beta^2/f^2(\beta) = \beta^2/(1 + a\beta^2)^2$, to which moments of inertia are proportional as seen from Eq. (9), plotted as a function of the nuclear deformation β for different values of the parameter a. See Section XI for further discussion.

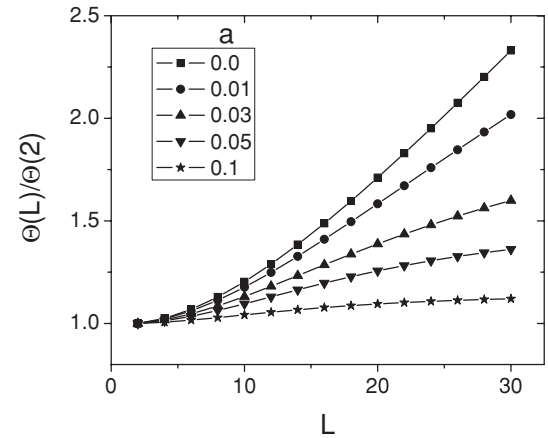

FIG. 2. Moments of inertia $\Theta(L)$ for the ground-state band, given by Eq. (120) and normalized to $\Theta(2)$, are shown for the specific values of $\beta_0 = 2$ and $c = 5$, and varying parameter a. See Section XI for further discussion.

TABLE III. Normalized [to the energy of the first excited state, $E(2_1^+)$] energy levels of the ground-state band (gsb) and the β_1 and γ_1 bands of ^{162}Dy and ^{238}U, obtained from the Bohr Hamiltonian with β-dependent mass for axially symmetric prolate deformed nuclei using the parameters given in Table II, compared to experimental data [40]. See Sec. XI B for further discussion.

L	^{162}Dy exp	^{162}Dy th	^{238}U exp	^{238}U th	L	^{162}Dy exp	^{162}Dy th	^{238}U exp	^{238}U th
	gsb	gsb	gsb	gsb		γ_1	γ_1	γ_1	γ_1
0	0.00	0.00	0.00	0.00	2	11.0	11.2	23.6	24.7
2	1.00	1.00	1.00	1.00	3	11.9	12.1	24.6	25.5
4	3.29	3.30	3.30	3.31	4	13.2	13.3	25.9	26.7
6	6.80	6.80	6.84	6.86	5	14.7	14.7	27.4	28.1
8	11.41	11.41	11.54	11.57	6	16.4	16.5	29.2	29.8
10	17.04	17.01	17.27	17.33	7	18.5	18.5	31.2	31.7
12	23.57	23.49	23.97	24.06	8	20.7	20.8	33.5	33.9
14	30.90	30.74	31.51	31.63	9	23.3	23.3	36.0	36.3
16	38.90	38.70	39.82	39.97	10	25.9	26.0	38.8	39.0
18	47.58	47.28	48.78	48.98	11	29.0	28.9	41.7	41.9
20			58.31	58.61	12	31.4	32.1	44.9	45.0
22			68.31	68.77	13	35.5	35.5	48.3	48.3
24			78.71	79.44	14	39.4	39.9	51.9	51.8
26			89.46	90.55	15			55.7	55.5
28			100.57	102.08	16			59.7	59.4
30			112.10	113.99	17			63.9	63.4
					18			68.2	67.7
	β_1	β_1	β_1	β_1	19			72.7	72.0
0	17.3	15.7	20.6	20.6	20			77.3	76.6
2	18.0	16.7	21.5	21.6	21			82.1	81.3
4	19.5	19.0	23.5	24.0	22			87.0	86.1
6	21.9	22.6			23			91.9	91.0
8	24.6	27.4			24			97.0	96.1
					25			102.1	101.3
					26			107.4	106.6
					27			112.7	112.0

β_0 term is needed in the potential, the fitting therefore leading to $\beta_0 = 0$ (i.e., to pure harmonic behavior).

(ii) In the next two isotopes (^{130}Xe and ^{128}Xe) the need to depart from the pure harmonic oscillator becomes clear, the fitting leading therefore to nonzero β_0 values. However, there is still no need of dependence of the mass on the deformation, the fitting still leading to $a = 0$.

(iii) Beyond ^{126}Xe both the β_0 term in the potential and the deformation dependence of the mass become necessary, leading to nonzero values of both β_0 and a.

Other chains of isotopes also show similar behavior.

B. Spectra of axially symmetric deformed nuclei

Fits of spectra of deformed rare earth and actinide nuclei are shown in Table II. The energy levels of the ground-state band and the β_1 band (both having $n_\gamma = 0$ and $K = 0$) are obtained from Eqs. (83) and (84), respectively, whereas the levels of the γ_1 band are obtained from Eq. (83) with $n_\gamma = 1$ and $K = 2$. Again, the choice $\delta = \lambda = 0$ was made, and it is seen that different choices for δ and λ lead to a renormalization of the parameter values a, β_0, and c, the predicted energy levels remaining exactly the same.

The quality of the fits obtained can also be seen in Table III, where the calculated energy levels of ^{162}Dy and ^{238}U are compared to experiment.

The following remarks apply.

(i) Both the bandheads and the spacings within bands are in general well reproduced. This is particularly true for the ground state and the γ_1 bands. The deviation in the gsb of ^{162}Dy reaches 0.6% at $L = 18$, while in the gsb of ^{238}U it reaches 1.7% at $L = 30$. The experimental levels of the γ_1 band of ^{162}Dy (up to $L = 14$) extend over 28.4 energy units, whereas the corresponding theoretical predictions spread over 28.7 units, the difference being of the order of 1%. Similarly in ^{238}U the experimental spread of the γ_1 band (up to $L = 27$) is 89.1 energy units, whereas the theoretical one is 87.3 units, the difference being of the order of 2%.

(ii) However we remark that the theoretical level spacings within the β_1 bands are larger than the experimental ones. This should be attributed to the shape of the Davidson potential, which raises to infinity at large β, pushing β bands higher and increasing their interlevel spacing. It is known that this problem can be avoided

TABLE IV. Comparison of experimental data [40] (upper line) for several $B(E2)$ ratios of γ-unstable nuclei to predictions (lower line) by the Bohr Hamiltonian with β-dependent mass (with $\delta = \lambda = 0$), for the parameter values shown in Table I. See subsection XI C for further discussion.

Nucleus	$\frac{4_1\to2_1}{2_1\to0_1}$	$\frac{6_1\to4_1}{2_1\to0_1}$	$\frac{8_1\to6_1}{2_1\to0_1}$	$\frac{10_1\to8_1}{2_1\to0_1}$	$\frac{2_2\to2_1}{2_1\to0_1}$	$\frac{2_2\to0_1}{2_1\to0_1}$ $\times10^3$	$\frac{0_2\to2_1}{2_1\to0_1}$	$\frac{2_3\to0_1}{2_1\to0_1}$ $\times10^3$
^{98}Ru	1.44(25)				1.62(61)	36.0(152)		
	1.82	2.62	3.42	4.22	1.82	0.0	1.36	3.60
^{100}Ru	1.45(13)				0.64(12)	41.1(52)	0.98(15)	
	1.72	2.40	3.07	3.73	1.72	0.0	1.05	10.89
^{102}Ru	1.50(24)				0.62(7)	24.8(7)	0.80(14)	
	1.78	2.54	3.28	4.01	1.78	0.0	1.27	8.70
^{104}Ru	1.18(28)				0.63(15)	35.0(84)	0.42(7)	
	1.63	2.18	2.71	3.21	1.63	0.0	0.79	22.41
^{102}Pd	1.56(19)				0.46(9)	128.8(735)		
	1.76	2.49	3.19	3.87	1.76	0.0	1.22	12.34
^{104}Pd	1.36(27)				0.61(8)	33.3(74)		
	1.74	2.45	3.15	3.85	1.74	0.0	1.11	8.13
^{106}Pd	1.63(28)				0.98(12)	26.2(31)	0.67(18)	
	1.85	2.67	3.49	4.28	1.85	0.0	1.49	5.98
^{108}Pd	1.47(20)	2.16(28)	2.99(48)		1.43(14)	16.6(18)	1.05(13)	1.90(29)
	1.75	2.45	3.12	3.75	1.75	0.0	1.20	15.82
^{110}Pd	1.71(34)				0.98(24)	14.1(22)	0.64(10)	
	1.76	2.43	3.01	3.51	1.76	0.0	1.31	26.24
^{106}Cd	1.78(25)				0.43(12)	93.0(127)		
	1.68	2.32	2.95	3.58	1.68	0.0	0.92	10.44
^{108}Cd	1.54(24)				0.64(20)	67.7(120)		
	1.85	2.69	3.52	4.35	1.85	0.0	1.49	4.06
^{110}Cd	1.68(24)				1.09(19)	48.9(78)		9.85(595)
	1.99	2.97	3.93	4.87	1.99	0.0	1.98	1.61
^{112}Cd	2.02(22)				0.50(10)	19.9(35)	1.69(48)	11.26(210)
	2.00	2.99	3.98	4.96	2.00	0.0	1.99	0.48
^{114}Cd	1.99(25)	3.83(72)	2.73(97)		0.71(24)	15.4(29)	0.88(11)	10.61(193)
	2.00	2.99	3.97	4.94	2.00	0.0	1.99	0.74
^{116}Cd	1.70(52)				0.63(46)	32.8(86)	0.02	
	1.74	2.46	3.17	3.90	1.74	0.0	1.11	4.42
^{118}Cd	$>$1.85						0.16(4)	
	1.71	2.39	3.06	3.74	1.71	0.0	1.00	5.88
^{118}Xe	1.11(7)	0.88(27)	0.49(20)	$>$0.73				
	1.67	2.28	2.85	3.39	1.67	0.0	0.95	21.93
^{120}Xe	1.16(14)	1.17(24)	0.96(22)	0.91(19)				
	1.60	2.11	2.60	3.08	1.60	0.0	0.67	21.51
^{122}Xe	1.47(38)	0.89(26)	$>$0.44					
	1.58	2.05	2.48	2.89	1.58	0.0	0.63	29.29
^{124}Xe	1.34(24)	1.59(71)	0.63(29)	0.29(8)	0.70(19)	15.9(46)		
	1.59	2.09	2.57	3.04	1.59	0.0	0.63	20.14
^{128}Xe	1.47(20)	1.94(26)	2.39(40)	2.74(114)	1.19(19)	15.9(23)		
	1.63	2.20	2.75	3.31	1.63	0.0	0.73	9.64
^{132}Xe	1.24(18)				1.77(29)	3.4(7)		
	2.00	3.00	4.00	5.00	2.00	0.0	2.00	0.00
^{130}Ba	1.36(6)	1.62(15)	1.55(56)	0.93(15)				
	1.56	2.01	2.41	2.77	1.56	0.0	0.61	34.54
^{132}Ba					3.35(64)	90.7(177)		
	1.68	2.30	2.90	3.50	1.68	0.0	0.92	15.21
^{134}Ba	1.55(21)				2.17(69)	12.5(41)		
	1.75	2.48	3.21	3.94	1.75	0.0	1.14	4.08
^{142}Ba	1.40(17)	0.56(14)						
	1.55	2.00	2.41	2.82	1.55	0.0	0.49	18.60
^{148}Nd	1.61(13)	1.76(19)			0.25(4)	9.3(17)	0.54(6)	32.82(816)
	1.63	2.17	2.68	3.15	1.63	0.0	0.81	26.86

TABLE IV. (*Continued*)

Nucleus	$\frac{4_1\to2_1}{2_1\to0_1}$	$\frac{6_1\to4_1}{2_1\to0_1}$	$\frac{8_1\to6_1}{2_1\to0_1}$	$\frac{10_1\to8_1}{2_1\to0_1}$	$\frac{2_2\to2_1}{2_1\to0_1}$	$\frac{2_2\to0_1}{2_1\to0_1}\times10^3$	$\frac{0_2\to2_1}{2_1\to0_1}$	$\frac{2_3\to0_1}{2_1\to0_1}\times10^3$
^{152}Gd	1.84(29)	2.74(81)			0.23(4)	4.2(8)	2.47(78)	
	1.98	2.92	3.81	4.65	1.98	0.0	1.95	4.51
^{154}Dy	1.62(35)	2.05(42)	2.27(62)	1.86(69)				
	1.91	2.79	3.64	4.46	1.91	0.0	1.70	5.41
^{156}Er	1.78(16)	1.89(36)	0.76(20)	0.88(22)				
	1.70	2.35	3.00	3.64	1.70	0.0	0.98	11.50
^{192}Pt	1.56(12)	1.23(55)			1.91(16)	9.5(9)		
	1.59	2.09	2.57	3.05	1.59	0.0	0.61	16.98
^{194}Pt	1.73(13)	1.36(45)	1.02(30)	0.69(19)	1.81(25)	5.9(9)	0.01	
	1.59	2.09	2.57	3.04	1.59	0.0	0.63	19.78
^{196}Pt	1.48(3)	1.80(23)	1.92(23)			0.4	0.07(4)	0.06(6)
	1.64	2.21	2.75	3.28	1.64	0.0	0.82	20.83
^{198}Pt	1.19(13)	>1.78			1.16(23)	1.2(4)	0.81(22)	1.56(126)
	1.82	2.60	3.36	4.08	1.82	0.0	1.41	10.09

by using a potential going to some finite value at large β [41], like the Morse potential [42].

C. $B(E2)$s of γ-unstable nuclei

$B(E2)$s within the ground-state band, as well as interband $B(E2)$s for which experimental data exist for several nuclei, have been calculated using the procedure described in Sec. X A. The results are shown in Table IV, the overall agreement being good.

D. $B(E2)$s of axially symmetric deformed nuclei

$B(E2)$s within the ground-state band, as well as interband $B(E2)$s for which experimental data exist for several nuclei, have been calculated using the procedure described in Sec. X B. The results are shown in Table V. The overall agreement is good for transitions within the ground-state band (gsb), as well as for transitions connecting the γ_1 band to the gsb, while transitions from the β_1 band to the gsb tend to be overpredicted. One should remember at this point that the β_1 band was the one giving poor results also in the case of the spectra (in Sec. XI B).

XII. CONNECTION TO EARLIER WORK

It is instructive to examine the relation between the present approach and earlier numerical work.

(i) The formalism of Sec. III B clarifies the relation between the present approach and the numerical solution of Kumar and Baranger [26], who used a matrix of the form (14) with g_{ij}, $i, j = 1, 2, 3$ the same as in Eq. (15), but with

$$g_{44} = B_{00}, \quad g_{55} = B_{2'2'}, \quad g_{45} = g_{54} = B_{02'}, \tag{122}$$

where B_{00}, $B_{2'2'}$, $B_{02'}$, as well as the moments of inertia $\mathcal{J}_i$ $(i = 1, 2, 3)$ and the potential V have been treated as seven arbitrary functions of the variables $\beta_0 = \beta\cos\gamma$ and $\beta_{2'} = \beta\sin\gamma$ [while in the Bohr formulation [1] $a_0 = \beta\cos\gamma$ and $a_2 = \beta\sin\gamma/\sqrt{2}$ are used]. On one hand, the present solution is a special case of Ref. [26] because it contains no nondiagonal terms $g_{45} = g_{54}$. On the other hand, in the present approach the above-mentioned quantities are interrelated by the overall symmetry in a specific way, greatly reducing the number of free parameters (down to two or three in total). It should be pointed out that the functional dependence of the mass on the deformation for the potential used is dictated by SUSYQM. Therefore, the successful prediction of the behavior of the moments of inertia, for example, provides credit for the present approach. What we see, independently of the parameter values, is that the increase of the moments of inertia as a function of deformation is moderated by the f^2 factor, which can be seen as a result of the dependence of the mass on the deformation, or, alternatively, as seen in Sec. III B, as a result of using a curved space.

(ii) It should be pointed out that in Ref. [9] the equivalence between the position-dependent mass case and the curved space approach was established in the special case of $\kappa = 2$ and $\delta = \lambda = 0$ [see Eq. (3) for the meaning of the symbols], which represents the BenDaniel and Duke Hamiltonian [43],

$$H_{\rm BD} = -\frac{\hbar^2}{2}\nabla f^2\nabla + V_{\rm BD}. \tag{123}$$

This resembles the collective Hamiltonian,

$$H_{\rm coll} = -\frac{\hbar^2}{2}\Sigma_{i,j}\frac{\partial}{\partial q_i}[\mathcal{M}(q)_{ij}]^{-1}\frac{\partial}{\partial q_j} + V(q), \tag{124}$$

used by Libert *et al.* [27] in mean-field calculations, in which a tensor mass appears.

XIII. CONCLUSION

In the present work analytical solutions are obtained for a Bohr Hamiltonian in which the mass was allowed to depend on the deformation.

TABLE V. Comparison of experimental data [40] (upper line) for several $B(E2)$ ratios of axially symmetric prolate deformed nuclei to predictions (lower line) by the Bohr Hamiltonian with β-dependent mass (with $\delta = \lambda = 0$), for the parameter values shown in Table II. See Sec. XI D for further discussion.

Nucleus	$\frac{4_1\to2_1}{2_1\to0_1}$	$\frac{6_1\to4_1}{2_1\to0_1}$	$\frac{8_1\to6_1}{2_1\to0_1}$	$\frac{10_1\to8_1}{2_1\to0_1}$	$\frac{2_\beta\to0_1}{2_1\to0_1}$ $\times10^3$	$\frac{2_\beta\to2_1}{2_1\to0_1}$ $\times10^3$	$\frac{2_\beta\to4_1}{2_1\to0_1}$ $\times10^3$	$\frac{2_\gamma\to0_1}{2_1\to0_1}$ $\times10^3$	$\frac{2_\gamma\to2_1}{2_1\to0_1}$ $\times10^3$	$\frac{2_\gamma\to4_1}{2_1\to0_1}$ $\times10^3$
^{154}Sm	1.40(5)	1.67(7)	1.83(11)	1.81(11)	5.4(13)		25(6)	18.4(34)		3.9(7)
	1.47	1.69	1.87	2.06	26.7	50.0	150	47.5	69.6	3.7
^{156}Gd	1.41(5)	1.58(6)	1.71(10)	1.68(9)	3.4(3)	18(2)	22(2)	25.0(15)	38.7(24)	4.1(3)
	1.48	1.73	1.95	2.18	29.7	59.1	191	62.5	92.4	4.9
^{158}Gd	1.46(5)		1.67(16)	1.72(16)	1.6(2)	0.4(1)	7.0(8)	17.2(20)	30.3(45)	1.4(2)
	1.46	1.66	1.82	1.98	25.7	45.9	127	64.0	93.0	4.8
^{158}Dy	1.45(10)	1.86(12)	1.86(38)	1.75(28)	12(3)	19(4)	66(16)	32.2(78)	103.8(258)	11.5(48)
	1.50	1.78	2.04	2.31	30.5	65.4	232	88.5	131.7	7.1
^{160}Dy	1.46(7)	1.23(7)	1.70(16)	1.69(9)	3.4(4)		8.5(10)	23.2(21)	43.8(42)	3.1(3)
	1.46	1.68	1.85	2.03	22.9	43.5	133	78.6	114.5	6.0
^{162}Dy	1.45(7)	1.51(10)	1.74(10)	1.76(13)				0.12(1)	0.20	0.02
	1.45	1.65	1.80	1.95	23.9	42.4	116	89.8	129.8	6.7
^{164}Dy	1.30(7)	1.56(7)	1.48(9)	1.69(9)				19.1(22)	38.3(39)	4.6(5)
	1.44	1.62	1.75	1.86	16.9	29.1	77	99.7	143.4	7.3
^{162}Er					8(7)		170(90)	32.5(28)	77.0(56)	9.4(69)
	1.49	1.75	1.99	2.24	27.8	58.3	202	91.1	134.8	7.2
^{164}Er	1.18(13)		1.57(9)	1.64(11)				23.9(35)	52.3(72)	7.8(12)
	1.47	1.70	1.89	2.09	28.3	53.5	162	103.8	151.2	7.9
^{166}Er	1.45(12)	1.62(22)	1.71(25)	1.73(23)				25.7(31)	45.3(54)	3.1(4)
	1.46	1.66	1.81	1.96	20.7	38.2	111	100.0	144.8	7.4
^{168}Er	1.54(7)	2.13(16)	1.69(11)	1.46(11)				23.2(15)	41.1(31)	3.0(3)
	1.45	1.65	1.79	1.93	27.7	47.2	120	100.6	145.1	7.4
^{170}Er			1.78(15)	1.54(11)	1.4(1)	0.2(2)	6.8(12)	17.7(9)		1.4(4)
	1.47	1.69	1.86	2.03	39.2	67.9	177	78.6	114.2	5.9
^{166}Yb	1.43(9)	1.53(10)	1.70(18)	1.61(80)						
	1.50	1.78	2.05	2.33	33.7	71.0	245	97.2	144.5	7.8
^{168}Yb					8.6(9)			22.0(55)	45.9(73)	8.6
	1.48	1.72	1.93	2.14	29.6	57.5	180	82.9	121.6	6.4
^{170}Yb			1.79(16)	1.77(14)	5.4(10)			13.4(34)	23.9(57)	2.4(6)
	1.47	1.71	1.91	2.12	30.6	58.2	176	66.2	97.1	5.1
^{172}Yb	1.42(10)	1.51(14)	1.89(19)	1.77(11)	1.1(1)	3.7(6)	12(1)	6.3(6)		0.6(1)
	1.46	1.67	1.83	1.99	32.2	55.9	147	51.6	75.0	3.9
^{174}Yb	1.39(7)	1.84(26)	1.93(12)	1.67(12)					12.4(29)	
	1.45	1.63	1.75	1.86	20.9	35.1	88	45.0	64.9	3.3
^{176}Yb	1.49(15)	1.63(14)	1.65(28)	1.76(18)				9.8		
	1.46	1.66	1.82	1.97	27.9	49.0	132	63.1	91.6	4.7
^{174}Hf					14(4)		9(3)	31.6(161)	48.7(124)	
	1.48	1.74	1.96	2.20	31.4	62.2	200	66.9	98.8	5.3
^{176}Hf					5.4(11)		31(6)	21.3(26)		
	1.47	1.70	1.89	2.09	30.8	57.3	169	57.9	84.9	4.5
^{178}Hf		1.38(9)	1.49(6)	1.62(7)	0.4(2)		2.4(9)	24.5(39)	27.7(28)	1.6(2)
	1.47	1.69	1.88	2.07	28.4	53.1	158	73.8	107.8	5.6
^{180}Hf	1.48(20)	1.41(15)	1.61(26)	1.55(10)				24.5(47)	32.9(56)	
	1.46	1.66	1.82	1.98	34.9	59.5	151	78.4	113.4	5.8
^{182}W	1.43(8)	1.46(16)	1.53(14)	1.48(14)	6.6(6)	4.6(6)	13(1)	24.8(12)	49.2(24)	0.2
	1.47	1.69	1.87	2.04	32.5	58.3	162	79.9	116.2	6.0
^{184}W	1.35(12)	1.54(9)	2.00(18)	2.45(51)	1.8(3)		24(3)	37.1(28)	70.6(51)	4.0(4)
	1.48	1.73	1.95	2.16	40.7	75.2	216	128.3	187.3	9.8
^{186}W	1.30(9)	1.69(12)	1.60(12)	1.36(36)				41.7(92)	91.0(201)	
	1.51	1.80	2.07	2.34	46.2	91.9	289	165.7	244.5	12.9
^{186}Os	1.45(7)	1.99(7)	1.89(11)	2.06(44)				109.4(71)	254.6(150)	13.0(47)
	1.53	1.87	2.20	2.55	39.7	90.2	335	164.9	247.4	13.4
^{188}Os	1.68(11)	1.75(11)	2.04(15)	2.38(32)				63.3(92)	202.5(304)	43.0(74)
	1.54	1.89	2.25	2.63	33.9	83.9	344	229.8	345.2	18.7

TABLE V. (*Continued*)

Nucleus	$\frac{4_1\to2_1}{2_1\to0_1}$	$\frac{6_1\to4_1}{2_1\to0_1}$	$\frac{8_1\to6_1}{2_1\to0_1}$	$\frac{10_1\to8_1}{2_1\to0_1}$	$\frac{2_\beta\to0_1}{2_1\to0_1}\times10^3$	$\frac{2_\beta\to2_1}{2_1\to0_1}\times10^3$	$\frac{2_\beta\to4_1}{2_1\to0_1}\times10^3$	$\frac{2_\gamma\to0_1}{2_1\to0_1}\times10^3$	$\frac{2_\gamma\to2_1}{2_1\to0_1}\times10^3$	$\frac{2_\gamma\to4_1}{2_1\to0_1}\times10^3$
^{230}Th	1.36(8)				5.7(26)		20(11)	15.6(59)	28.1(100)	1.8(11)
	1.47	1.70	1.90	2.09	30.0	56.4	168	63.6	93.2	4.9
^{232}Th	1.44(15)	1.65(14)	1.73(12)	1.82(15)	14(6)	2.6(13)	17(8)	14.6(28)	36.4(56)	0.7
	1.46	1.67	1.84	2.01	25.8	47.1	135	57.0	83.0	4.3
^{234}U								12.5(27)	21.1(44)	1.2(3)
	1.45	1.64	1.78	1.90	20.7	36.1	97	42.7	61.8	3.2
^{236}U	1.42(11)	1.55(11)	1.59(17)	1.46(17)						
	1.45	1.63	1.76	1.87	19.3	33.2	87	44.7	64.5	3.3
^{238}U			1.45(23)	1.71(22)	1.4(6)	3.6(14)	12(5)	10.8(8)	18.9(17)	1.2(1)
	1.45	1.63	1.75	1.86	18.9	32.3	83	37.7	54.5	2.8
^{238}Pu					14(4)		11(4)			
	1.44	1.62	1.73	1.84	19.1	31.7	78	41.6	59.9	3.0
^{250}Cf								6.8(17)	10.9(25)	0.6(1)
	1.44	1.61	1.72	1.81	15.0	24.9	61	40.0	57.5	2.9

From the mathematical point of view, this is achieved through the use of techniques of supersymmetric quantum mechanics [10,11], involving a deformed shape invariance condition [12]. Analytical expressions for the spectra and wave functions have been obtained.

From the physics point of view, spectra and $B(E2)$ transition rates have been calculated for γ-unstable, axially symmetric prolate deformed, and triaxial nuclei, implementing the usual approximations in each case, and compared to experimental data for the first two cases. The main new result is that the dependence of the mass on the deformation moderates the increase of the moment of inertia with the deformation, removing an important drawback [4] of the model. It should be emphasized that the functional dependence of the mass on the deformation for the potential used is dictated by SUSYQM, thus the correction in the behavior of the moments of inertia is a general effect, independent of any specific parameter value combinations.

However, certain discrepancies with experimental data remain, especially related to the β_1 band and its interband transitions. It should be remembered at this point that in the present study separation of variables was achieved by *assuming* that the potential either is independent of the γ variable, or it has the exactly separable form of Eq. (34). Furthermore, the approximations related to Eqs. (35) and (44) have been implemented. Recently, the numerical solution of the Bohr Hamiltonian for any value of β and γ, avoiding all these approximations, was achieved in the framework of the powerful algebraic collective model [44–46]. The detailed study of discrepancies from experimental data both in the SUSYQM framework and in the context of the algebraic model, especially for multiphonon excitations [47], could shed light on the origins of these discrepancies.

As it was already mentioned, the form of the dependence of the mass on the deformation is dictated by SUSYQM for the potential used in the β degree of freedom. In the present work, the Davidson potential was used, called the deformation-dependent mass (DDM) Davidson model. The application of the SUSYQM approach to the Bohr Hamiltonian with the Kratzer potential [48,49] is receiving attention.

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APPENDIX

When using Eq. (112) in numerical calculations, problems can appear because of $\Gamma(x)$ functions with large x. These problems can be avoided by using Eq. (6.1.16) of Ref. [36],

$$\Gamma(n+z) = (n-1+z)(n-2+z)\dots(1+z)\Gamma(1+z). \tag{A1}$$

In the normalization factors we need the ratio of

$$\Gamma\left(n+\frac{\Delta_1+\Delta_2}{2}+1\right), \tag{A2}$$

over

$$\Gamma\left(n+\frac{\Delta_2}{2}+1\right). \tag{A3}$$

Let us call I the integer part of $\Delta_2/2$ and r the rest of it, that is,

$$\frac{\Delta_2}{2} = I + r. \tag{A4}$$

Then we have

$$\Gamma\left(n+\frac{\Delta_1+\Delta_2}{2}+1\right)=\Gamma\left(I+(n+\Delta_1/2+r+1)\right)=(I-1+n+\Delta_1/2+r+1)(I-2+n+\Delta_1/2+r+1)$$
$$\times\cdots(1+n+\Delta_1/2+r+1)\Gamma(1+n+\Delta_1/2+r+1), \tag{A5}$$

$$\Gamma(n+\frac{\Delta_2}{2}+1)=\Gamma(I+(n+r+1))=(I-1+n+r+1)(I-2+n+r+1)\ldots(1+n+r+1)\Gamma(1+n+r+1). \tag{A6}$$

Their ratio becomes

$$\begin{aligned}\frac{\Gamma\left(n+\frac{\Delta_1+\Delta_2}{2}+1\right)}{\Gamma\left(n+\frac{\Delta_2}{2}+1\right)}&=\frac{(I-1+n+\Delta_1/2+r+1)}{(I-1+n+r+1)}\frac{(I-2+n+\Delta_1/2+r+1)}{(I-2+n+r+1)}\\&\times\cdots\frac{(1+n+\Delta_1/2+r+1)}{(1+n+r+1)}\frac{\Gamma(1+n+\Delta_1/2+r+1)}{\Gamma(1+n+r+1)}\\&=\left(1+\frac{\Delta_1/2}{(I-1+n+r+1)}\right)\quad\left(1+\frac{\Delta_1/2}{(I-2+n+r+1)}\right)\\&\times\ldots\left(1+\frac{\Delta_1/2}{(1+n+r+1)}\right)\frac{\Gamma(1+n+\Delta_1/2+r+1)}{\Gamma(1+n+r+1)},\end{aligned} \tag{A7}$$

in which one does not have to calculate $\Gamma(x)$ functions with large x. The only large numbers appear in denominators of fractions accompanying 1, which do not pose any problem.

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