Pions are neither perturbative nor nonperturbative: Wilsonian renormalization-group analysis of nuclear effective field theory including pions

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Nuclear effective field theory (NEFT) including pions in the two-nucleon sector is examined from the Wilsonian renormalization group point of view. The pion exchange is cut off at the floating cutoff scale, Λ, with the short-distance part being represented as contact interactions in accordance with the general principle of renormalization. We derive the nonperturbative renormalization group equations in the leading order of the nonrelativistic approximation in the operator space up to including $\mathcal{O}(p^2)$, and find the nontrivial fixed points in the $^1S_0$ and $^3S_1$-$^3D_1$ channels which are identified with those in the pionless NEFT. The scaling dimensions, which determine the power counting, of the contact interactions at the nontrivial fixed points are also identified with those in the pionless NEFT. We emphasize the importance of the separation of the pion exchange into the short-distance and the long-distance parts, since a part of the former is nonperturbative while the latter is perturbative.

PACS number(s): 13.75.Cs, 11.10.Hi, 12.39.Fe, 21.45.Bc

I. INTRODUCTION

Nuclear effective field theory (NEFT) is the low-energy effective field theory [1] of nucleons based on symmetries of QCD and is expected to give a model-independent description of nuclear phenomena at low energies. Since the seminal articles of Weinberg [2–4], a lot of applications have been done with success for more than two decades. See Refs. [5–9] for reviews. It still has a wide range of phenomena to be explored.

In spite of these successes, however, there are unsettled issues at the fundamental level, even in the simplest two-nucleon sector, specifically regarding whether the pions should be treated perturbatively and how to treat the contact interactions. Because the understanding of the two-nucleon scattering is essential also for other phenomena, this issue is of central importance in NEFT.

The original power counting due to Weinberg [2–4] for the "effective potential" is nothing but the naive dimensional analysis. The "effective potential" is plugged into the Lippmann-Schwinger equation and the scattering amplitude is calculated, so the pion exchange is iterated infinite times. In this Weinberg scheme the pions are thus treated nonperturbatively. Kaplan, Savage, and Wise (KSW) [10] pointed out that the Weinberg power counting scheme is inconsistent in the sense that the higher-order counterterms are required to cancel the divergences at the given order. They (and, independently,

van Kolck) proposed an alternative power counting scheme in which only the nonderivative contact interaction, the C_0 term, is treated nonperturbatively so the pions are treated perturbatively [11–13]. This scheme (widely known as KSW scheme) is free from the inconsistency, but Fleming, Mehen, and Stewart [14] showed that the KSW scheme does not lead to the converging results in the channels in which the singular tensor part of the pion exchange contributes. Beane, Bedaque, Savage, and van Kolck [15] proposed a remedy, known as the hybrid approach, in which the amplitudes are expanded around the chiral limit so the $1/r^3$ part (which survives in the limit) is treated nonperturbatively. The treatment of the tensor part of the one-pion exchange (OPE) is the source of controversy. See also Refs. [16–21] for more details.

Recently, Beane, Kaplan, and Vuorinen (BKV) [22] considered Pauli-Villars–type regularization for the OPE in order to separate the short-distance part of the tensor interaction from the long-distance part. The short-distance part is represented by contact interactions. They employed the power divergence subtraction (PDS) renormalization and obtained the convergent results. Note that not all the $1/r^3$ part of the OPE is singular. It is the *short-distance part* that is singular and spoils the convergence of the amplitude as found in Ref. [14]. It means that all the $1/r^3$ part of the OPE does not need to be iterated.

The idea of separating the singular short-distance part of the tensor interaction from the long-distance part and representing the former by contact interactions (BKV prescription) is essentially the Wilsonian renormalization group (RG) idea of effective interactions [23,24]. The separation scale in Ref. [22]

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may be viewed as an analog of the floating cutoff in the Wilsonian RG.

The Wilsonian RG is a useful tool to investigate the effects of the short-distance physics on the long-distance physics. When the quantum fluctuations with momenta higher than the floating cutoff Λ are integrated out, their effects are simulated by a series of local operators, which serve as low-momentum effective interactions. The coupling constants thus depend on the cutoff Λ, while the physical quantities such as scattering amplitudes do not.

The Wilsonian RG has been applied to NEFT in order to examine the power counting issue [19,25–30]. (See also Refs. [31–33] for other use of the Wilsonian RG in NEFT.) The existence of the nontrivial fixed points of the RG equations (RGEs) in the S waves explains unnaturally large scattering lengths. The power counting can be determined by the scaling dimensions of operators around the fixed points. In particular, it is relevant operators that should be resummed to all orders.

Birse [19] examines the NEFT with pions by using the "distorted-wave RG [26]" and claims that the scaling dimensions are shifted from those of the pionless theory due to the singularity of the tensor force in the spin-triplet channel. Also, his results imply that the effects of pions do not decouple even at very low momenta.

It sounds strange, however, that pions do not decouple at the momenta where the pionless NEFT is valid. The effects should be eventually represented by contact interactions at very low momenta. One should expect that the transition from the NEFT including pions to the pionless NEFT is smooth. A formulation of the Wilsonian RG which permits the smooth transition is desired.

In this article, we perform the Wilsonian RG analysis of the NEFT including pions in the two-nucleon sector and examine the effects of pions. In the Wilsonian RG approach, there is a single scale Λ which separates the long-distance physics from the short-distance physics so one does not need to implement an additional regularization for the short-distance part of the OPE, as does in the BKV prescription.

The key idea here is the separation of the short-distance part from the long-distance part. NEFT, as a low-energy effective theory, has a finite range of applicability, specified by the physical cutoff Λ_0. It is a general principle of EFT that the physics with the momentum scale larger than Λ_0 is represented as local operators. The short-distance part of the OPE (S-OPE) should also be represented as local operators (contact interactions). (In the Wilsonian RG analysis, the floating cutoff Λ plays the role of Λ_0, after integrating the modes with momenta between Λ and Λ_0.)

Note that, although the contact interactions are attributed to the effects of the heavy particles in the usual EFT lore, this is not precise. They arise also from the short-distance physics of light particles, such as pion exchanges. Note also that the representation of short-distance physics by local interactions is a general treatment and has nothing to do with how singular the S-OPE is.

We find nontrivial fixed points in the $^1S_0$ and $^3S_1$-$^3D_1$ channels, which are responsible for the large scattering lengths. Importantly these fixed points are identified with those found in the Wilsonian RG analysis in the pionless

NEFT. Thus the scaling dimensions at the fixed points are also the same. There is one relevant operator in each channel. That is, the pion interactions do not alter the scaling dimensions.

We emphasize that the question of whether the pions are perturbative or nonperturbative is not well posed. Since the S-OPE is represented as contact interactions in the Wilsonian analysis, only the long-distance part of the OPE (L-OPE) is the proper interactions due to OPE. On the other hand, the S-OPE cannot be distinguished from other contributions, such as heavier meson (ρ, ω, etc.) exchanges. (Because of the cutoff, we do not have enough resolution.) The RGEs tell us that, at the nontrivial fixed point, the L-OPE should be treated perturbatively, while there is a relevant operator (a part of which is the S-OPE) that should be resummed to all orders, i.e., nonperturbatively. The existence of the relevant operator at the nontrivial fixed point sharpens the distinction between the S-OPE and the L-OPE. If there were no relevant operator, the separation would not make much difference.

Our Wilsonian RG permits the smooth transition from the NEFT including pions to the pionless NEFT. The nontrivial fixed points do not change nor the scaling dimensions. The L-OPE transmutes into contact interactions which represent the S-OPE as the cutoff is lowered. At the value of Λ lower than the pion mass, the most of the L-OPE has been changed into contact interactions, thus the system is well described by the pionless NEFT.

The structure of the article is as follows. In Sec. II we recapitulate the main points of the previous articles in order to introduce the notations and the main concepts in our analysis. In Sec. III we discuss the nonrelativistic approximation. Starting with the nonrelativistic nucleons and the relativistic pions, we estimate the order of magnitude of a various kind of diagrams contributing to the RGEs and determine the leading terms. In Sec. IV we present the RGEs and examine the structure of the flows in the $^1S_0$ and $^3S_1$-$^3D_1$ channels. The nontrivial fixed points are found to be identified with those of the pionless theory. Section V is devoted to the summary and the comments on the related works. In Appendix A the RGEs for the case of $\Lambda < m_\pi$ are presented and compared with the pionless case. In Appendix B we discuss the similarity and the difference between the NEFT with pions and QED.

II. WILSONIAN RG ANALYSIS OF PIONLESS NEFT IN TWO-NUCLEON SECTOR

In previous articles [28–30], we have explained the basic concepts of the Wilsonian RG and its relevance to the power counting in the NEFT. Here, we briefly recapitulate it, give some remarks, and introduce the notations used in later sections.

A. What is the use of the Wilsonian RG in NEFT?

The most basic idea behind the power counting is the order of magnitude estimate based on dimensional analysis. In an EFT with the physical cutoff Λ_0, the dimensional analysis

is usually based on this scale. At the classical level, the (canonical) mass dimension of an operator is determined with respect to the kinetic term. For example, a Dirac field ψ with a kinetic term $\mathcal{L}_{\text{kin}} = \bar{\psi} i \partial\!\!\!/ \psi$ has mass dimension three halves, $[\psi] = 3/2$. The dimensions of other operators are determined accordingly. The operator $(\bar{\psi}\psi)^2$ has dimension six, so it enters in the Lagrangian as $(c/\Lambda_0^2)(\bar{\psi}\psi)^2$, where c is a dimensionless constant. The coupling constant c/Λ_0^2 associated with the operator $(\bar{\psi}\psi)^2$ has dimension -2, which counts the power of Λ_0.

Quantum fluctuations may, in general, change the classical dimensional analysis. The quantum counterpart of the (canonical) dimension is called the *scaling dimension*, which can be obtained by the RG analysis. Wilsonian RG is a nonperturbative tool to handle the quantum fluctuations.

An operator whose coupling has a negative (scaling) dimension is called *irrelevant* because it becomes less important at lower energies. An operator whose coupling has a positive (scaling) dimension is called *relevant* because it becomes more important at lower energies. An operator whose coupling is dimensionless is called *marginal*. The scaling dimension of an operator is the measure of how important it is. It is therefore natural to consider the power counting on the basis of the scaling dimensions.

In the S-wave scattering of two nucleons, the scattering lengths are known to be much larger than the "natural" size, $1/\Lambda_0$. [In the case of the pionless NEFT, the physical cutoff is of order of the pion mass, $\Lambda_0 \sim \mathcal{O}(m_\pi)$.] From the RG point of view, the "fine-tuning" is related to the existence of a nontrivial fixed point (and a critical surface) of the RG flow.

Around the nontrivial fixed points the scaling dimensions differ drastically from the canonical dimensions. It has been shown [25,29,30] that the coupling which corresponds to the scattering length becomes relevant, although it is irrelevant at the classical level.

There are values of coupling constants with which the scattering length is infinite. This set of coupling constants forms a critical surface. It separates the weak- and strong-coupling phases. For the two-nucleon system, the spin-singlet channel is considered to be in the weak-coupling phase, while the spin-triplet channel is considered to be in the strong-coupling phase, because of the (non-)existence of a bound state in these channels.

To summarize: the two-nucleon system with large S-wave scattering lengths is governed by the existence of nontrivial fixed points, and the Wilsonian RG is a systematic tool to study the scaling dimensions on which the power counting should be based, around the nontrivial fixed points.

B. Remarks on Wilsonian RGEs with Galilean invariance

There are several formulations for the Wilsonian RG analysis [34–39], which are, however, essentially equivalent. The most popular one is the functional RG method. (See Refs. [40,41] for reviews.) In this formulation, a cutoff function is introduced for each propagator to suppress the low-frequency fluctuations. The effective averaged action $\Gamma_\Lambda[\Phi]$, which interpolates the bare classical action $S[\Phi]$ ($\Lambda = \Lambda_0$)

and the effective action $\Gamma[\Phi]$ ($\Lambda = 0$), depends on the floating cutoff scale, Λ, as

$$\frac{d\Gamma_\Lambda}{d\Lambda} = \frac{1}{2} \text{Tr} \left[\frac{dR_\Lambda}{d\Lambda} (\Gamma_{(2)} + R_\Lambda)^{-1} \right], \qquad (2.1)$$

where Φ is the classical field, $\Gamma_{(2)}$ stands for the second derivative of the averaged action Γ_Λ with respect to Φ, and R_Λ is the cutoff function which suppresses the fluctuations with $p \lesssim \Lambda$. Note that, although it is an "one-loop" equation, it contains all the nonperturbative information.

A straightforward application of this formulation to a nonrelativistic system, however, encounters difficulties. In the usual formulation for a relativistic system, one considers the theory in Euclidean space. The cutoff is imposed on the magnitude of the four-momentum of the propagator. On the other hand, in a nonrelativistic system, space and time should be treated differently, and thus the Euclidean formulation cannot be used. One might rather like to consider a cutoff imposed on the three-momentum in the propagator. But such a cutoff necessarily breaks Galilean invariance of the nonrelativistic system. There is no obvious way to impose a Galilean invariant cutoff at the averaged action level. This is a general feature independent of the choice of the cutoff function. Furthermore, if the cutoff function is not smooth enough, nonanalytic terms in momenta arise.[1] See Ref. [27] for an example in a similar context.

In a system of *two* nonrelativistic particles, there is a simple and physically sensible way out of this problem: it is to impose a cutoff on the *relative* three-momentum of the two particles, which is Galilean invariant. In addition, the results are very insensitive to the choice of the cutoff function. See Appendices of Refs. [29,30]. In particular, the results with a sharp cutoff are the same as those with a smooth one. It is a technical advantage that a sharp cutoff can be used because it simplifies the calculations considerably.

C. Pionless NEFT up to $\mathcal{O}(p^2)$ in the $^1S_0$ and $^3S_1$-$^3D_1$ channels

In Ref. [29], we consider the pionless NEFT without isospin breaking. The relevant degrees of freedom are nonrelativistic nucleons, which interact with themselves only through contact interactions. In the two-nucleon sector, they are four-nucleon operators with an arbitrary number of derivatives. An operator with more derivatives has higher canonical dimensions than the ones with less derivatives.

Since there are infinitely many operators involved in the flow equation, one needs to introduce a truncation of the space of operators in the averaged action in order to solve it. We retain only the operators with derivatives up to a certain order. We simply count the number of spatial derivatives ($\nabla \sim p$) and a time derivative is counted as two spatial derivatives ($\partial_t \sim p^2$). We consider the following ansatz for the averaged action up to $\mathcal{O}(p^2)$,

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[1]If one considers only a few leading order terms in the derivative expansion, typically in the local potential approximation, nonsmooth cutoff does not cause the problem. It is the reason why the problem of nonanalyticity is not revealed in many applications.

$$\Gamma_\Lambda^{(\not{\pi})} = \int d^4x \left[N^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) N \begin{cases} -C_0^{(S)}\mathcal{O}_0^{(S)} + C_2^{(S)}\mathcal{O}_2^{(S)} + 2B^{(S)}\mathcal{O}_2^{(SB)} \right], & (^1S_0 \text{ channel}) \\ -C_0^{(T)}\mathcal{O}_0^{(T)} + C_2^{(T)}\mathcal{O}_2^{(T)} + 2B^{(T)}\mathcal{O}_2^{(TB)} + C_2^{(SD)}\mathcal{O}_2^{(SD)} \right], & (^3S_1\text{-}^3D_1 \text{ channel}) \end{cases}$$

$$(2.2)$$

where the operators in the $^1S_0$ are given by

$$\mathcal{O}_0^{(S)} = \left(N^T P_a^{(S)} N \right)^\dagger \left(N^T P_a^{(S)} N \right), \tag{2.3a}$$

$$\mathcal{O}_2^{(S)} = \left[\left(N^T P_a^{(S)} N \right)^\dagger \left(N^T P_a^{(S)} \overleftrightarrow{\nabla}^2 N \right) + \text{H.c.} \right], \tag{2.3b}$$

$$\mathcal{O}_2^{(SB)} = \left[\left\{ N^T P_a^{(S)} \left(i\partial_t + \frac{\nabla^2}{2M} \right) N \right\}^\dagger \left(N^T P_a^{(S)} N \right) + \text{H.c.} \right], \tag{2.3c}$$

and in the $^3S_1$-$^3D_1$ channel,

$$\mathcal{O}_0^{(T)} = \left(N^T P_i^{(T)} N \right)^\dagger \left(N^T P_i^{(T)} N \right), \tag{2.4a}$$

$$\mathcal{O}_2^{(T)} = \left[\left(N^T P_i^{(T)} N \right)^\dagger \left(N^T P_i^{(T)} \overleftrightarrow{\nabla}^2 N \right) + \text{H.c.} \right], \tag{2.4b}$$

$$\mathcal{O}_2^{(SD)} = \left[\left(N^T P_i^{(T)} N \right)^\dagger \left\{ N^T \left(\overleftrightarrow{\nabla}_i \overleftrightarrow{\nabla}_j - \tfrac{1}{3}\delta_{ij}\overleftrightarrow{\nabla}^2 \right) P_j^{(T)} N \right\} + \text{H.c.} \right], \tag{2.4c}$$

$$\mathcal{O}_2^{(TB)} = \left[\left\{ N^T P_i^{(T)} \left(i\partial_t + \frac{\nabla^2}{2M} \right) N \right\}^\dagger \left(N^T P_i^{(T)} N \right) + \text{H.c.} \right], \tag{2.4d}$$

where we have introduced the notation $\overleftrightarrow{\nabla}^2 \equiv \overleftarrow{\nabla}^2 + \overrightarrow{\nabla}^2 - 2\overleftarrow{\nabla}\cdot\overrightarrow{\nabla}$ and the projection operators [14],

$$P_a^{(S)} \equiv \frac{1}{\sqrt{8}}\sigma^2\tau^2\tau^a, \qquad P_k^{(T)} \equiv \frac{1}{\sqrt{8}}\sigma^2\sigma^k\tau^2, \tag{2.5}$$

for the $^1S_0$ (spin singlet) channel and the $^3S_1$ (spin triplet) channel, respectively. The nucleon field $N(x)$ with mass M is an isospin doublet nonrelativistic two-component spinor. Pauli matrices σ^i and τ^a act on spin indices and isospin indices, respectively. The two channels are completely decoupled, and thus we can consider each channel separately. Note that the possible forms of the operators are restricted by Galilean invariance. Note also that we have included "redundant operators," $\mathcal{O}_2^{(SB)}$ and $\mathcal{O}_2^{(TB)}$, because they are necessary in a consistent Wilsonian RG analysis [28].

We introduce the following dimensionless coupling constants,

$$x \equiv \frac{M\Lambda}{2\pi^2}C_0^{(S)}, \quad y \equiv \frac{M\Lambda^3}{2\pi^2}4C_2^{(S)}, \quad z \equiv \frac{\Lambda^3}{2\pi^2}B^{(S)}, \tag{2.6}$$

for the spin-singlet channel and

$$x' \equiv \frac{M\Lambda}{2\pi^2}C_0^{(T)}, \quad y' \equiv \frac{M\Lambda^3}{2\pi^2}4C_2^{(T)},$$

$$z' \equiv \frac{\Lambda^3}{2\pi^2}B^{(T)}, \quad w' \equiv \frac{M\Lambda^3}{2\pi^2}\frac{4}{3}C_2^{(SD)}, \tag{2.7}$$

for the spin-triplet channel. With the sharp cutoff on the relative momenta, the flow equations that determine the dependence on $t = \ln(\Lambda_0/\Lambda)$ of the coupling constants can be written as [30]

$$\frac{dx_C}{dt} + d_C x_C = \sum_{A,B} x_A x_B \frac{M\Lambda}{2\pi^2} \left. \frac{F_A(p_i,\Lambda)F_B(\Lambda,p_f)}{1-\tilde{A}(p_i)} \right|_C, \tag{2.8}$$

where x_C stands for one of the dimensionless coupling constants and d_C is the power of Λ in the definition of the dimensionless coupling constant. In the following, we call $-d_C$ the canonical dimension[2] of the coupling. $\tilde{A}(P)$ is defined as

$$A(P) = P^0 - \frac{\boldsymbol{P}^2}{4M}, \quad \tilde{A}(P) = MA(P)/\Lambda^2, \tag{2.9}$$

where $P = (P^0, \boldsymbol{P})$ is the total momentum of the system. $F_A(p_i, p_f)$ is the momentum-dependent factor associated with the coupling x_A:

$$F_x = \frac{-2\pi^2}{M\Lambda}, \quad F_y = \frac{-2\pi^2}{4M\Lambda^3}(r_{12}+r_{34}),$$

$$F_z = \frac{2\pi^2}{\Lambda^3}\sum_{i=1}^4 S_i, \tag{2.10}$$

for the spin-singlet channel and

$$F_{x'} = \frac{-2\pi^2}{M\Lambda}, \quad F_{y'} = \frac{-2\pi^2}{4M\Lambda^3}(r_{12}+r_{34}),$$

$$F_{z'} = \frac{2\pi^2}{\Lambda^3}\sum_{i=1}^4 S_i, \tag{2.11}$$

$$F_{w'}^{ij} = \frac{3}{4}\left(\frac{-2\pi^2}{M\Lambda^3}\right)\left[p_{12}^i p_{12}^j + p_{34}^i p_{34}^j - \frac{1}{3}\delta^{ij}(r_{12}+r_{34}) \right],$$

for the spin-triplet channel, with

$$S_i = p_i^0 - \frac{\boldsymbol{p}_i^2}{2M}, \quad \boldsymbol{p}_{ij} = \boldsymbol{p}_i - \boldsymbol{p}_j, \quad r_{ij} = (\boldsymbol{p}_i - \boldsymbol{p}_j)^2, \tag{2.12}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the incoming momenta of the nucleons to the vertex and $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ are the outgoing momenta from the vertex. The notation $|_C$ stands for the operation of taking the coefficient of $F_C(p_i, p_f)$ in the sum.

By using the formula Eq. (2.8) the RGEs are obtained. The explicit expressions are not presented here but can be read easily from the RGEs [Eqs. (4.6) and Eqs. (4.12)] for the theory with pions discussed in Sec. IV.

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[2]This definition of the canonical dimension reflects the nonrelativistic scaling property. See Ref. [42].

There is a nontrivial fixed point in each channel, which is relevant to the physical two-nucleon system:

$$(x^\star, y^\star, z^\star) = \left(-1, -\tfrac{1}{2}, \tfrac{1}{2}\right), \qquad (2.13)$$

in the spin-singlet channel, and

$$(x'^\star, y'^\star, z'^\star, w'^\star) = \left(-1, -\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \qquad (2.14)$$

in the spin-triplet channel. At the nontrivial fixed point, the operators get large anomalous dimensions and there is one operator that becomes *relevant* with the scaling dimension of the coupling constant being one in each channel.

III. NONRELATIVISTIC APPROXIMATION, IR ENHANCEMENT, AND THE LEADING ORDER IN Λ/M

In this section, we consider the inclusion of pions as dynamical degrees of freedom. It extends the range of applicability of NEFT to higher momenta beyond the pion mass scale. The contact interactions in the pionless theory resolve into the effects by pion propagation and the rest. The physical cutoff Λ_0 is now larger than m_π, and we suppose that Λ_0 is of order 400 MeV.

A. Chiral symmetry and nonrelativistic nucleons

The most important feature of the theory is chiral symmetry, $SU(2)_L \times SU(2)_R$ spontaneously broken to $SU(2)_V$. It is convenient to introduce the field Σ, which transforms linearly as

$$\Sigma(x) \to L\Sigma(x)R^\dagger, \qquad (3.1)$$

where L and R are the elements of $SU(2)_L$ and $SU(2)_R$, respectively. The pion field $\pi^a(x)$ may be defined through

$$\Sigma(x) = \exp[i\pi^a(x)\tau^a/f], \qquad (3.2)$$

where f is the pion decay constant in the chiral limit. The nucleon field transforms as

$$N(x) \to U(x)N(x), \qquad (3.3)$$

where $U(x)$ is a function of L, R, and $\Sigma(x)$ and defined through

$$\xi(x) \to L\xi(x)U(x)^\dagger = U(x)\xi(x)R^\dagger, \qquad (3.4)$$

where $\xi^2(x) = \Sigma(x)$, i.e., $\xi(x) = \exp[i\pi^a(x)\tau^a/2f]$.

The chiral invariant Lagrangian for the nonrelativistic nucleon interacting with pions is given as

$$\begin{aligned}
\mathcal{L}_{\rm NR} = {} & N^\dagger \left[iD_0 + \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{D})^2}{2M} \right] N + g_A N^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{A} N \\
& - C_0^{(c)} \mathcal{O}_0^{(c)} + C_2^{(c)} \mathcal{O}_2^{(c)} + 2B^{(c)} \mathcal{O}_2^{(cB)} \\
& - D_2^{(c)} \frac{m_\pi^2}{2} {\rm Tr}(\Sigma + \Sigma^\dagger) \mathcal{O}_0^{(c)} + \cdots,
\end{aligned} \qquad (3.5)$$

where the chiral covariant derivative D_μ is defined as

$$D_\mu N = (\partial_\mu + V_\mu)N, \qquad (3.6)$$

and V_μ and A_μ are defined as

$$V_\mu \equiv \tfrac{1}{2}(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger), \qquad (3.7)$$

$$A_\mu \equiv \tfrac{i}{2}(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger). \qquad (3.8)$$

The superscript (c) denotes the channel to be specified, and the summation is also implied, i.e., the term $C_2^{(c)} \mathcal{O}_2^{(c)}$ for the spin-triplet channel contains $C_2^{(T)} \mathcal{O}_2^{(T)}$ and $C_2^{(SD)} \mathcal{O}_2^{(SD)}$. Note that the derivatives in the four-nucleon operators should be replaced by the covariant derivatives, though we do not explicitly show them here.

The ellipsis in Eq. (3.5) denotes other terms, e.g., the higher-order operators, including six-nucleon operators, four-nucleon operators with more than two derivatives, etc. It also contains the counterterms of the form $N^\dagger N$ and $N^\dagger \nabla^2 N$, which play an important role in the renormalization of the nucleon mass. Chiral symmetry does not prevent the appearance of the operators that contain more than one A, such as $N^\dagger A^2 N$. But it is not generated by nonrelativistic reduction from the simple (chiral invariant) Dirac action. It implies that the operators have smaller coefficients than $1/M$. Furthermore, we will see that it either does not contribute to the RGEs for the four-nucleon operators or gives only suppressed contributions to the order we are working.

The Lagrangian for the pions is given by

$$\mathcal{L}_\pi = \frac{f^2}{4}\left[{\rm Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + m_\pi^2 {\rm Tr}(\Sigma^\dagger + \Sigma) \right] + \cdots. \qquad (3.9)$$

B. Order of magnitude estimation of the contributions to the Wilsonian RGEs

In order to perform the Wilsonian RG analysis, one usually needs to use a cutoff function that preserves all the symmetries of the theory. Unfortunately, there does not seem to exist a manifestly chiral invariant cutoff function which controls all the fluctuations, because of the nonlinearity of the transformation [43]. Furthermore, it is known that perturbation theory generates apparently noninvariant terms (ANTs), even with the lattice and the dimensional regularization, which preserve chiral symmetry [44,45]. ANTs are also expected to appear in the Wilsonian RG analysis, but it is not obvious how to treat them. In addition, the inclusion of pions makes the notion of relative momentum obscure. Furthermore, because pions are relativistic, Galilean invariance does not make good sense as a constraint. Fortunately, however, it turns out that these problems do not interfere with the leading-order calculations in the nonrelativistic approximation. The argument is based on the order of magnitude estimate of the contribution of each diagram to the Wilsonian RGEs.

We are going to obtain the RGEs for the two-nucleon sector in the next-to-leading order in momentum expansion [to $\mathcal{O}(p^2)$] of the averaged action in the nonrelativistic formulation. Let us first note the following things:

(i) The contributions to the Wilsonian RGEs only come from one-loop diagrams.

(ii) Because of the nonrelativistic feature, diagrams with antinucleon lines are absent. Thus, the diagrams are

divided into sectors, each of them is specified by the nucleon number. The n-nucleon sector has n nucleons at each time slice.

(iii) The contributions to the two-nucleon sector do not come from n-nucleon operators with $n \geqslant 6$.

(iv) Our primary concern is the renormalization of the four-nucleon operators with no pion emission and absorption. Chiral symmetry constrains the renormalization of the operators in which the pion fields appear through covariant derivatives.

(v) Supplementarily, one needs to consider the self-energy diagram of the nucleon and the diagrams for the nucleon-pion vertex that contribute to the RGEs for the four-nucleon operators.

Note also that there is no contributions to the pion self-energy to this order.

1. Four-nucleon operators

According to the above-mentioned remarks, we need to consider the diagrams given in Fig. 1 for the four-nucleon operators.

In order to obtain the RGEs, we need to evaluate the contributions from the so-called "shell-mode," the loop contributions with the magnitude of the loop (relative) three-momentum $k = |\mathbf{k}|$ is between $\Lambda - d\Lambda$ and Λ. The argument is similar to the "power counting" with the scale Q found in the literature, but here, (i) we do not consider the amplitude but
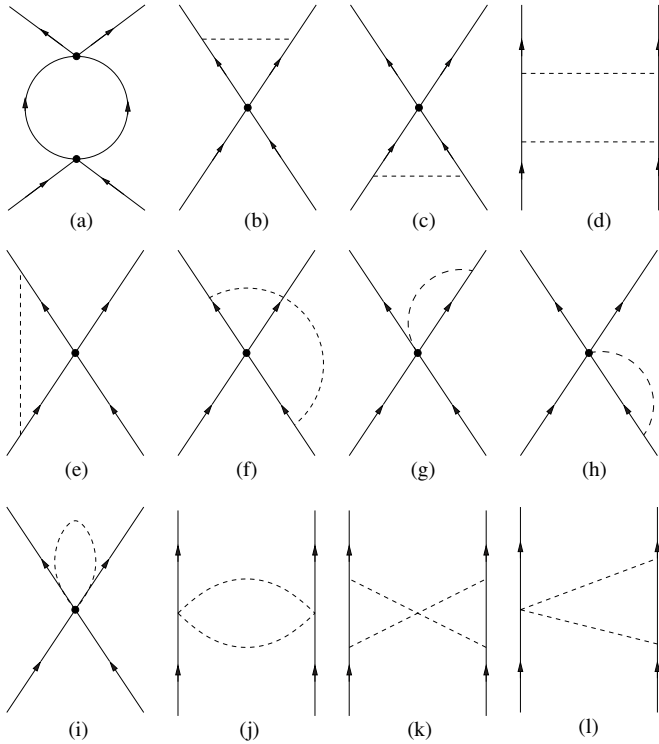

FIG. 1. Contributions to the four-nucleon operators. The four-nucleon vertices represent contact interactions collectively. The dotted lines represent the pion propagators. For the diagrams (e), (f), (g), (h), and (l), there are also mirrored diagrams with the left and the right interchanged.

the contribution to the RGEs and (ii) the magnitude of the momentum in the loop is actually Λ, while in the "Q-counting" it is *assumed* that the dominant contributions come from the loop momentum of order Q.

To see how the dominant contributions to the RGEs emerge in certain diagrams, let us consider Fig. 1(b) as an illustrative example. At the moment, we assume that the floating cutoff Λ is larger than m_π for simplicity. The diagram contains the loop integral

$$\int_{\text{shell}} \frac{d^4k}{(2\pi)^4} \frac{i\mathbf{k}^2}{k^2 - m_\pi^2 + i\epsilon} \frac{i}{(p^0 + k^0) - (\mathbf{p} + \mathbf{k})^2/2M + i\epsilon}$$
$$\times \frac{i}{(p'^0 - k^0) - (\mathbf{p}' - \mathbf{k})^2/2M + i\epsilon}, \quad (3.10)$$

where $\int_{\text{shell}}$ denotes the shell mode integral with the restriction on the magnitude of the relative three-momentum. There are four poles in the complex k^0 plane, one nucleon pole and one pion pole in both the upper and the lower half-planes. One can evaluate the integral by the residues of the poles in either half-plane and find that the nucleon pole gives a dominant contributions,

$$\int_{\text{shell}} \frac{d^3k}{(2\pi)^3} \frac{-\mathbf{k}^2}{[-p^0 + (\mathbf{p} + \mathbf{k})^2/2M]^2 - \omega_k^2}$$
$$\times \frac{1}{(p^0 + p'^0) - (\mathbf{p} + \mathbf{k})^2/2M - (\mathbf{p}' - \mathbf{k})^2/2M}$$
$$\sim \int_{\text{shell}} \frac{d^3k}{(2\pi)^3} \frac{-(\mathbf{k} - (\mathbf{p} - \mathbf{p}')/2)^2}{[-p^0 + (\mathbf{k} + \mathbf{P}/2)^2/2M]^2 - \omega_{k-(p-p')/2}^2}$$
$$\times \frac{1}{E - \mathbf{P}^2/4M - \mathbf{k}^2/M}, \quad (3.11)$$

where $\omega_k \equiv \sqrt{\mathbf{k} + m_\pi^2}$, and $E \equiv p^0 + p'^0$ and $\mathbf{P} \equiv \mathbf{p} + \mathbf{p}'$ are the total energy and the total three-momentum of the two nucleons, respectively. In going to the second line, we have made a shift of the integration momentum so $\mathbf{k}$ is now the relative momentum. Since $M \gg |\mathbf{k}| = \Lambda \gg |\mathbf{P}|$, p^0, and $E - \mathbf{P}^2/4M \ll \Lambda^2/M$, one may estimate the loop integral as

$$\sim \frac{-1}{2\pi^2} M d\Lambda. \quad (3.12)$$

The pion pole gives $\sim \Lambda d\Lambda/4\pi^2$, which is smaller than the dominant contribution by a factor of Λ/M.

On the other hand, the diagram in Fig. 1(e) contains

$$\int_{\text{shell}} \frac{d^4k}{(2\pi)^4} \frac{i\mathbf{k}^2}{k^2 - m_\pi^2 + i\epsilon} \frac{i}{(p^0 - k^0) - (\mathbf{p} - \mathbf{k})^2/2M + i\epsilon}$$
$$\times \frac{i}{(p'^0 - k^0) - (\mathbf{p}' - \mathbf{k})^2/2M + i\epsilon}, \quad (3.13)$$

which is similar to Eq. (3.10), but there is a crucial difference. No nucleon pole appears in the lower half-plane. Thus the contribution can be evaluated by the residue of the pion pole in the lower half-plane and gives $\sim -\Lambda d\Lambda/4\pi^2$. This is suppressed by a factor of Λ/M compared with the diagram in Fig. 1(b).

It is a general feature that the two-nucleon reducible (2NR) diagram acquires the "IR enhancement," first noted by Weinberg [3], where the residue of the nucleon pole gets
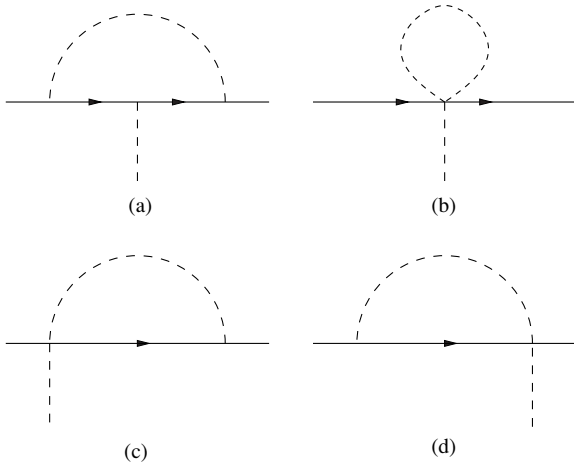
FIG. 2. Contributions to the g_A vertex.

an enhancement factor M/Λ. Note that the IR enhancement is a consequence of the nonrelativistic kinematics. In addition, with the residue of the nucleon pole, the pion propagator

$$\frac{i}{[-p^0 + (\boldsymbol{p} + \boldsymbol{k})^2/2M]^2 - \omega_k^2} \ . \tag{3.14}$$

Noting $p^0 - \boldsymbol{p}^2/2M \ll \Lambda^2/M \ll \Lambda$, it is approximated by

$$\sim \frac{-i}{\boldsymbol{k}^2 + m_\pi^2} \tag{3.15}$$

for $k \sim \Lambda$. That is, the effects of the pion can be represented as the instantaneous potential.

Although we have shown that the mechanism of "IR enhancement" works in the case of $m_\pi < \Lambda$, it is actually independent of this assumption, and it also works in the case of $\Lambda < m_\pi$.

2. Vertex corrections and the nucleon self-energy

We also need to consider the renormalization of the g_A term. The relevant diagrams are depicted in Fig. 2. There are contributions to the self-energy of the nucleon, shown in Fig. 3. These are potentially important to the renormalization of the four-nucleon operators.

The examples in the previous subsection imply that the nucleon self-energy diagram and the contributions to the nucleon-pion vertex do not have IR enhancement. We will see shortly that these contributions can be neglected in our leading-order calculations. The key point is that the appropriate dimensionless coupling constant for the nucleon-pion coupling is, as
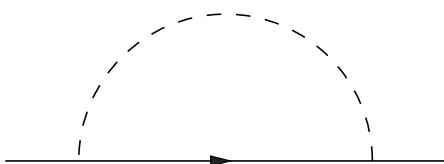

FIG. 3. Contributions to the self-energy of nucleon.

we will explain in the next section (Sec. IV A), given by γ,

$$\gamma \equiv \frac{M\Lambda}{2\pi^2}\left(\frac{g_A}{2f}\right)^2. \tag{3.16}$$

The contributions from the nucleon-pion vertex to γ should be written in terms of γ.

There are four diagrams contributing to the nucleon-pion vertex shown in Sec. III B. The tadpole diagram [Fig. 2(b)] can be absorbed in the redefinition of the coupling constant g_A. Each of the diagrams in Figs. 2(c) and 2(d) has an extra $1/M$ factor because the two-pion vertex comes from the kinetic term $\boldsymbol{D}^2/2M$ so they are suppressed compared to the diagram in Fig. 2(a). Figure 2(a) gives rise to the second term of the right-hand side of the following RGE expressed in terms of γ,

$$\frac{d\gamma}{dt} = -\gamma - 3\left(\frac{\Lambda}{M}\right)\gamma^2. \tag{3.17}$$

(The contributions from the nucleon wave-function renormalization is not considered here.) Note that the last term has an explicit Λ dependence. In Eq. (4.7) the last term has been neglected because Λ/M is much smaller than 1.

The self-energy diagram $\Sigma(p)$ shown in Fig. 3 gives contributions

$$\Lambda \frac{d\Sigma(p)}{d\Lambda} = A + Bp^0 + C\boldsymbol{p}^2 + \mathcal{O}(\Lambda^2/M^2), \tag{3.18}$$

where A, B, and C are the constants that depend on the couplings and the cutoff Λ. The constants A and C are canceled by the counterterms, leaving the pole of the nucleon propagator intact. After doing so, the propagator with this shell-mode contribution becomes

$$\left(1 - B\frac{d\Lambda}{\Lambda}\right)^{-1}\frac{i}{p^0 - \frac{\boldsymbol{p}^2}{2M} + i\epsilon}, \tag{3.19}$$

so the contributions to the wave-function renormalization constant for the nucleon field, Z_N, can be written as

$$\frac{dZ_N}{dt} = B. \tag{3.20}$$

The order of magnitude estimate of B gives

$$B \sim \left(\frac{\Lambda}{M}\right)\gamma, \tag{3.21}$$

so the inclusion of the effect of wave-function renormalization does not alter the results of the leading-order calculations.

C. Averaged action with L-OPE

We have seen that the leading contributions to the RGEs consist of the one-loop 2NR diagrams with contact interactions and/or the instantaneous pion exchanges. Actually these contributions can be generated by a much simpler action than that in Eq. (3.5). Thus we start with the following ansatz for the averaged action Γ_Λ,

$$\Gamma_\Lambda = \Gamma_\Lambda^{(\not\pi)} + \int d^4x \left\{ - D_2^{(c)}m_\pi^2 \mathcal{O}_0^{(c)} \right\} + \frac{g_A^2}{4f^2} \int dt \int d^3x \, d^3y$$

$$\times \left[\mathcal{O}^{(S)}(x, y)\nabla_x^2 + \mathcal{O}_{ii}^{(T)}(x, y)\nabla_x^2 \right.$$

$$\left. - 6\mathcal{O}_{ij}^{(T)}(x, y)\left(\partial_i^x \partial_j^x - \frac{1}{3}\delta_{ij}\nabla_x^2\right)\right] Y(|\boldsymbol{x} - \boldsymbol{y}|), \tag{3.22}$$

where we have introduced

$$\mathcal{O}^{(S)}(x, y) = \left[N^T(x)P_a^{(S)}N(y)\right]^\dagger \left[N^T(y)P_a^{(S)}N(x)\right], \quad (3.23)$$

$$\mathcal{O}_{ij}^{(T)}(x, y) = \left[N^T(x)P_i^{(T)}N(y)\right]^\dagger \left[N^T(y)P_j^{(T)}N(x)\right], \quad (3.24)$$

and

$$Y(r) = \frac{1}{4\pi}\frac{e^{-m_\pi r}}{r}. \quad (3.25)$$

Note that all the derivatives are now the usual ones, not the covariant derivatives. It means that chiral symmetry is broken explicitly, but the breaking is of higher order in Λ/M, as is seen from the derivation. Note also that the operator corresponds to $D_2^{(c)}$ is the same as that to $C_0^{(c)}$, but the former is a part of the operator that emits/absorbs pions and is of higher order in the p expansion than the latter.

The effects of pions are represented as one-pion-exchange interactions. It is L-OPE because the averaged action is defined with the cutoff Λ, so the S-OPE (with the momenta larger than Λ) is included in the contact interactions. The last term in Eq. (3.22) represents the tensor force of L-OPE in the spin-triplet channel.

IV. WILSONIAN RGEs FOR THE NEFT WITH PIONS TO $\mathcal{O}(p^2)$ IN THE $^1S_0$ AND $^3S_1$-$^3D_1$ CHANNELS

We are now ready to calculate the RGEs for the two-nucleon system in the S waves. The formula Eq. (2.8) can be used with slight modification.

(i) We introduce the dimensionless coupling constants γ defined in Eq. (3.16) for L-OPE, and u and u' for $D_2^{(c)}$, defined by

$$u \equiv \frac{M\Lambda^3}{2\pi^2}D_2^{(S)}, \quad u' \equiv \frac{M\Lambda^3}{2\pi^2}D_2^{(T)}. \quad (4.1)$$

Note that g_A and f appear in Eq. (3.22) only through the combination of $(g_A/f)^2$.

(ii) The momentum-dependent factors associated with γ, u, and u' are given by

$$F_{\gamma S}(p_f, p_i) = \left(\frac{-2\pi^2}{M\Lambda}\right)\frac{1}{2}\left[\frac{r_{13}}{r_{13}+m_\pi^2}+\frac{r_{14}}{r_{14}+m_\pi^2}\right], \quad (4.2)$$

$$F_{\gamma T}^{ij}(p_f, p_i) = \left(\frac{-6\pi^2}{M\Lambda}\right)\frac{1}{2}\left[\frac{\delta^{ij}r_{13} - 2p_{13}^i p_{13}^j}{r_{13} + m_\pi^2}\right.$$
$$\left. + \frac{\delta^{ij}r_{14} - 2p_{14}^i p_{14}^j}{r_{14} + m_\pi^2}\right], \quad (4.3)$$

$$F_u = F_{u'} = \frac{-2\pi^2 m_\pi^2}{M\Lambda^3}. \quad (4.4)$$

(iii) Since the $F_{\gamma S}$ and $F_{\gamma T}^{ij}$ have a bit more complicated momentum dependence than those in the pionless theory, the formula contains nontrivial integrations over angular variables. The part $F_A(p_i, \Lambda)F_B(\Lambda, p_f)$ in Eq. (2.8) is replaced by $\langle F_A(p_i, \Lambda)F_B(\Lambda, p_f)\rangle$, where

$\langle\cdots\rangle$ is defined as

$$\langle\cdots\rangle = \frac{1}{4\pi}\int d\Omega_{\hat{k}}(\cdots), \quad (4.5)$$

where $\Omega_{\hat{k}}$ stands for the angular variables of $\boldsymbol{k}$. See the original derivation of the formula for the details [30].

In the following, we assume that $m_\pi < \Lambda$ and expand in powers of m_π/Λ. The case of $\Lambda < m_\pi$ is discussed in Appendix A.

A. Spin-singlet channel

In the spin-singlet channel, we have the following RGEs:

$$\frac{dx}{dt} = -x - [x^2 + 2xy + y^2 + 2xz + 2yz + z^2]$$
$$- 2(x + y + z)\gamma - \gamma^2, \quad (4.6a)$$

$$\frac{dy}{dt} = -3y - \left[\frac{1}{2}x^2 + 2xy + \frac{3}{2}y^2 + yz - \frac{1}{2}z^2\right]$$
$$- (x + 2y)\gamma - \frac{1}{2}\gamma^2, \quad (4.6b)$$

$$\frac{dz}{dt} = -3z + \left[\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 - xz - yz - \frac{3}{2}z^2\right]$$
$$+ (x + y - z)\gamma + \frac{1}{2}\gamma^2, \quad (4.6c)$$

$$\frac{du}{dt} = -3u - 2(x + y + z)(u - \gamma) - 2u\gamma + 2\gamma^2, \quad (4.6d)$$

and for γ,

$$\frac{d\gamma}{dt} = -\gamma. \quad (4.7)$$

The first lines of Eqs. (4.6)–(4.6c) are the same as those in the pionless calculations obtained in Ref. [29]. The terms in the second lines express how the L-OPE is rearranged into the S-OPE when the floating cutoff is lowered.

We emphasize the choice made here of the dimensionless coupling constant γ. There are several ways to make a dimensionless quantity from the combination $(g_A/f)^2$, M, and Λ. Our choice is the one for which the RGEs for the coupling constants of the contact interactions do not have explicit Λ dependence. If the explicit Λ dependence were present in the RGEs, the iterative property (self-similarity) would be lost and the concept of fixed points would become obscure. Since the unnaturally large scattering lengths in the S-wave scattering are believed to be related to the nontrivial fixed points, one needs to use such dimensionless variables that allow fixed points.

The nontrivial fixed point of the RGEs (4.6) and Eq. (4.7) relevant to the real two-nucleon system is found to be

$$(x^\star, y^\star, z^\star, u^\star, \gamma^\star) = \left(-1, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right), \quad (4.8)$$

which is identified with that found in the pionless NEFT, given in Eq. (2.13).

We linearize the RGEs around the fixed point and find the following eigenvalues (scaling dimensions) and the

corresponding eigenvectors:

$$
\nu_1 = +1: u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \nu_2 = -1: u_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\nu_3 = -2: u_3 = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad \nu_4 = -1: u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.9}
$$

The eigenvalues ν_1, ν_2, and ν_3 and corresponding eigenvectors u_1, u_2, and u_3 can be identified with those found in the pionless theory. Therefore, the power counting is not modified by the inclusion of the pions; only the one relevant operator (u_1) should be resummed to all orders.

Note that the eigenvalue problem is five dimensional and one thus expects five pairs of eigenvalues and eigenvectors. It is easily seen that the eigenvalue -1 is triply degenerate. Two of the (nonzero) eigenvectors are u_2 and u_4, but the third one does not exist. This is not a mathematical inconsistency, however. Although we do not understand very well the reason why the third eigenvector is missing, it clearly has something to do with the γ direction, because the vector in the γ direction cannot be expressed as a linear combination of u_i's ($i = 1, \ldots, 4$).

Equation (4.7) shows that the L-OPE is irrelevant. It implies that the L-OPE should be treated as a perturbation. Note that there is a typical scale in the NEFT with pions, Λ_{NN},

$$
\Lambda_{NN} = \frac{4\pi}{M} \left(\frac{2f}{g_A} \right)^2, \tag{4.10}
$$

and our γ is related to it as

$$
\gamma(\Lambda) = \frac{2}{\pi} \left(\frac{\Lambda}{\Lambda_{NN}} \right). \tag{4.11}
$$

Kaplan, Savage, and Wise [11,12] regard p/Λ_{NN} as an expansion parameter. Our finding is consistent with their approach.

B. Spin-triplet channel

In the spin-triplet channel, we have the following RGEs:

$$
\frac{dx'}{dt} = -x' - [x'^2 + 2x'y' + y'^2 + 2x'z' + 2y'z' + z'^2 + 2w'^2]
$$
$$
- 2(x' + y' + z' - 4w')\gamma - 9\gamma^2, \tag{4.12a}
$$

$$
\frac{dy'}{dt} = -3y' - \left[\frac{1}{2}x'^2 + 2x'y' + \frac{3}{2}y'^2 + y'z' - \frac{1}{2}z'^2 + w'^2 \right]
$$
$$
- (x' + 2y')\gamma + \frac{7}{2}\gamma^2, \tag{4.12b}
$$

$$
\frac{dz'}{dt} = -3z' + \left[\frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 - x'z' - y'z' - \frac{3}{2}z'^2 + w'^2 \right]
$$
$$
+ (x' + y' - z' - 4w')\gamma + \frac{9}{2}\gamma^2, \tag{4.12c}
$$

$$
\frac{dw'}{dt} = -3w' - [x'w' + y'w' + z'w']
$$
$$
+ \frac{1}{5}(2x' + 2y' + 2z' - 9w')\gamma + 2\gamma^2, \tag{4.12d}
$$

$$
\frac{du'}{dt} = -3u' - 2(x' + y' + z')u' + 2(x' + y' + z' - 4w')\gamma
$$
$$
- 2u'\gamma + 18\gamma^2. \tag{4.12e}
$$

Here, again, the first lines of Eqs (4.12a)–(4.12d) are the same as those in the pionless calculations, obtained in Ref. [29], while the second lines are the contributions from L-OPE.

Note that the magnitude of the coefficients of γ^2 is large compared to those for the spin-singlet channel. This is the effect of the tensor part of L-OPE.

The nontrivial fixed point of the RGEs (4.12) and Eq. (4.7) relevant to the real two-nucleon system is found to be

$$
(x^\star, y^\star, z^\star, w^\star, u^\star, \gamma^\star) = \left(-1, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0 \right), \tag{4.13}
$$

which is the same as that found in the pionless NEFT, given in Eq. (2.14).

The RGEs linearized around the fixed point lead to the following set of eigenvalues and the eigenvectors:

$$
\nu_1 = +1: u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nu_2 = -1: u_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\nu_3 = -2: u_3 = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nu_4 = -2: u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.14}
$$

$$
\nu_5 = -1: u_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
$$

Here, again, the eigenvalues ν_1, ν_2, ν_3, and ν_4 and corresponding eigenvectors u_1, u_2, u_3, and u_4 can be identified with those found in the pionless theory. Therefore, in spite of the presence of the tensor force in this channel, the power counting is not modified. As in the spin-singlet channel, only the one relevant (u_1) should be resummed to all orders. As in the spin-singlet channel, the eigenvalue -1 is triply degenerate, but one eigenvector is missing.

C. The effects of pions

In this section, we discuss the effects of pions to the RGEs and thus to the power counting.

As we discussed in Sec. II C, the scaling dimensions at the nontrivial fixed point determine the power counting of NEFT in the S waves in the two-nucleon sector. In the previous sections, we have seen that the nontrivial fixed points as well as the scaling dimensions at them in both $^1S_0$ and $^3S_1$-$^3D_1$ channels remain the same even when pions are included as explicit degrees of freedom. Therefore the power counting for the contact operators in the theory including pions is the same as that of the pionless theory.

The L-OPE does not affect the location of the fixed points and the scaling dimensions there, but the details of the flows. The "strong" tensor force in the spin-triplet channel does not change the properties of the nontrivial fixed point but gives rise to the strong γ dependence [because of the large factor "6" in Eq. (3.22)] in the channel and modifies the flow considerably compared to the spin-singlet channel.

The effects can be read off from the γ dependence of the RGEs. To see the effects in more detail, let us introduce new variables,

$$X = x + y + z, \quad Y = y + z, \quad Z = y - z, \quad (4.15)$$

and rewrite the RGEs in Eq. (4.6) for the spin-singlet channel as follows:

$$\frac{dX}{dt} = -X - 2Y - XY - X^2 - (2X + Y)\gamma - \gamma^2, \quad (4.16a)$$

$$\frac{dY}{dt} = -(3 + X + \gamma)Y, \quad (4.16b)$$

$$\frac{dZ}{dt} = -3Z - X(X - Y + 2Z) - (2X - Y + 2Z)\gamma - \gamma^2, \quad (4.16c)$$

$$\frac{du}{dt} = -3u - 2Xu - 2(X + u)\gamma + 2\gamma^2. \quad (4.16d)$$

FIG. 5. (Color online) Typical RG flow lines in the spin-singlet channel. The intervals between points indicate how fast they flow.

We see that the RGEs for (X, Y, γ) form a closed subset and can be solved without knowing the flow for Z and u. In these variables, the nontrivial fixed point is given by $(X^\star, Y^\star, \gamma^\star) = (-1, 0, 0)$.

Similarly, we introduce

$$X' = x' + y' + z', \quad Y' = y' + z', \quad Z' = y' - z', \quad (4.17)$$

in the spin-triplet channel, and the RGEs can be rewritten as

$$\frac{dX'}{dt} = -X' - 2Y' - X'Y' - X'^2 - 2w'^2 \\ - (2X' + Y' - 4w')\gamma - \gamma^2, \quad (4.18a)$$

$$\frac{dY'}{dt} = -(3 + X')Y' - (Y' + 4w')\gamma + 8\gamma^2, \quad (4.18b)$$

$$\frac{dZ'}{dt} = -3Z' - X'(X' - Y' + 2Z') - 2w'^2 \\ - (2X' - Y' + 2Z' - 4w')\gamma - \gamma^2, \quad (4.18c)$$

$$\frac{dw'}{dt} = -3w' - X'w' + \frac{1}{5}(2X' - 9w')\gamma + 2\gamma^2, \quad (4.18d)$$

$$\frac{du'}{dt} = -3u' - 2Xu' + 2(X' - u' - 4w')\gamma + 18\gamma^2. \quad (4.18e)$$
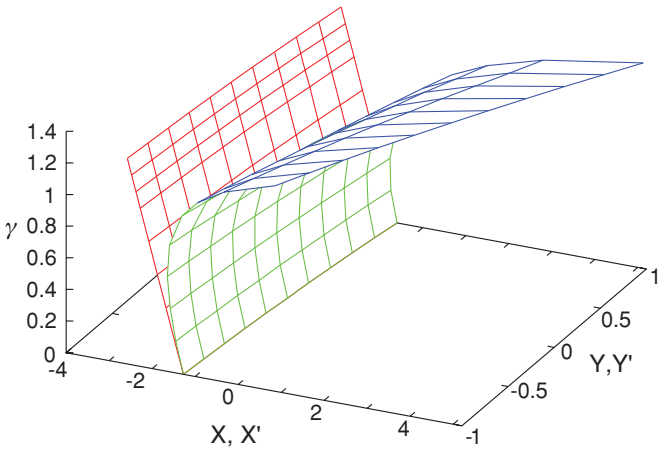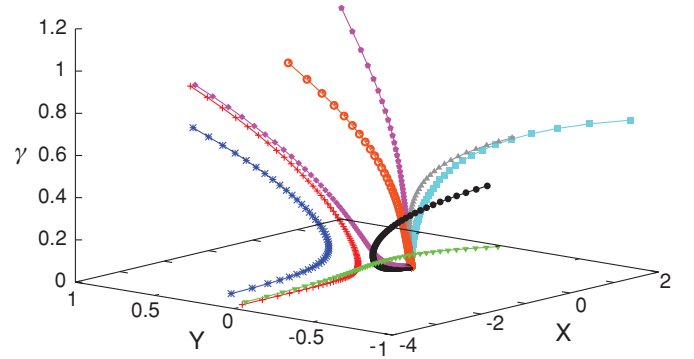

FIG. 4. (Color online) The surfaces which separate the initial values into those from which the flows go to the strong-coupling phase (the left side) and those to the weak-coupling phase (the right side). The surface on the right is for the spin-triplet channel and the other on the left for the spin-singlet channel. All the points have the initial values with $w' = 0$ for the spin-triplet channel.
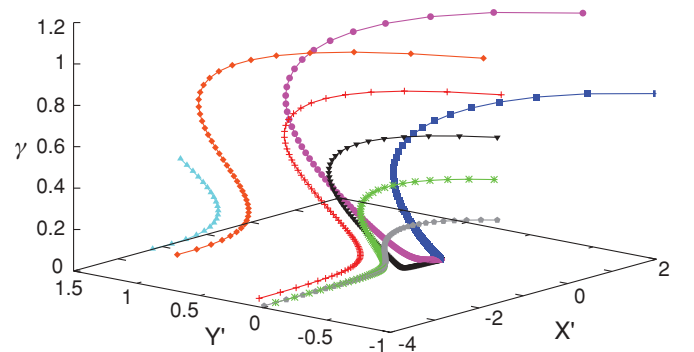

FIG. 6. (Color online) Typical RG flow lines in the spin-triplet channel with the initial value of w' set equal to zero. The intervals between points indicate how fast they flow. Note the strong "dragging force" in the Y' direction.

The RGEs for (X', Y', w', γ) form a closed subset, and the nontrivial fixed point is given by $(X'^\star, Y'^\star, w'^\star, \gamma^\star) = (-1, 0, 0, 0)$.

Thanks to the similarity of the RGEs in both channels, we can project the flows onto the (X, Y, γ) and (X', Y', γ) space and compare them in the single plot. In Fig. 4, we have drawn the surfaces which separate the initial values into two regions: the region from which the flows go to the strong-coupling phase and the other to the weak-coupling phase. It is evident that the region of initial values to the strong-coupling phase is larger in the spin-triplet channel than in the spin-singlet channel.

In Figs. 5 and 6, we show some typical flows. Note that the large coefficient of the γ^2 term of Eq. (4.18b), in comparison with Eq. (4.17), results in the large bending of the flow lines in the Y' direction.

V. SUMMARY AND DISCUSSIONS

A. Summary

In this article, we consider the NEFT including pions in the two-nucleon sector in the S waves in the leading order of the nonrelativistic approximation in order to study the power counting issue from the Wilsonian RG point of view. We show that the leading-order contributions to the RGEs come from the two-nucleon-reducible diagrams, and the pion propagators are dominated by the instantaneous Yukawa potential.

The separation of the pion-exchange contributions into the L-OPE and the S-OPE is emphasized on the basis of the general effective field theory philosophy. The L-OPE is expressed as the Yukawa potential in the averaged action, while the S-OPE is included in the contact interactions along with the other short-distance effects.

We derive the RGEs for the spin-singlet and spin-triplet channels from the effective averaged action ansatz up to including the $\mathcal{O}(p^2)$ in the expansion of momenta and the pion mass. The nontrivial fixed point of physical importance is found to be the same as that in the pionless NEFT in each channel. The eigenvalues (scaling dimensions) and the corresponding eigenoperator of the linearized RGEs around the fixed point are also shown to be the same. That is, there is one relevant contact operator to be resummed. The other operators should be treated as perturbations. The L-OPE is also treated as a perturbation. A part of the S-OPE contained in the relevant operator is resummed to all orders.

We emphasize that the effects of pions do not alter the scaling dimensions and hence the power counting. The pions affect the details of the RG flows. We show that in the spin-triplet channel the flow is more affected by the pions and the region of the initial values that flow to the strong-coupling phase is larger than in the spin-singlet channel.

We believe that the difference between these channels eventually leads to the existence of the bound state (the deuteron) in the spin-triplet channel, and the nonexistence in the spin-singlet channel.

B. Comments on the related works

In the following, we discuss the relation of the present article to the relevant works in the literature.

As stated in the Introduction, our work is very closely related to the work by Beane, Kaplan, and Vuorinen [22], who, working with the PDS renormalization scheme, introduce a regularization mass scale λ to separate the pion exchange into its long-distance and short-distance parts, and the short-distance part is represented as contact interactions. The separation of the pion exchange into two parts is essential to improve the convergence of the EFT expansion in the spin-triplet channel and is similar to our Wilsonian RG analysis, with the regularization scale λ corresponding to our floating cutoff Λ.

There are, however, several points to be addressed: (i) Even though they regard Λ as a low-energy scale, they actually consider high-momentum values in their numerical calculation, ranging from 600 to 1000 MeV. (ii) They employ the PDS renormalization scheme simply assuming the modification does not affect the power counting. (iii) The renormalization scale μ and the regularization scale Λ seem to play a similar role in reordering the EFT expansion but they are treated independently. As a result, Λ becomes just a new parameter. (iv) They consider the regularization only for the spin-triplet channel because of the singular $1/r^3$ potential but not for the spin-singlet channel.

From our point of view, these may be seen as follows: (i) Our separation scale Λ is smaller than the physical cutoff $\Lambda_0 \approx 400$ MeV, so it can be consistently regarded as a low-energy scale. (ii) We show in this article that the nontrivial fixed points as well as the scaling dimensions are the same as those in the pionless theory. We have shown that the scaling dimensions near the nontrivial fixed points in the pionless theory correspond to those of the PDS ones [30]. (iii) In our Wilsonian RG analysis, there is only one scale Λ, which plays the role of μ, renormalization scale, in the RGEs, and simultaneously the role of λ, the separation scale. (iv) We think that one should consider such a regularization for both channels from a general EFT point of view.

The effects of pions in the Wilsonian RG context have been studied by Birse [19,46]. He shows that the range of momenta in which the expansion converges is very small in the spin-triplet channel and claims that the pion exchanges should be resummed. He then employs the distorted-wave RG [26] and finds that the scaling dimensions shift by $-1/2$ (in our convention) in the spin-triplet channel at the nontrivial fixed point. Even at the trivial fixed point, the scaling dimensions are found to shift from the canonical values. He also notes that the scale λ_π [which is the same as Λ_{NN} in Eq. (4.10)] is numerically small ($\sim 2m_\pi$) so the resummation is justified.

We note that he does not introduce the separation scale Λ for the pion exchange consistently with the contact interactions but introduces an additional regularization scale (R) and keeps it fixed when studying the RG flows. Thus his distorted-wave RGE does not take into account the contributions from the OPE to the contact interactions [Figs. 1(b) to 1(d)] at all. As a result, his RGE does not have a smooth transition to that of the pionless theory, where all the pion-exchange effects are represented as contact interactions. That is, the pion exchanges never decouple. In contrast, our RGEs have a smooth transition to those of the pionless theory, as shown in Appendix A, and the pions decouple as they should. In addition, because

of the iterative property of the RGEs, the definition of the dimensionless coupling, γ, is uniquely determined, and it leads to the perturbative treatment of L-OPE, as explained at the end of Sec. IV A.

Fleming *et al.* [14] also report the nonconvergence of the EFT expansion in the P waves (the $^3P_0$ channel). Although we have not explicitly calculated the RGEs in this channel, we think that a similar machinery also works there. The difference between the S waves and the P waves is that the physical NN system is near the nontrivial fixed point in the S waves, while in the P waves it seems to be near the trivial fixed point. Although the scaling dimensions obtained in the PDS scheme with the pole at $D = 3$ subtracted are shown to be the same as those at the nontrivial fixed point in the S waves, they correspond neither to those at the nontrivial fixed point nor to those at the trivial fixed point in the P waves [30]. Thus the simple PDS with the pole at $D = 3$ subtracted should not be used.

C. Prospects of future research

Finally we would like to make a comment on a possible implementation of the findings of the present article into a more tractable way of calculating the physical amplitudes to higher orders. The Wilsonian RG method with the momentum cutoff is theoretically transparent but practically too complicated to do higher-order calculations. A simple but powerful scheme that is also consistent with our results is desired. Such a scheme would employ the dimensional regularization. Since dimensional regularization does not have a natural separation scale in itself, one should introduce it by hand. Thus it would be very similar to the BKV prescription. It seems necessary, however, to make a connection between the renormalization scale μ and the separation scale λ so we have a consistent RGEs with those obtained in the present article. Work in this direction is now in progress.

ACKNOWLEDGMENTS

APPENDIX A: RGEs FOR THE CASE OF $\Lambda < m_\pi$

In Sec. IV, we have derived the RGEs for the case $m_\pi < \Lambda$ by expanding the contributions in powers of m_π/Λ. But the NEFT with pions is valid also for the case $\Lambda < m_\pi$. In this Appendix we present the RGEs for the case $\Lambda < m_\pi$. The diagrams which contribute to the RGEs are the same. The difference is that the contributions are now expanded in powers of Λ/m_π.

In the spin-singlet channel, we have

$$\frac{dx}{dt} = [\text{first line of Eq. (4.6a)}]$$
$$- 2(x + y + z + \tilde{u})\tilde{\gamma} - \tilde{\gamma}^2, \qquad (A1a)$$

$$\frac{dy}{dt} = [\text{first line of Eq. (4.6b)}]$$
$$- (2x + 3y + z + 2\tilde{u})\tilde{\gamma} - \frac{3}{2}\tilde{\gamma}^2, \qquad (A1b)$$

$$\frac{dz}{dt} = [\text{first line of Eq (4.6c)}]$$
$$+ (x + y - z + \tilde{u})\tilde{\gamma} + \frac{1}{2}\tilde{\gamma}^2, \qquad (A1c)$$

$$\frac{d\tilde{u}}{dt} = -\tilde{u} - 2(x + y + z)\tilde{u} - 2\tilde{u}\tilde{\gamma}, \qquad (A1d)$$

where we have introduced new notations,

$$\tilde{\gamma} = \frac{\Lambda^2}{m_\pi^2}\gamma, \quad \tilde{u} = \frac{m_\pi^2}{\Lambda^2}u. \qquad (A2)$$

Now the RGE for $\tilde{\gamma}$ is given by

$$\frac{d\tilde{\gamma}}{dt} = -3\tilde{\gamma}. \qquad (A3)$$

Similarly, in the spin-triplet channel, we have

$$\frac{dx'}{dt} = [\text{first line of Eq. (4.12a)}]$$
$$- 2(x' + \tilde{u}' + y' + z' - 4w')\tilde{\gamma} - 9\tilde{\gamma}^2, \qquad (A4a)$$

$$\frac{dy'}{dt} = [\text{first line of Eq. (4.12b)}]$$
$$- (2x' + 2\tilde{u}' + 3y' + z' - 4w')\tilde{\gamma} - \frac{11}{2}\tilde{\gamma}^2, \qquad (A4b)$$

$$\frac{dz'}{dt} = [\text{first line of Eq. (4.12c)}]$$
$$+ (x' + \tilde{u}' + y' - z' - 4w')\tilde{\gamma} + \frac{9}{2}\tilde{\gamma}^2, \qquad (A4c)$$

$$\frac{dw'}{dt} = [\text{first line of Eq. (4.12d)}]$$
$$+ (2x' + 2\tilde{u}' + 2y' + 2z' - w')\tilde{\gamma} + 2\tilde{\gamma}^2, \qquad (A4d)$$

$$\frac{d\tilde{u}'}{dt} = -\tilde{u}' - 2(x' + y' + z')\tilde{u}' - 2\tilde{u}'\tilde{\gamma}. \qquad (A4e)$$

In the case of $\Lambda < m_\pi$, the operator corresponding $D_2^{(c)}$ cannot be distinguished with the operator corresponding to $C_0^{(c)}$, because these operators are of the same form to this order and the pion mass is not a small parameter to be expanded. Thus they appear only though the combinations $C_0^{(c)} + m_\pi^2 D_2^{(c)}$, or equivalently, $\chi \equiv x + \tilde{u}$ and $\chi' \equiv x' + \tilde{u}'$. In terms of these variables, the RGEs can be rewritten as

$$\frac{d\chi}{dt} = [\text{first line of Eq. (4.6a) with } x \to \chi]$$
$$- 2(\chi + y + z)\tilde{\gamma} - \tilde{\gamma}^2, \qquad (A5a)$$

$$\frac{dy}{dt} = [\text{first line of Eq. (4.6b) with } x \to \chi]$$
$$- (2\chi + 3y + z)\tilde{\gamma} - \frac{3}{2}\tilde{\gamma}^2, \qquad (A5b)$$

$$\frac{dz}{dt} = [\text{first line of Eq. (4.6c) with } x \to \chi]$$
$$+ (\chi + y - z)\tilde{\gamma} + \frac{1}{2}\tilde{\gamma}^2, \qquad (A5c)$$

for the spin-singlet channel, and

$$\frac{d\chi'}{dt} = [\text{first line of Eq. (4.12a) with } x' \to \chi]$$
$$- 2(\chi' + y' + z' - 4w')\tilde{\gamma} - 9\tilde{\gamma}^2, \quad \text{(A6a)}$$

$$\frac{dy'}{dt} = [\text{first line of Eq. (4.12b) with } x' \to \chi]$$
$$- (2\chi' + 3y' + z' - 4w')\tilde{\gamma} - \frac{11}{2}\tilde{\gamma}^2, \quad \text{(A6b)}$$

$$\frac{dz'}{dt} = [\text{first line of Eq. (4.12c) with } x' \to \chi]$$
$$+ (\chi' + y' - z' - 4w')\tilde{\gamma} + \frac{9}{2}\tilde{\gamma}^2, \quad \text{(A6c)}$$

$$\frac{dw'}{dt} = [\text{first line of Eq. (4.12d) with } x' \to \chi]$$
$$+ (2\chi' + 2y' + 2z' - w')\tilde{\gamma} + 2\tilde{\gamma}^2, \quad \text{(A6d)}$$

for the spin-triplet channel. Here we have now included the other terms that we neglected in the ansatz to the order $\mathcal{O}(p^2)$, such as terms proportional to m_π^4. In a similar way, we may consider that χ and χ' contain the terms proportional to m_π^2 but also terms of all order in the expansion in m_π^2. These χ and χ' should be compared to the couplings x and x' in the pionless NEFT.

The new coupling $\tilde{\gamma}$ is a natural measure of the strength of the pion exchange for $\Lambda < m_\pi$, as γ is for $m_\pi < \Lambda$. Note that the RGE for the $\tilde{\gamma}$, Eq. (A3), shows that the L-OPE is more irrelevant and the effects of pions on the contact interactions thus become negligible very rapidly in this region. This implies that one may put $\tilde{\gamma} = 0$ as a good approximation. In this way, the RGEs of the NEFT with pions is smoothly connected to those of the pionless NEFT.

APPENDIX B: THE CASE OF QED

In this Appendix, we briefly discuss the case of QED in the nonrelativistic region (NRQED). As a concrete example, we have a hydrogen atom (or electron-proton scattering) in mind, and we are interested in the low-energy region where even the electron behaves as a nonrelativistic particle.

The system consists of a nonrelativistic proton and an electron interacting by exchanging photons. The Lagrangian is similar to that of NEFT with propagating pions, Eq. (3.5). Note that we also include contact interactions of protons and electrons. By a similar analysis, one easily finds that the IR enhancement takes place for the proton-electron reducible diagrams, giving the leading-order contributions to the RGEs. There (the time-time component of) the photon propagator is replaced with the instantaneous Coulomb potential. (It is independent of the choice of the gauge.) Effectively, the RGEs are generated by the averaged action consisting of the contact interactions and the instantaneous Coulomb interaction, such as

$$-\frac{e^2}{2} \int dt \int d^3x \, d^3y \, [e^T(t, \boldsymbol{x})\sigma_2 p(t, \boldsymbol{y})]^\dagger [e^T(t, \boldsymbol{x})\sigma_2 p(t, \boldsymbol{y})]$$
$$\times C(|\boldsymbol{x} - \boldsymbol{y}|), \quad \text{(B1)}$$

where $C(r)$ is the Coulomb potential,

$$C(r) = \frac{1}{4\pi r}, \quad \text{(B2)}$$

similar to that of NEFT, Eq. (3.22). The Coulomb interaction here should be considered as the long-distance part of the Coulomb interaction (L-Coulomb), while the short-distance part (S-Coulomb) is included in the contact interactions.

The difference between the NEFT with pions and the NRQED comes from the following facts: (i) the photon is massless while the pion is massive, (ii) the Coulomb interaction is not a derivative coupling, while the $NN\pi$ is, and (iii) the electromagnetic coupling e^2 is dimensionless, whereas the $NN\pi$ coupling $(g_A/2f)^2$ is not.

For the NRQED RGEs, we introduce the dimensionless coupling constant γ_{QED},

$$\gamma_{\text{QED}} \equiv e^2 \frac{M}{\Lambda}, \quad \text{(B3)}$$

where M is the reduced mass of the proton and the electron. This particular Λ dependence is required by the condition that the RGEs for the contact interactions do not contain the explicit Λ dependence, as in Sec. IV A for the NEFT with pions. The RGE for γ_{QED} in the leading order is given by

$$\frac{d\gamma_{\text{QED}}}{dt} = +\gamma_{\text{QED}}. \quad \text{(B4)}$$

(Note that the usual β function contribution vanishes, $de^2/dt = 0$, because we are in a region where Λ is smaller than the electron mass.) We see that the L-Coulomb interactions become more important in the infrared, in strong contrast to the L-OPE. Photons do not decouple as they should not. There is no "photonless" QED.

The contact interactions lead to similar RGEs as to those of the NEFT. There is a nontrivial fixed point similar to the one found in the NEFT. Since there seems to be no fine-tuning for NRQED, it is natural to consider that the physical system is close to the trivial fixed point. Therefore, the S-Coulomb is irrelevant and should be treated as a perturbation. In the NEFT case, the L-OPE becomes rapidly irrelevant for $\Lambda < m_\pi$, as is shown in Appendix A. On the other hand, NRQED does not have such a region where the L-Coulomb becomes irrelevant because the photon is massless.

Therefore, we conclude that the L-Coulomb interaction is nonperturbative, and the contact interactions including the S-Coulomb interaction are perturbative. It is interesting to compare this with the case of the NEFT, where the L-OPE is perturbative while (a part of) the S-OPE should be resummed to all orders.

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