Rigorous pion electromagnetic form factor behavior in the spacelike region

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Precise experimental information on $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-)$ is transferred into the spacelike region by taking advantage of the analyticity. As a result, rigorous pion electromagnetic form factor behavior in spacelike region is obtained. The latter is compared with some existing model predictions.

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The pion electromagnetic (EM) form factor (FF) $F_\pi(Q^2)$ with the squared four-momentum transfer $Q^2 = -t$ is one of the most simple objects of investigations in strong interaction physics. In spite of this fact, there is no theory able to explain all its known features. Even in the spacelike region, where the pion EM FF behavior is expected to be represented by a simple smooth decreasing curve between the norm $F_\pi(0) = 1$ and the pQCD asymptotic behavior [1–3],

$$F_\pi(Q^2)_{Q^2\to\infty} \sim \frac{64\pi^2 f_\pi^2}{(11 - 2/3n_f)Q^2 \ln Q^2/\Lambda^2}, \quad (1)$$

all known attempts (see [4–8]) to reach experimentally measurable region do not give uniform results.

In this paper, it is shown how pion EM FF can be reconstructed in the spacelike region with the help of the accurate data on the total cross section $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-) \equiv \sigma_{\rm tot}(t)$ in the elastic $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$ region, which plays a dominant role in our prediction. On the basis of the Phragmen-Lindelöf theorem, the assumption is made that the asymptotically pion EM FF in the Minkowski region has a form analogous to that in the Euclidean one. As a result, the asymptotic form of the imaginary part of the pion EM FF in the timelike region is found to be helpful to specify correct parametrization of corrections from the interval $(m_{\pi^0} + m_\omega)^2 \leqslant t \leqslant +\infty$. All these ingredients are linked up together via dispersion integrals, and as a result, a prediction for the pion EM FF in the spacelike region can be achieved.

Really, the analytic properties of the pion EM FF, by means of the Cauchy formula and assuming the validity of the pQCD asymptotic behavior (1) in all directions of the complex t plane, can be transformed into the dispersion relation without any subtractions:

$$F_\pi(Q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{{\rm Im}^E\, F_\pi(t')}{t' + Q^2} dt' + \frac{1}{\pi}\int_{t_{\pi^0\omega}}^{\infty} \frac{{\rm Im}^A\, F_\pi(t')}{t' + Q^2} dt'. \quad (2)$$

Using the normalization condition $F_\pi(0) = 1$ for $Q^2 = 0$ in Eq. (2), one gets the sum rule for the pion FF imaginary parts

$$1 = \frac{1}{\pi}\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{{\rm Im}^E\, F_\pi(t')}{t'} dt' + \frac{1}{\pi}\int_{t_{\pi^0\omega}}^{\infty} \frac{{\rm Im}^A\, F_\pi(t')}{t'} dt'. \quad (3)$$

Another superconvergence sum rule for the same imaginary parts, namely

$$0 = \frac{1}{\pi}\int_{4m_\pi^2}^{t_{\pi^0\omega}} {\rm Im}^E\, F_\pi(t')dt' + \frac{1}{\pi}\int_{t_{\pi^0\omega}}^{\infty} {\rm Im}^A\, F_\pi(t')dt', \quad (4)$$

can be derived by an application of the Cauchy theorem to $F_\pi(t)$ in the complex t plane and its pQCD asymptotics (1).

In all previous three integral relations, we have automatically separated the elastic region $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$ contributions of ${\rm Im}^E\, F_\pi(t)$ (therefore superscript E) which for $F_\pi(Q^2)$ at $Q^2 = -t = 0$ represents up to 90% of the total ${\rm Im}\, F_\pi(t) = {\rm Im}^E\, F_\pi(t) + {\rm Im}^A\, F_\pi(t)$, as one can see from further considerations.

In order to evaluate the first integral in Eq. (2), one can apply the following method of extracting ${\rm Im}^E\, F_\pi(t)$ from $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-) \equiv \sigma_{\rm tot}(t)$, which is the foremost quantity for obtaining of experimental values of the pure isovector pion EM FF in the timelike region.

As the electron-positron annihilation into two charged pions is of the EM nature, one can treat it in the one-photon-exchange approximation, and as a result, there are no model ingredients in the extraction of $|F_\pi(t)|$ from the measured cross section. Since two final-state pions with total orbital moment $l = 1$ (due to the spin of the photon) have the isospin $I = 1$ and a positive G parity, the pion EM FF is of the pure isovector nature and all resonances to be seen in the pion EM FF data can be only isovectors (the ρ-meson family) with $G = +1$ and with all other quantum numbers of the photon, like $J = 1$ and negative intrinsic and charge parities.

Nevertheless, in [9] one finds also isoscalar vector meson isospin-violating decays into two charged pions, ω (782) $\to \pi^+\pi^-$ with fraction $(\Gamma_i/\Gamma) = 1.53\%$ and $\Phi(1020) \to \pi^+\pi^-$ with fraction $(\Gamma_i/\Gamma) = 7.3 \times 10^{-5}\%$, which contribute through higher order corrections to the $e^+e^- \to \pi^+\pi^-$ process, and experimentalists are unable to eliminate them from final results.

In order to obtain the pure isovector pion EM FF experimental information from existing data on the $e^+e^- \to \pi^+\pi^-$ process, we write its total cross section in the form

$$\sigma_{\rm tot}(t) = \frac{\pi\alpha^2\beta_\pi^3}{3t}|F_{\pi\rho}(t) + \xi \cdot \exp(i\alpha)F_{\pi\omega}(t)|^2, \qquad \beta_\pi = \left[\left(t - 4m_\pi^2\right)/t\right]^{1/2}, \quad (5)$$

where $F_{\pi\rho}(t)$ and $F_{\pi\omega}(t)$ represent ρ- and ω-meson contributions to the $e^+e^- \to \pi^+\pi^-$ process, respectively, and the Φ-meson contribution is neglected, as we are interested only in $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-)$ in the elastic region $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$. The so-called ρ-ω interference amplitude ξ can be expressed through the partial decay width $\Gamma(\omega \to \pi^+\pi^-)$ by the relation

$$\xi = \frac{6}{\alpha m_\omega}\left(\frac{m_\omega^2}{m_\omega^2 - 4m_\pi^2}\right)^{3/4}[\Gamma(\omega \to e^+e^-) \cdot \Gamma(\omega \to \pi^+\pi^-)]^{1/2}, \tag{6}$$

and the ρ-ω interference phase α is

$$\alpha = \arctan\frac{m_\rho\Gamma_\rho}{m_\rho^2 - m_\omega^2}. \tag{7}$$

Because the ω-vector meson is a very narrow resonance, one can approximate the ω-meson contribution to the $e^+e^- \to \pi^+\pi^-$ process in (5) by the Breit-Wigner form

$$F_{\pi\omega}(t) = \frac{m_\omega^2}{m_\omega^2 - t - i m_\omega\Gamma_\omega}. \tag{8}$$

Further, first we exploit the pion FF phase representation $F_{\pi\rho}(t) = |F_\pi(t)| \cdot \exp(i\delta_\pi)$ in Eq. (5) and subsequently the pion FF phase identity $\delta_\pi \equiv \delta_1^1$ with the P-wave isovector $\pi\pi$ scattering phase shift $\delta_1^1(t)$ for $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$. The latter follows just from the elastic pion FF unitarity condition $\mathrm{Im}^E\, F_\pi(t) = |F_\pi(t)|e^{i\delta_\pi(t)}e^{-i\delta_1^1(t)}\sin\delta_1^1(t)$. As a result, the quadratic equation for the absolute value of pure isovector pion FF data is obtained [10]:

$$|F_{\pi\rho}(t)|^2 + 2Z(t)|F_{\pi\rho}(t)| + \left[\frac{\xi^2 m_\omega^4}{\left(m_\omega^2 - t\right)^2 + m_\omega^2\Gamma_\omega^2} - \frac{3t}{\pi\alpha^2\beta_\pi^3}\sigma_{\rm tot}(t)\right] = 0, \tag{9}$$

with the physical solution

$$|F_{\pi\rho}(t)| = -Z(t) + \left[Z^2(t) + \frac{3t}{\pi\alpha^2\beta_\pi^3}\sigma_{\rm tot}(t) - \frac{\xi^2 m_\omega^4}{\left(m_\omega^2 - t\right)^2 + m_\omega^2\Gamma_\omega^2}\right]^{1/2} \tag{10}$$

and

$$Z(t) = \frac{\xi m_\omega^2}{\left(m_\omega^2 - t\right)^2 + m_\omega^2\Gamma_\omega^2}\left[\left(m_\omega^2 - t\right)\cos\left(\alpha - \delta_1^1\right) - m_\omega\Gamma_\omega\sin\left(\alpha - \delta_1^1\right)\right]. \tag{11}$$

The data on $\mathrm{Im}^E F_{\pi\rho}(t)$ with errors for $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$ are then determined by the relation

$$\mathrm{Im}^E\, F_{\pi\rho}(t) = |F_{\pi\rho}(t)|\sin\delta_1^1, \tag{12}$$

using experimental information on ξ, α, m_ω, and Γ_ω, the data recently measured by Frascati [11] for the radiative return and by Novosibirsk [12,13] for improved experimental information on $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-)$, and the suitable parametrization [10] of $\delta_1^1(t)$. Then the first integral of Eq. (2) as a function of Q^2 is a smoothly decreasing curve and the first integrals of Eqs. (3) and (4) give

$$\frac{1}{\pi}\int_{4m_\pi^2}^{t_{\pi^0\omega}}\frac{\mathrm{Im}^E\, F_\pi(t')}{t'}dt' = 0.8995 \tag{13}$$

and

$$\frac{1}{\pi}\int_{4m_\pi^2}^{t_{\pi^0\omega}}\mathrm{Im}^E\, F_\pi(t')dt' = 0.5023, \tag{14}$$

respectively, where we have already identified $\mathrm{Im}^E F_{\pi\rho}(t)$ with $\mathrm{Im}^E F_\pi(t)$.

In order to estimate the second integral in Eq. (2) as a function of Q^2, one has to know something about the asymptotic $\mathrm{Im}^A\, F_\pi(t)$ for $(m_{\pi^0} + m_\omega)^2 \leqslant t < +\infty$. The analytic continuation of Eq. (1) to the upper boundary of the pion FF cut on the positive real axis of the $t = -Q^2$ plane leads to the pion FF imaginary part

$$\mathrm{Im}\, F_\pi(t)_{t\to\infty} \sim -\pi\frac{\left(64\pi^2 f_\pi^2\right)}{(11 - 2/3n_f)t\ln^2 t/\Lambda^2}. \tag{15}$$

The positivity of all data on $\mathrm{Im}^E\, F_\pi(t)$ following from Eq. (12) for $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0} + m_\omega)^2$ and the asymptotic form (15) can be satisfied simultaneously only if $\mathrm{Im}^A\, F_\pi(t)$ in Eqs. (2)–(4) acquires at least one zero value at t_z for $t > (m_{\pi^0} + m_\omega)^2$ and vanishes asymptotically from the negative values as $t \to +\infty$. The simplest function reflecting all these required properties is

$$\mathrm{Im}^A\, F_\pi(t) = \pi\frac{64\pi^2 f_\pi^2}{(11 - 2/3n_f)}\frac{t_z - t}{(t - C)^2\ln^2 t/\Lambda^2}, \tag{16}$$

with the parameter values

$$\Lambda = 0.7226\ \mathrm{GeV}, \quad C = -9.7255\ \mathrm{GeV}^2, \quad t_z = 4.6975\ \mathrm{GeV}^2, \quad n_f = 13.2517 \tag{17}$$

to be determined from conditions

$$\begin{aligned} \mathrm{Im}^E\, F_\pi(t)_{|_{t=t_{\pi^0\omega}}} &= \mathrm{Im}^A\, F_\pi(t)_{|_{t=t_{\pi^0\omega}}}, \\ \frac{d}{dt}\mathrm{Im}^E\, F_\pi(t)_{|_{t=t_{\pi^0\omega}}} &= \frac{d}{dt}\mathrm{Im}^A\, F_\pi(t)_{|_{t=t_{\pi^0\omega}}}, \\ 0.1005 &= \frac{64\pi^2 f_\pi^2}{(11 - 2/3n_f)} \times \int_{t_{\pi^0\omega}}^{\infty}\frac{t_z - t'}{t'(t' - C)^2\ln^2 t'/\Lambda^2}dt', \\ -0.5023 &= \frac{64\pi^2 f_\pi^2}{(11 - 2/3n_f)} \times \int_{t_{\pi^0\omega}}^{\infty}\frac{t_z - t'}{(t' - C)^2\ln^2 t'/\Lambda^2}dt', \end{aligned} \tag{18}$$

obtained by using also the values of the integrals (13) and (14), respectively.

The pion EM FF spacelike region behavior calculated by the dispersion relation (2) is displayed in Fig. 1 (solid line), where also recent theoretical predictions [4–7] are presented for comparison.

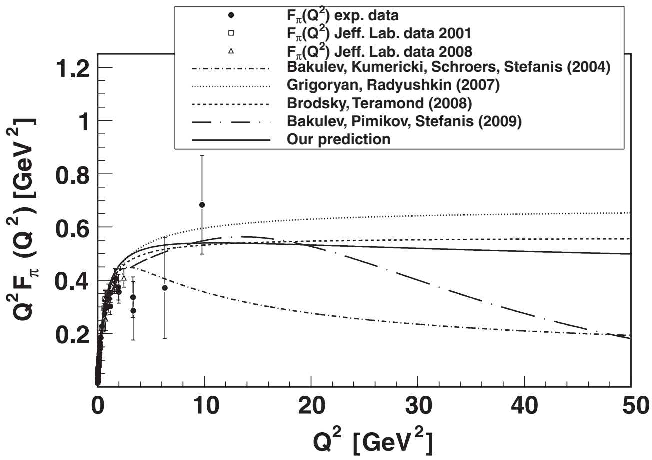

FIG. 1. Theoretical predictions of the pion EM FF behavior in the spacelike region and their comparison with existing data.

In Fig. 2, we draw the ratio of the second integral in Eq. (2) to the first one as a function of Q^2 in order to demonstrate our approach to be more or less model independent. Really, as one can clearly see from Fig. 2, the correction of the weakly model dependent parametrization (16) of the $\mathrm{Im}^A F_\pi(t)$ for $(m_{\pi^0}+m_\omega)^2 \leqslant t < +\infty$ becomes negligible with increased values of Q^2. As a result, our prediction of the pion EM FF in Fig. 1 with increased values of Q^2 is more and more model independent.

We defend the reliability of our prediction for the pion EM FF in the spacelike region, presented in Fig. 1, also by a prediction of the complex pion EM FF on the upper boundary of the cut in the timelike region and compare it with existing data. In this region, predictions seem to be more sensitive to the analytic approximations and the issues discussed previously and their comparison with the accurate timelike data surely will be a more self-consistent test of the whole elaborate approach.

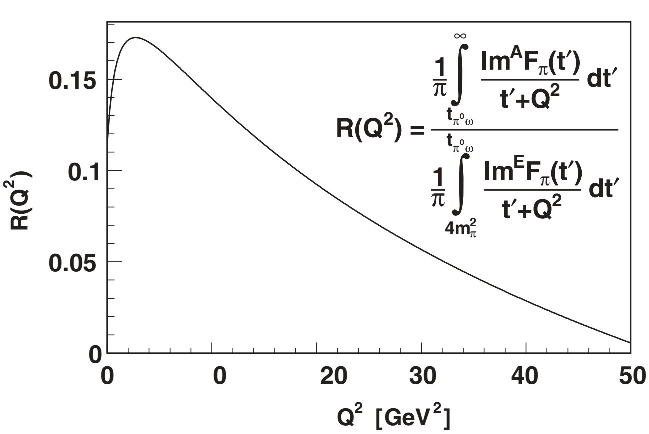

FIG. 2. The ratio of the $\mathrm{Im}^A F_\pi(t)$ contribution to the $\mathrm{Im}^E F_\pi(t)$ one in predicted pion EM FF behavior in the spacelike region.

We start with the dispersion relation

$$F_\pi(t) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{\mathrm{Im}^E\, F_\pi(t'+i\varepsilon)}{t'-t-i\varepsilon}dt' + \int_{t_{\pi^0\omega}}^{\infty} \frac{\mathrm{Im}^A\, F_\pi(t'+i\varepsilon)}{t'-t-i\varepsilon}dt'\right\}, \tag{19}$$

where the first integral in brackets is singular if the interval $4m_\pi^2 \leqslant t \leqslant (m_{\pi^0}+m_\omega)^2$ is considered and the second integral in brackets is singular if the complex pion EM FF is calculated within the interval $t_{\pi^0\omega} < t < +\infty$.

For an evaluation of the singular integrals, one can use the well-known symbolic so-called Sokhotsky-Plemelj formula from the theory of functions of complex variables,

$$\lim_{\varepsilon\to 0}\frac{1}{t'-t\mp i\varepsilon} = P\frac{1}{t'-t} \pm i\pi\delta(t'-t). \tag{20}$$

Then considering the first integral in Eq. (19) to be singular, one practically obtains

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{\mathrm{Im}^E\, F_\pi(t'+i\varepsilon)}{t'-t-i\varepsilon}dt' = \frac{1}{\pi}P\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{\mathrm{Im}^E\, F_\pi(t')}{t'-t}dt' + i\int_{4m_\pi^2}^{t_{\pi^0\omega}} \mathrm{Im}^E\, F_\pi(t')\delta(t'-t)dt', \tag{21}$$

where P denotes that the Cauchy principal value

$$\frac{1}{\pi}P\int_{4m_\pi^2}^{t_{\pi^0\omega}} \frac{\mathrm{Im}^E\, F_\pi(t')}{t'-t}dt' = \frac{1}{\pi}\lim_{\delta\to 0}\left\{\int_{4m_\pi^2}^{t-\delta} \frac{\mathrm{Im}^E\, F_\pi(t')}{t'-t}dt' + \int_{t+\delta}^{t_{\pi^0\omega}} \frac{\mathrm{Im}^E\, F_\pi(t')}{t'-t}dt'\right\} \equiv \mathrm{Re}^E\, F_\pi(t) \tag{22}$$

has to be taken and the second integral in (21) gives just $\mathrm{Im}^E\, F_\pi(t)$, by means of which the dominant part of the pion EM FF spacelike behavior is found.

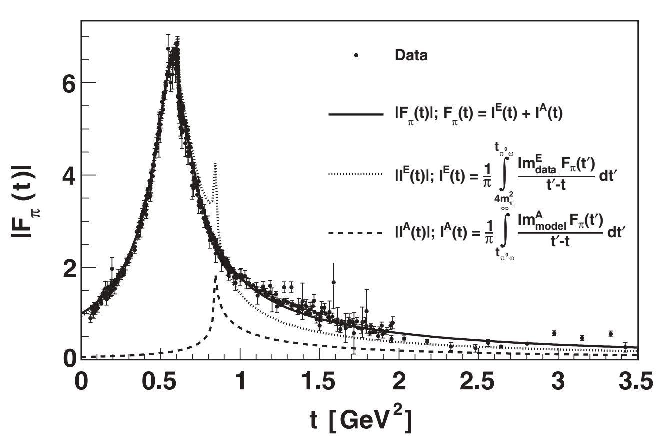

FIG. 3. Self-consistent reconstruction of the absolute value of the pion EM FF behavior in the timelike region with the help of the accurate experimental information on $\sigma_{tot}(e^+e^- \to \pi^+\pi^-)$.

In a similar way, a contribution of the second singular integral in Eq. (19) to the complex pion EM FF on the upper boundary of the cut in the timelike region can be evaluated.

Numerical predictions for the absolute values of both integrals in Eq. (19), as well as the absolute value of the whole complex pion EM FF in the timelike region and its comparison with existing data up to $t = 3.5\ \mathrm{GeV}^2$, are presented in Fig. 3.

Agreement (see Fig. 3) of the predicted absolute value of the pion EM FF in the timelike region with existing data confirms a reliability of our prediction of $F_\pi(t)$ in the spacelike region as it is presented in Fig. 1.

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