Bulk viscosity and relaxation time of causal dissipative relativistic fluid dynamics

Xu-Guang Huang,[1,2] Takeshi Kodama,[3] Tomoi Koide,[1] and Dirk H. Rischke[1,2]
[1]*Frankfurt Institute for Advanced Studies, D-60438 Frankfurt am Main, Germany*
[2]*Institut für Theoretische Physik, J.W. Goethe-Universität, D-60438 Frankfurt am Main, Germany*
[3]*Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postale 68528, 21945-970, Rio de Janeiro, Brazil*
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The microscopic formulas of the bulk viscosity ζ and the corresponding relaxation time τ_Π in causal dissipative relativistic fluid dynamics are derived by using the projection operator method. In applying these formulas to the pionic fluid, we find that the renormalizable energy-momentum tensor should be employed to obtain consistent results. In the leading-order approximation in the chiral perturbation theory, the relaxation time is enhanced near the QCD phase transition, and τ_Π and ζ are related as $\tau_\Pi = \zeta/[\beta\{(1/3 - c_s^2)(\varepsilon + P) - 2(\varepsilon - 3P)/9\}]$, where ε, P, and c_s are the energy density, pressure, and velocity of sound, respectively. The predicted ζ and τ_Π should satisfy the so-called causality condition. We compare our result with the results of the kinetic calculation by Israel and Stewart and the string theory, and confirm that all three approaches are consistent with the causality condition.

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I. INTRODUCTION

Relativistic hydrodynamics is an important tool for describing high energy flow phenomena in different areas, such as heavy-ion collisions, relativistic astrophysics, and cosmology, although its theoretical foundation has not yet been fully established, in particular, when dissipative processes are involved [1]. The simplest formulation is a relativistic covariant extension of the nonrelativistic Navier-Stokes equation. It is, however, known that the relativistic Navier-Stokes theory suffers the problem of relativistic acausality and instability [2–4]. The importance of the modification of the relativistic Navier-Stokes theory is discussed also in Ref. [5].

Since the seminal work by Israel and Stewart [6], many different approaches of relativistic hydrodynamics which are consistent with causality have been proposed [7–13]. In the following, we call these the causal dissipative relativistic hydrodynamics (CDR) theories [1]. The crucial difference of any CDR theories from the Navier-Stokes theory can be characterized by the introduction of finite relaxation times in the definitions of irreversible currents. This aspect is somehow overlooked but has an important consequence. That is, any relativistically causal fluids will be *non-Newtonian* in the sense that irreversible currents are no longer simply proportional to the corresponding thermodynamic forces while their *Newtonian* counterparts are.

In hydrodynamics, all transport coefficients such as shear viscosity and bulk viscosity are inputs and should be determined from a microscopic theory. In Navier-Stokes theory, the coefficients are usually calculated by using two different approaches. One is the kinetic approach based mainly on the Boltzmann equation, and the other is the microscopic approach using the Green-Kubo-Nakano (GKN) formula.

Strictly speaking, the kinetic approach is applicable only to rarefied gas and not reliable in practice for calculating the transport coefficients for finite density systems. For example, the density expansion of the shear viscosity $\eta(\rho)$ of a classical fluid in three-dimensional space is given by [14]

$$\eta(\rho) = \eta_0 + \eta_1\rho + \eta_2\rho^2 \ln\rho + \cdots. \quad (1)$$

What we can calculate from the Boltzmann equation is only the first term η_0. This is because the Boltzmann equation is the lowest order approximation of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equation and does not contain the information of multiple collisions, which violates the important assumption of the Boltzmann equation, that is, molecular chaos. To calculate the further expansion coefficients η_1 and η_2 in the kinetic approach, we have to use, for example, the Bogoliubov-Choh-Uhlenbeck equation which is a generalized Boltzmann equation [14]. Unfortunately, a systematic generalization of the relativistic Boltzmann equation is not yet known.

On the other hand, the GKN formula does not have such a limitation with respect to the finiteness of density as far as we know, because it is derived from the microscopic theory, quantum field theory [15]. However, for a relativistic fluid, we have to be careful because the Newtonian property of a fluid is assumed to derive the GKN formula. Thus new formalism is needed to calculate the transport coefficients of CDR.

Transport phenomena such as viscosities, diffusion, and heat conduction in hydrodynamics are related rather to the changes in boundary conditions than to the responses of the system to an external mechanical perturbation. For this reason, the required formulation should be different from the ordinary linear response theory. This fact was already emphasized by Kubo [16]. As a matter of fact, the well-known expressions of transport coefficients of relativistic hydrodynamics are obtained by using the nonequilibrium statistical operator method by Zubarev [17,18]. So far, several different approaches have been proposed to calculate the transport coefficients: indirect Kubo method, Langevin-Fokker-Planck method, regression hypothesis based method, local equilibrium approach, external reservoir method, prediction theory, and so on. See Ref. [19] for details.

Recently, we proposed a new microscopic formula to calculate shear viscosity and the corresponding relaxation time of CDR using the projection operator method, which belongs to the Langevin-Fokker-Planck method in the classification mentioned above [20]. Our formula is expressed in terms of the time correlation functions of microscopic currents and is a natural extension of the GKN formula. We showed that, for shear viscosity, it reproduces the GKN results in the leading order. When applied to a Navier-Stokes fluid and nonrelativistic diffusion processes, our approach reproduces the well-known results, as is discussed in Refs. [21,22].

The purpose of this paper is to derive the microscopic formulas of the bulk viscosity ζ and the corresponding relaxation time τ_Π in the framework of the projection operator method [23]. We apply the result to a pionic fluid and calculate in the leading-order approximation in the chiral perturbation theory with the renormalizable energy-momentum tensor. We find that the relaxation time τ_Π is enhanced around the temperature near the QCD phase transition.

We further discuss the differences among our formalism, the kinetic theory [6,24–26] and the string theory [27]. In a CDR, the values of ζ and τ_Π should satisfy the causality condition, which is derived by employing that the propagating speed of a physical signal should not exceed the speed of light [3,4]. If this condition is violated, relativistic fluids become unstable [3,4]. We confirm that the values of transport coefficients obtained by all three approaches are consistent with the causality condition.

It should be noted that the bulk viscosity and the corresponding relaxation time are important not only for heavy-ion collision physics, but also for astrophysics, for example, the stability windows in parameter space of rotating compact stars [28], driving inflation in the early universe, and the associated entropy production [29].

This paper is organized as follows. First we give a brief review of the projection operator method in Sec. II for the sake of later convenience. In Sec. III, we apply the method and derive expressions for bulk viscosity and relaxation time. We calculate the formulas in the leading-order approximation and discuss the consistency with the causality condition in Sec. IV. Possible other generalizations of our result is discussed in Sec. V. The summary of our results concludes this work in Sec. VI. Throughout this paper, we will use metric $g = \mathrm{diag}(+,-,-,-)$ and the natural units $\hbar = c = k_B = 1$.

II. PROJECTION OPERATOR METHOD

For later convenience, let us briefly review the projection operator method [30–32]. It should be emphasized that the projection operator method was first proposed by Nakajima [33], although it is often referred to as the Mori-Zwanzig formalism due to the extensive use and developments done by these authors [34,35].

Many dynamical variables of practical interest, such as the conserved quantities, usually vary slowly in time compared to other microscopic quantities. We call them gross variables.[1] In order to discuss the dynamics of these slowly varying relevant variables, we need to introduce a coarse-graining procedure to smooth out the microscopic dynamics. The projection operator method provides a systematic way to extract the information of the relevant coarse-grained dynamics from the underlying microscopic theories.

In the case of a quantum system, the full microscopic dynamics is described by the Heisenberg equation of motion,

$$\partial_t O(t) = i[H, O(t)] \equiv iLO(t), \tag{2}$$

where O is an arbitrary operator and H is the Hamiltonian. For simplicity, here we have assumed that H is independent of time. See Refs. [37,38] for the case of a system with time-dependent Hamiltonian. The second equality defines the Liouville operator L. In order to project out the irrelevant information associated with variables of microscopic (short) time scales, we introduce a time-independent projection operator P and its complementary operator $Q = 1 - P$, which satisfy the following general properties,

$$P^2 = P, \quad PQ = QP = 0. \tag{3}$$

With the help of these operators, the Heisenberg equation of motion can be re-expressed as [35,37,38]

$$\frac{\partial}{\partial t}O(t) = e^{iLt}PiLO(0) + \int_0^t d\tau e^{iL(t-\tau)}PiLQe^{iLQ\tau}iLO(0) + Qe^{iLQt}iLO(0). \tag{4}$$

This is called the time-convolution (TC) equation and its right-hand side is composed of three distinct parts. The first term is called the streaming term and usually corresponds to collective oscillations such as plasma wave and spin wave. The second term is called the memory term which turns into the dissipation term after a coarse-graining procedure. The third term is identified with the noise term after implementing coarse-graining of time, as we will see later. Thus this equation is considered as a generalized Langevin equation. As a matter of fact, the memory term and the noise term are related through the fluctuation-dissipation theorem of second kind [35,39]. Note that the TC equation is very general and still equivalent to the Heisenberg equation of motion.

The choice of the most appropriate projection operator depends on the specific properties of a given system and also on the coarse-graining procedure which we wish to introduce. If we choose the projection operator so as to extract all the relevant gross (in our case, hydrodynamic) variables, we can, in principle, derive hydrodynamic equations from the TC equation. For this purpose, we use the Mori projection operator [35]. Let

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_n \end{pmatrix} \tag{5}$$

[1]When a phase transition is present, in principle, we should consider also the corresponding order parameters and soft modes as the candidates of the gross (hydrodynamic) variables. We do not, however, discuss such a case in this work [36].

be an n-dimensional vector formed by n time-independent operators corresponding to the gross variables $\{\bar{A}_i\}$, where the notation $\bar{A}$ is used to distinguish the Schrödinger operator from its Heisenberg form, $A = A(t)$. We choose $\bar{A} = A(0)$, so that $A(t) = e^{iLt}\bar{A}$.

Then the time-independent Mori projection operator P is defined as [35]

$$PO = \sum_{i=1,}^{n} c_i \bar{A}_i, \tag{6}$$

where O is an arbitrary operator, and the coefficient c_i is given by

$$c_i = \sum_{j=1}^{n} (O, \bar{A}_j^\dagger) \cdot (\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1}_{ji}. \tag{7}$$

Here (X, Y) denotes the inner product of two arbitrary operators X and Y (see below), and $(\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1}_{ji}$ denotes ji element of the inverse matrix of $(\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)$, i.e.,

$$\sum_j (\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1}_{ij} \cdot (\bar{A}_j, \bar{A}_k^\dagger) = \delta_{i,k}. \tag{8}$$

In this way, we expect that the relevant part of an arbitrary operator is expressed as a function of the gross variables by operating this projection operator. In order to follow the dynamics of the gross variables in time, we would need the time-dependent projections [40], but for the present purpose of calculating the transport coefficients, the time-independent projection is sufficient.

We are still left with the freedom to choose the definition of the inner product. Here, following Ref. [35], we use Kubo's canonical correlation,

$$(X, Y) = \int_0^\beta \frac{d\lambda}{\beta} \mathrm{Tr}[\rho_{\mathrm{eq}} e^{\lambda H} X e^{-\lambda H} Y], \tag{9}$$

where $\rho_{\mathrm{eq}} = e^{-\beta H}/\mathrm{Tr}[e^{-\beta H}]$ with β being the inverse of temperature. One can see that, if it is a classical system, Kubo's canonical correlation is reduced to the usual classical thermal expectation value. Thus Kubo's canonical correlation is the quantum generalization of the classical expectation values. It is easy to confirm that

$$(iLX, Y) = -(X, iLY). \tag{10}$$

Finally, the TC equation for the gross variable $\mathbf{A}(t)$ in the Heisenberg picture can be expressed as

$$\frac{\partial}{\partial t}\mathbf{A}(t) = i\Delta\,\mathbf{A}(t) - \int_0^t d\tau\, \Xi(\tau)\mathbf{A}(t-\tau) + \xi(t), \tag{11}$$

where Δ and Ξ are operators of $(n \times n)$ matrices and ξ is an n vector whose elements are given by

$$i\Delta_{ij} = \sum_k (iL\bar{A}_i, \bar{A}_k^\dagger)(\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1}_{kj}, \tag{12}$$

$$\Xi_{ij}(t) = -\theta(t)\sum_k (iLQe^{iLQt} iL\bar{A}_i, \bar{A}_k^\dagger)(\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1}_{kj}, \tag{13}$$

$$\xi_i(t) = Qe^{iLQt} iL\bar{A}_i. \tag{14}$$

If the set of n-gross variables $\{\bar{A}_i\}$ is appropriately chosen so as to extract all the dynamics associated with the slow hydrodynamic time scale, we expect that the dynamical variation time scale of the last term $\xi_i(t)$ of Eq. (11) should be very small compared to the hydrodynamic time scale, because the projection operator Q projects out components only orthogonal to $\{\bar{A}_i\}$. For this reason the term $\xi_i(t)$ is called the noise term.

III. GENERAL FORMULAS FOR BULK VISCOSITY AND RELAXATION TIME

In this section, we derive the microscopic formulas for the bulk viscosity ζ and the corresponding relaxation time τ_Π. Our strategy is as follows. We derive the evolution equation of the bulk viscous pressure Π from the TC equation, and compare the derived microscopic equation with the phenomenological one to extract the microscopic formulas. For our purpose of obtaining the transport coefficient, it is sufficient to consider small deviation from the stationary background fluid in thermal equilibrium.

The phenomenological equation of the bulk viscous pressure Π in CDR is given by [7,8]

$$\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu, \tag{15}$$

where ζ, τ_Π, and u^μ are the bulk viscosity, relaxation time, and fluid velocity, respectively. The first term on the left-hand side represents the retardation effect of Π which is necessary to satisfy relativistic causality. For $\tau_\Pi = 0$, Eq. (15) is reduced to the usual Navier-Stokes constructive equation. In general, as is predicted from the kinetic theory, it is possible to introduce more nonlinear terms in Eq. (15) but, as mentioned above, here we discuss only the lowest order equation consistent with CDR.

To avoid any possible influence from the shear stress tensor, we consider a perturbation in an infinite fluid in thermal equilibrium having a planar symmetry in the (y, z) plane. All the quantities associated with the perturbed fluid dynamics vary spatially only along the x direction. In this case, the fluid velocity points to the x direction (if one wants to discuss only the shear viscosity, one can choose the fluid velocity to point to the x direction but varying spatially along the y direction, as is done in Refs. [20,30]). Then, the equation of continuity of the energy-momentum tensor $T^{\mu\nu}$ in momentum space is given by

$$\partial_t T^{x0}(k^x, t) = -ik_x T^{xx}(k^x, t), \tag{16}$$

where k^x denotes the x component of the momentum vector $\mathbf{k}$. On the other hand, Eq. (15) is simplified as

$$\tau_\Pi \partial_t \Pi(k^x, t) + \Pi(k^x, t) = -\zeta i k^x u^x(k^x, t). \tag{17}$$

We use this equation as the definition of ζ and τ_Π. Here $\Pi(k^x, t)$, $T^{x0}(k^x, t)$, and $T^{xx}(k^x, t)$ are the Fourier transforms of $\Pi(x, t)$, $T^{x0}(x, t)$, and $T^{xx}(x, t)$, respectively.

To obtain the microscopic expressions of ζ and τ_Π, we derive the equation for Π from the TC equation. For this purpose, we have to choose appropriate gross variables included in Eq. (17) to define the projection operator. Among T^{x0}, T^{xx}, Π, and u^x, the bulk viscous pressure Π is the deviation from the equilibrium pressure in the diagonal components of the energy-momentum tensor. Thus we use the following operator representation,

$$\Pi(k^x) = -\tfrac{1}{3}T^{\mu}_{\mu}(k^x) + \tfrac{1}{3}\langle T^{\mu}_{\mu}(k^x)\rangle_{\rm eq}, \qquad (18)$$

where $\langle\cdots\rangle_{\rm eq}$ represents the equilibrium expectation value. Furthermore, as we will discuss later, u^x can be regarded as linearly dependent on T^{0x} in the lowest order of the perturbation. Then, Eq. (17) contains basically two independent gross variables,

$$\bar{\mathbf{A}}(k^x) = \begin{pmatrix} \bar{T}^{0x}(k^x) \\ \bar{\Pi}(k^x) \end{pmatrix}, \qquad (19)$$

where the bar notation, for example, $\bar{\Pi}$, refers to the operator value of $\Pi(t)$ at $t = 0$.

According to Eq. (6), the projection operator P is now defined by

$$PO = \frac{(O, \bar{T}^{0x}(-k^x))}{(\bar{T}^{0x}(k^x), \bar{T}^{0x}(-k^x))}\bar{T}^{0x}(k^x) + \frac{(O, \bar{\Pi}(-k^x))}{(\bar{\Pi}(k^x), \bar{\Pi}(-k^x))}\bar{\Pi}(k^x), \qquad (20)$$

Substituting it into Eq. (11), we obtain the following two equations,

$$\partial_t T^{0x}(k^x, t) = -ik^x\Pi(k^x, t), \qquad (21)$$

$$\partial_t \Pi(k^x, t) = -ik^x R_{k^x} T^{0x}(k^x, t) - \int_0^t d\tau\, \Xi_{22}(k^x, \tau) \times \Pi(k^x, t-\tau) + \xi(k^x, t), \qquad (22)$$

where

$$R_{k^x} = \frac{(\bar{\Pi}(k^x), \bar{\Pi}(-k^x))}{(\bar{T}^{0x}(k^x), \bar{T}^{0x}(-k^x))}, \qquad (23)$$

and we used

$$(iL\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)(\bar{\mathbf{A}}, \bar{\mathbf{A}}^\dagger)^{-1} = \begin{pmatrix} 0 & -ik^x \\ -ik^x R_{k^x} & 0 \end{pmatrix}. \qquad (24)$$

Note that we consider homogeneous energy density and pressure. Then the first equation is nothing but just Eq. (16), that is, the equation of continuity, and the second equation describes the nontrivial evolution of Π.

The exact expression of the memory function Ξ_{22} is given in Ref. [39]. However, if we are interested in the expression in the low k^x limit, we can calculate it more easily. From Eq. (22), the evolution of $(\Pi(k^x, t), \bar{\Pi}(-k^x))$ is given by

$$\begin{aligned}\partial_t(\Pi(k^x, t), \bar{\Pi}(-k^x)) &= -ik^x R^B_{k^x}(T^{0x}(k^x, t), \bar{\Pi}(-k^x)) \\ &\quad - \int_0^t d\tau\, \Xi_{22}(k^x, t-\tau)(\Pi(k^x, \tau), \bar{\Pi}(-k^x)). \end{aligned} \qquad (25)$$

Here we used $(\xi(k^x, t), \bar{\Pi}(-k^x)) = 0$, which is calculated from the definition of $\xi(k^x, t)$. Then the Laplace transform of the memory function at low k^x is

$$\Xi^L_{22}(k^x, s) = \frac{1 - sX^L(k^x, s)}{X^L(k^x, s)}, \qquad (26)$$

where

$$X^L(k^x, s) \equiv \int_0^\infty dt\, e^{-st}\frac{(\Pi(k^x, t), \bar{\Pi}(-k^x))}{(\bar{\Pi}(k^x), \bar{\Pi}(-k^x))}. \qquad (27)$$

From the final value theorem of the Laplace transform, we can show that

$$\begin{aligned}\lim_{s\to 0^+} sX^L(k^x, s) &= \lim_{t\to\infty}\frac{(\Pi(k^x, t), \bar{\Pi}(-k^x))}{(\bar{\Pi}(k^x), \bar{\Pi}(-k^x))} \\ &= \frac{\langle\Pi(k^x, \infty)\rangle_{\rm eq}\langle\bar{\Pi}(-k^x)\rangle_{\rm eq}}{(\bar{\Pi}(k^x), \bar{\Pi}(-k^x))} = 0. \end{aligned} \qquad (28)$$

Here we used the mixing property of the ergodic theory. Finally, the memory function in the low k^x and s limit is given by

$$\Xi^L_{22}(k^x, s) = \frac{1}{X^L(k^x, s)}. \qquad (29)$$

To extract the phenomenological equation (17), we have to violate the time-reversal symmetry. For this purpose, we implement the coarse-graining of time. Let us introduce a macroscopic time scale τ_M as

$$\tau_M = \epsilon t, \qquad (30)$$

where ϵ is a scale parameter and less than unity. Then the time-convolution integral is expressed as

$$\int_0^{\tau_M/\epsilon} d\tau\ \Xi_{22}(k^x, \tau)\Pi(k^x, \tau_M/\epsilon - \tau). \qquad (31)$$

When the microscopic and macroscopic time scales are clearly separated, we can take the vanishing ϵ limit. Then the integral is given by

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^{\tau_M/\epsilon} d\tau\ \Xi_{22}(k^x, \tau)\Pi(k^x, \tau_M/\epsilon - \tau) \\ &\quad = \int_0^\infty d\tau\ \Xi_{22}(k^x, \tau)\Pi(k^x, t). \end{aligned} \qquad (32)$$

We call this coarse-graining the time-convolutionless (TCL) approximation. Note that this approximation is very similar to the so-called Markov approximation. In the present case, however, there is still the memory effect for Π even after the TCL approximation, and we cannot call it the Markov approximation. With this approximation, Eq. (22) is expressed as

$$\begin{aligned}\partial_t\Pi(k^x, t) &\approx -ik^x R_{k^x} T^{0x}(k^x, t) - \int_0^\infty d\tau\, \Xi_{22}(k^x, \tau)\Pi(k^x, t) \\ &= -ik^x R_{k^x} T^{0x}(k^x, t) - \frac{1}{\tau^\Pi(k^x)}\Pi(k^x, t) \\ &\approx -ik^x R_{k^x}(\varepsilon + P)u^x(k^x, t) - \frac{1}{\tau^\Pi(k^x)}\Pi(k^x, t). \end{aligned} \qquad (33)$$

Here, the noise term is neglected. The function $\tau^\Pi(k^x)$ in the second line is defined by

$$\tau^\Pi(k^x) = X^L(k^x, s = 0). \qquad (34)$$

From the second line to the third line, we used the following replacement

$$T^{0x}(k^x,t) \simeq (\varepsilon + P)u^x(k^x,t), \quad (35)$$

which comes from the expression of the phenomenological energy-momentum tensor,

$$T^{0x}(x,t) = [\varepsilon + P + \Pi(x,t)]u^x(x,t), \quad (36)$$

and is justified near the local rest frame. Because we defined the projection operator with Π and u^x by neglecting nonlinear terms, we cannot predict the coefficients of nonlinear terms correctly in the present calculation. Thus, for the sake of consistency, we neglect the nonlinear term $\Pi(x,t)u^x(x,t)$ in Eq. (36). The validity of the TCL approximation and the general comment for the derivation of the nonlinear term are discussed in Sec. V.

By comparing Eq. (33) with Eq. (17), we obtain the following correspondences:

$$\tau_\Pi = \lim_{s,\mathbf{k}\to 0} X^L(\mathbf{k},s), \quad (37)$$

$$\zeta = (\varepsilon + P)R_{\mathbf{0}} \lim_{s,\mathbf{k}\to 0} X^L(\mathbf{k},s). \quad (38)$$

For the sake of convenience, we express these expressions in terms of the retarded Green's functions. Note that the correlation function $X^L(\mathbf{k},s)$ can be re-expressed as

$$X^L(\mathbf{k},\mathbf{s}) = -\frac{1}{\beta}\int_0^\infty dt \int d^3\mathbf{x}\, e^{-st-i\mathbf{k}\cdot\mathbf{x}} \int_t^\infty d\tau \langle \Pi(\mathbf{x},\tau)\bar{\Pi}(\mathbf{0})\rangle_{\text{ret}} \times \left[\int d^3\mathbf{x}_1\, e^{-i\mathbf{k}\cdot\mathbf{x}_1}(\bar{\Pi}(\mathbf{x}_1),\bar{\Pi}(\mathbf{0}))\right]^{-1}, \quad (39)$$

where the retarded Green's function is defined by

$$\begin{aligned}\langle \Pi(\mathbf{x},t)\Pi(\mathbf{x}_1,\tau)\rangle_{\text{ret}} &= -i\theta(t-\tau)\mathrm{Tr}\{\rho_{\text{eq}}[\Pi(\mathbf{x},t),\Pi(\mathbf{x}_1,\tau)]\} \\ &= \int_{-\infty}^\infty \frac{d\omega\, d^3\mathbf{k}}{(2\pi)^4} G^R_\Pi(\omega,\mathbf{k}) e^{i\omega(t-\tau)} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}_1)}. \quad (40)\end{aligned}$$

See Appendix A for details.

Then, finally, the formulas of ζ and τ_Π are given by

$$\begin{aligned}\frac{\zeta}{\beta(\varepsilon+P)} &= \frac{\zeta_{\text{GKN}}}{\beta^2\int d^3\mathbf{x}(\bar{T}^{0x}(\mathbf{x}),\bar{T}^{0x}(\mathbf{0}))} \quad (41)\\ &= -\frac{i}{\beta}\frac{\lim_{\omega\to 0}\lim_{\mathbf{k}\to\mathbf{0}} \partial G^R_\Pi(\omega,\mathbf{k})/\partial\omega}{\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\omega\to 0} G^R_{T^{0x}}(\omega,\mathbf{k})}, \quad (42)\end{aligned}$$

$$\begin{aligned}\frac{\tau_\Pi}{\beta} &= \frac{\zeta_{\text{GKN}}}{\beta^2\int d^3\mathbf{x}(\bar{\Pi}(\mathbf{x}),\bar{\Pi}(\mathbf{0}))} \quad (43)\\ &= -\frac{i}{\beta}\frac{\lim_{\omega\to 0}\lim_{\mathbf{k}\to\mathbf{0}} \partial G^R_\Pi(\omega,\mathbf{k})/\partial\omega}{\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\omega\to 0} G^R_\Pi(\omega,\mathbf{k})}. \quad (44)\end{aligned}$$

Here we have introduced the usual expression of bulk viscosity for the relativistic Navier-Stokes fluid in the GKN formula (obtained with the Zubarev method),

$$\begin{aligned}\zeta_{\text{GKN}} &= -\int d^3\mathbf{x}\int_0^\infty dt \int_t^\infty d\tau \langle \Pi(\mathbf{x},\tau)\bar{\Pi}(\mathbf{0})\rangle_{\text{ret}} \\ &= i\lim_{\omega\to 0}\lim_{\mathbf{k}\to\mathbf{0}} \frac{\partial G^R_\Pi(\omega,\mathbf{k})}{\partial\omega}, \quad (45)\end{aligned}$$

and one more retarded Green's function,

$$G^R_{T^{0x}}(\omega,\mathbf{k}) = \int_{-\infty}^\infty dt\, d^3\mathbf{x} \langle T^{0x}(\mathbf{x},t)\bar{T}^{0x}(\mathbf{0})\rangle_{\text{ret}}\, e^{-i\omega t} e^{i\mathbf{k}\mathbf{x}}. \quad (46)$$

Equations (41)–(44) are our main results. The bulk viscosity and it relaxation time are expressed by the ratios of Green's functions and different orderings of limits.

IV. APPLICATIONS TO HOT PIONIC FLUID

As an application of our microscopic formulas (42) and (44), we will calculate the bulk viscosity ζ and relaxation time τ_Π for hot pion fluid in confined phase within an effective model.

Let ϕ be the scalar field for pions (we simply use a real scalar field to present pions since the charge does not affect the results). The usual definition of the energy-momentum tensor of this field is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}, \quad (47)$$

where $\mathcal{L}$ is a Lagrangian density. In this case, the bulk viscous pressure of Eq. (18) for the noninteracting case would become order

$$\begin{aligned}\Pi(\mathbf{x},t) &= \tfrac{1}{3}\{[\partial\phi(\mathbf{x},t)]^2 - 2M^2\phi^2(\mathbf{x},t)\} \\ &\quad - \tfrac{1}{3}\langle[\partial\phi(\mathbf{x},t)]^2 - 2M^2\phi^2(\mathbf{x},t)\rangle_{\text{eq}}, \quad (48)\end{aligned}$$

where M is the pion mass. However, these expressions are not adequate for our purpose. First, note that the above bulk viscous pressure does not vanish even in the massless limit $M = 0$, which does not reflect the conformal property of the Lagrangian in this limit. Furthermore, the energy-momentum tensor (47) is not renormalizable, i.e., its matrix elements depend directly on the cutoff of the renormalized perturbation theory, as discussed in Ref. [41]. Thus we introduce the renormalizable energy-momentum tensor $\theta^{\mu\nu}$ following Ref. [41] as

$$\theta^{\mu\nu}(\mathbf{x},t) = T^{\mu\nu}(\mathbf{x},t) - \tfrac{1}{6}(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2(\mathbf{x},t). \quad (49)$$

Then the corresponding bulk viscous pressure for the noninteracting case is given by

$$\begin{aligned}\Pi(\mathbf{x},t) &= -\frac{1}{3}\theta^\mu_\mu(\mathbf{x},t) + \frac{1}{3}\langle\theta^\mu_\mu(\mathbf{x},t)\rangle_{\text{eq}} \\ &= -\frac{M^2}{3}[\phi^2(\mathbf{x},t) - \langle\phi^2(\mathbf{x},t)\rangle_{\text{eq}}], \quad (50)\end{aligned}$$

which recovers the conformal nature of the system in the vanishing limit of M. Note here that for the fermion and gauge fields, the usual definition of energy-momentum tensor is already renormalizable and no redefinition of the energy-momentum tensor is needed.

However, because of the reason which will be discussed in the end of this section, this is still not the definition of the bulk viscous pressure which is used in the following calculation. We recall that the behavior of the retarded Green's function G^R_Π in the low momentum limit is not changed by adding an additional term which is proportional to the energy density in the definition of the bulk viscous pressure. Finally, we added an additional term which is proportional to the energy density to define the bulk viscous pressure instead of Eq. (18),

$$\begin{aligned}\Pi(\mathbf{x},t) &= -\frac{1}{3}\theta^{\mu}_{\mu}(\mathbf{x},t) + \left(\frac{1}{3}-c_s^2\right)\theta^{00}(\mathbf{x},t) \\ &\quad + \left\langle \frac{1}{3}\theta^{\mu}_{\mu}(\mathbf{x},t) - \left(\frac{1}{3}-c_s^2\right)\theta^{00}(\mathbf{x},t)\right\rangle_{\mathrm{eq}} \\ &= -\frac{M^2}{3}\phi^2(\mathbf{x},t) + \left(\frac{1}{3}-c_s^2\right)\theta^{00}(\mathbf{x},t) \\ &\quad + \left\langle \frac{M^2}{3}\phi^2(\mathbf{x},t) - \left(\frac{1}{3}-c_s^2\right)\theta^{00}(\mathbf{x},t)\right\rangle_{\mathrm{eq}}, \end{aligned} \tag{51}$$

where c_s is the velocity of sound. This is the same definition of the bulk viscous pressure discussed in Refs. [17,42]. One can easily see that this bulk viscous pressure still vanishes in the massless limit.

As the lightest particles, pions dominate the transport properties of QCD in hadronic phase. From Eqs. (41) and (43), once the leading-order result of ζ_{GKN} is obtained, the corresponding leading-order results for ζ and τ_Π are obtained by substituting the denominators on the right-hand side by their noninteracting counterparts. A straightforward calculation leads to

$$\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\omega\to 0} G^R_{T^{0x}}(\omega,\mathbf{k}) = -\varepsilon - P, \tag{52}$$

$$\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\omega\to 0} G^R_{\Pi}(\omega,\mathbf{k}) = -\left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) + 2\frac{\varepsilon-3P}{9}, \tag{53}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+M^2}$. The energy density and pressure of the free pion gas are, respectively, given by

$$\varepsilon = \frac{N_\pi}{V}\sum_{\mathbf{p}} E_{\mathbf{p}} f(E_{\mathbf{p}}), \tag{54}$$

$$P = \frac{N_\pi}{3V}\sum_{\mathbf{p}} \frac{\mathbf{p}^2}{E_{\mathbf{p}}} f(E_{\mathbf{p}}), \tag{55}$$

where $f(x)$ is the Bose-Einstein distribution function $1/(e^{\beta x}-1)$ and the prefactor $N_\pi = 3$ counts the degeneracy of π^+, π^-, and π^0.

The left-hand side of Eq. (53) has a term which is ultraviolet divergent. In order to obtain the finite result of the right-hand side, we have renormalized out this vacuum term. However, it should be noted that the renormalization of this divergence is not trivial, as discussed in Ref. [43].

In short, in the leading-order approximation, the bulk viscosity and relaxation time are given by [23]

$$\frac{\zeta}{\beta(\varepsilon+P)} = \frac{\zeta_{\mathrm{GKN}}}{\beta(\varepsilon+P)}, \tag{56}$$

$$\frac{\tau_\Pi}{\beta} = \frac{\zeta_{\mathrm{GKN}}}{\beta\left[\left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) - 2\frac{\varepsilon-3P}{9}\right]}. \tag{57}$$

The first equation shows that the bulk viscosity ζ is reduced to the GKN bulk viscosity ζ_{GKN}, similary to the case of the shear viscosity in the leading-order calculation [20]. There already exist several calculations for ζ_{GKN} [44]. Thus we will not discuss its behavior here.

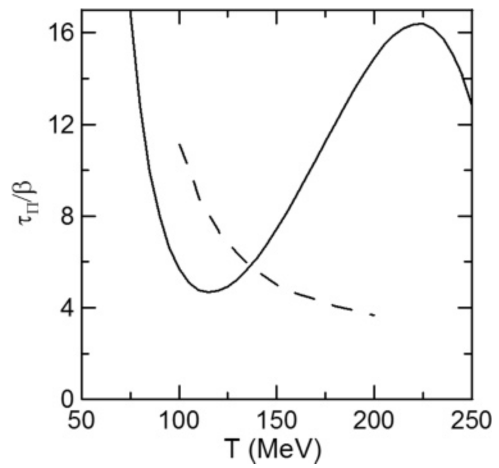

FIG. 1. Temperature dependence of the relaxation time of the bulk viscous pressure τ_Π/β of the hadron phase. The solid line represents the result of the leading-order approximation in chiral perturbation theory. For the sake of comparison, the result of the 14 moment approximation from Ref. [45] is shown by the dashed line.

By adopting the result of ζ_{GKN} calculated in the chiral perturbation theory, we plot the temperature dependence of the dimensionless ratio τ_Π/β in the hadron phase in Fig. 1. For the sake of comparison, the relaxation time calculated from the Boltzmann equation with Grad's moment method is shown by the dashed line [45]. The order of magnitude is same as the relaxation time of the shear viscosity, τ_π/β, which is shown in Fig. 1 in Ref. [20]. However, the temperature dependence of τ_Π/β is nontrivial. As shown in Ref. [20], τ_π/β is a monotonically decreasing function of temperature in the hadronic phase. On the other case, τ_Π/β, which is a decreasing function at low temperature, starts to increase around $T = 100$ MeV and shows maximum near the QCD phase transition. This comes from the enhancement of ζ_{GKN}. As discussed in Ref. [46], this enhancement is attributed to two mechanisms, the trace anomaly and the unitarity correction. The latter is related to the consideration of heavier excited states in chiral perturbation theory.

Now we compare our result with the results from Grad's method with the 14 moment approximation [6,24] and string theory [27]. For this purpose, it is convenient to consider the ζ/τ_Π ratio, because this quantity is independent of the choice of the collision term in the Boltzmann equation. In our leading-order result for pions, this ratio is given by

$$\frac{\zeta}{\tau_\Pi(\varepsilon+P)} = R_{\mathbf{0}} = \frac{\left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) - \frac{2}{9}(\varepsilon-3P)}{(\varepsilon+P)}. \tag{58}$$

In the result of the string theory, this ratio is given by [27]

$$\frac{\zeta}{\tau_\Pi(\varepsilon+P)} = \frac{\left(1/3-c_s^2\right)}{2-\ln 2}. \tag{59}$$

In the 14 moment approximation, this ratio is calculated by using the function β_0 which is defined by Eq. (7.8c) of Ref. [6].

The temperature dependence of the ζ/τ_Π ratio for pions is shown in Fig. 2. The solid, dashed, and dot-dashed lines represent our microscopic formula (58), the 14 moment approximation, and the string theory, respectively. The string

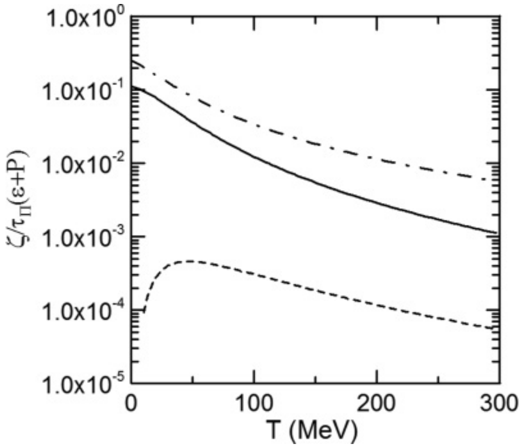

FIG. 2. Temperature dependence of the ζ/τ_Π ratio. The dot-dashed, solid, and dashed lines represents the results of string theory, our formula (58), and the 14 moment approximation, respectively.

theory predicts the largest value of the ratio, whereas the 14 moment approximation estimates the smallest value. At high temperature, the three lines are monotonically decreasing functions of temperature. The qualitative difference is observed at low temperature. The ratios of our formula and the string theory are still finite at $T = 0$ but that of Grad's method vanishes. The meaning of this difference is related to quantum fluctuation which is not included in the Boltzmann equation. This result is reported in another paper comparing our result with a new kinetic calculation based on the Boltzmann equation [23,26].

It is worth mentioning that this ratio is closely related to the propagation velocity of signals in CDR, which is given by [3,8]

$$v_g = \sqrt{c_s^2 + \frac{\zeta}{\tau_\Pi(\varepsilon + P)}}. \qquad (60)$$

Here, we neglected the contribution from the shear stress tensor. As discussed in Refs. [3,4], for hydrodynamics being causal and stable, this group velocity should be smaller than the speed of light. Thus this ratio should satisfy the constraint

$$\frac{\zeta}{\tau_\Pi(\varepsilon + P)} \leqslant 1 - c_s^2. \qquad (61)$$

This is the so-called causality condition [3,4,7]. One can easily see from Fig. 2 that all three calculations satisfy the causality condition.

As mentioned before, we used Eq. (51) instead of Eq. (50) as the definition of the bulk viscous pressure. Then we pointed out that both definitions should give the same result. However, this is not trivial when there is a UV divergence. As a matter of fact, the ζ/τ_Π ratio calculated with Eq. (50) is different from the solid line in Fig. 2 when we adapt the simple subtraction of the vacuum term as renormalization. In this calculation, we believe the calculation with Eq. (51) is more reliable than that with Eq. (50) for the following three reasons: (1) When we use Eq. (50) and employ the simple renormalization, the calculated ζ/τ_Π violates the causality condition. (2) As pointed out in Ref. [42], if Eq. (50) is used, the perturbative calculation collapses because of divergence. This problem is solved by using Eq. (51). (3) We can show that the ζ/τ_Π calculated with Eq. (51) is consistent with the result from a new kinetic calculation, as discussed in Ref. [23].

V. OTHER POSSIBLE GENERALIZATION

A. Another approximation to the memory function

In the derivation of Eq. (33), we replaced the time-convolution integral of the memory term with the time-convolutionless integral by assuming that the macroscopic time scale is clearly separated from the microscopic one. On the other hand, this coarse-graining may be formulated as the following expansion of the memory term:

$$\int_0^t d\tau\, \Xi_{22}(k_x,\tau)\Pi(k_x,t-\tau) = \int_0^t d\tau\, \Xi_{22}(k_x,\tau)\left[\Pi(k_x,t) - \tau\frac{\partial}{\partial t}\Pi(k_x,t) + \cdots\right]. \qquad (62)$$

When the higher order correction becomes important, the evolution equation of the bulk viscous pressure is modified as

$$\partial_t\Pi(k_x,t) \approx -ik_x R^B_{k_x}(\varepsilon+P)u^x(k_x,t) - \int_0^\infty d\tau\,\Xi_{22}(k_x,\tau) \times\left[\Pi(k_x,t) - \tau\frac{\partial}{\partial t}\Pi(k_x,t)\right] \longrightarrow \tilde{\tau}_\Pi\partial_t\Pi(k_x,t) + \Pi(k_x,t) = -ik_x\zeta u^x(k_x,t). \qquad (63)$$

Here, we replaced the upper limit of the integral by ∞, assuming that the dominant contribution of the memory function still comes from $\tau = 0$. The expression of ζ is not changed by this correction. However, the relaxation time is modified by $\tilde{\tau}_\Pi$ which is given by

$$\tilde{\tau}_\Pi = \tau_\Pi\left(1 + \lim_{s,\mathbf{k}\to 0}\frac{\partial\,\Xi^L(\mathbf{k},s)}{\partial s}\right). \qquad (64)$$

This can be expressed in terms of the retarded Green's function as

$$\tilde{\tau_\Pi} = \tau_\Pi\frac{\left[\lim_{\omega,\mathbf{k}\to 0}\partial^2 G^R_\Pi(\omega,\mathbf{k})/\partial\omega^2\right]\left[\lim_{\mathbf{k},\omega\to 0}G^R_\Pi(\omega,\mathbf{k})\right]}{2\left[\lim_{\omega,\mathbf{k}\to 0}\partial G^R_\Pi(\omega,\mathbf{k})/\partial\omega\right]^2}. \qquad (65)$$

For example, let us assume an exponential form for the memory function,

$$\Xi_{22}(\mathbf{0},t) = A\Gamma e^{-\Gamma t}, \qquad (66)$$

where A and Γ are parameters. Then we obtain that

$$\tilde{\tau_\Pi} = \tau_\Pi\left(1 - \frac{A}{\Gamma}\right). \qquad (67)$$

The result with the TCL approximation is reproduced in the infinite Γ limit.

However, for the following reasons, we consider that τ_Π is more reliable than $\tilde{\tau}_\Pi$. Let us consider the equation

$$\partial_t J(t) = F(t) + \int_0^t d\tau\,\Xi(t-\tau)J(\tau). \qquad (68)$$

From the initial and final value theorem of the Laplace transform, we can calculate the initial and final values of $J(t)$ as follows:

$$\lim_{t\to 0} J(t) = \lim_{s\to\infty} sJ^L(s) = \lim_{s\to\infty} s\frac{J(t=0) - F^L(s)}{s - \Xi^L(s)}$$
$$= J(t=0) - F^L(\infty), \tag{69}$$
$$\lim_{t\to\infty} J(t) = \lim_{s\to 0} sJ^L(s) = 0. \tag{70}$$

Here we assumed that $\Xi^L(0)$ is finite because of the existence of the finite relaxation time. Next, we approximate the time-convolution integral of the above equation by using the Taylor expansion,

$$\partial_t J(t) = F(t) + AJ(t) + B\partial_t J(t). \tag{71}$$

Then the initial and final values are

$$\lim_{t\to 0} J(t) = \frac{J(t=0) - BJ^L(t=0) - F^L(\infty)}{1-B}, \tag{72}$$
$$\lim_{t\to\infty} J(t) = \lim_{s\to 0} sJ^L(s) = 0. \tag{73}$$

To reproduce Eq. (69), we have to set $B = 0$.

As another reason, we consider the calculation of the transport coefficients of the diffusion equation. As shown by Kadanoff and Martin [47] and later by one of the present authors [48], the ratio of the diffusion coefficient D and the corresponding relaxation time τ_D is exactly determined from a sum rule. See Eqs. (B.10), (B.16), and (B.28) of Ref. [48]. Then we obtain

$$\frac{D}{\tau_D} = \frac{\int d^3\mathbf{x}(\mathbf{J}(\mathbf{x}), \mathbf{J}(\mathbf{0}))}{\int d^3\mathbf{x}(n(\mathbf{x}), n(\mathbf{0}))}, \tag{74}$$

where n is the conserved number density and $\partial_t n + \nabla \mathbf{J} = 0$. In the projection operator method, this result is reproduced only when the TCL approximation is applied.

That is, if we regard the TCL approximation as the lowest order of the Taylor expansion and the next-order correction is considered, we obtain a result which is inconsistent with the initial value theorem and the sum rule.

When we observe with the time scale where the structure of the memory function in Eq. (68) cannot be neglected, it means that there still exist macroscopic degrees of freedom in the memory function. To perform the program of coarse-graining in the projection operator method, we have to redefine the projection operator so as to extract this remaining macroscopic degree of freedom. Then the form of hydrodynamics itself is changed and the obtained equation is not given by Eq. (15) anymore. In other words, once we assume that the hydrodynamic equation is given by Eq. (15), we must observe with the time scale where the structure of the memory function is negligible, and hence, we must use the TCL approximation instead of the Taylor expansion.

As an example of the time dependence of the memory function, see Fig. 4 in Ref. [36]. One can see that the memory function has finite values only around $t = 0$.

B. Nonlinear terms

In this work, we used the phenomenological equation (15) as the definition of the bulk viscosity ζ and the relaxation time τ_Π. On the other hand, it is possible to introduce nonlinear terms to the phenomenological equation. For example, it might be possible to derive the hydrodynamic equation for the rarefied gas from the Boltzmann equation, by using Grad's moment method with the 14 moment approximation. In particular, the equation of the bulk viscous pressure is given by [24–26]

$$\tau_\Pi \frac{d}{d\tau}\Pi + \Pi = -\zeta\partial_\mu u^\mu + a_1\Pi\partial_\mu u^\mu + a_2\nu^\mu\frac{du_\mu}{d\tau} + a_3\partial_\mu\nu^\mu + a_4\nu^\mu\Delta_{\mu\nu}\partial^\nu\alpha + a_5\pi_{\mu\nu}\Delta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta, \tag{75}$$

where

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \tag{76}$$
$$\Delta^{\mu\nu\alpha\beta} = \tfrac{1}{2}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\alpha}\Delta^{\nu\beta} - \tfrac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right). \tag{77}$$

To obtain the microscopic expressions of these a_i's in the projection operator method, the definition of the projection operator must be generalized so as to collect all gross variables that appear in this nonlinear equation. This will be presented in future work.

VI. CONCLUDING REMARKS

In this paper, we derived the microscopic formulas of the bulk viscosity ζ and the corresponding relaxation time τ_Π in causal dissipative relativistic fluid dynamics by using the projection operator method. Applying these formulas to the pionic fluid and calculating in the leading-order approximation in chiral perturbation theory, we found that the relaxation time is enhanced around the temperature near the QCD phase transition, and there is a simple relation $\tau_\Pi = \zeta/[\beta\{(1/3 - c_s^2)(\varepsilon + P) - 2(\varepsilon - 3P)/9\}]$ between ζ and τ_Π. We compared our result with the results of Grad's moment method with the 14 moment approximation and the string theory by calculating the ζ/τ_Π ratio which is independent of the choice of the collision term of the Boltzmann equation. The ratio must be smaller than $(1 - c_s^2)(\varepsilon + P)$ to satisfy the causality condition. We confirmed that all three approaches are consistent with the causality condition. Finally, we discussed that the time-convolutionless approximation, which is used to derive the transport coefficients, is consistent with exact results, and we should not consider corrections to this approximation.

It should be emphasized that Grad's moment method with the 14 moment approximation is not a unique method for calculating the transport coefficients of CDR from the Boltzmann equation. Recently, a new calculation method based on the Boltzmann equation was developed, and it was found that the calculated transport coefficients are different from those of the 14 moment approximation [26]. These new results are completely consistent with the leading-order results of our formulas. This result was reported in another paper [23].

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APPENDIX: RELATION BETWEEN KUBO'S CANONICAL CORRELATION AND RETARDED GREEN'S FUNCTION

In this Appendix, we give a relation connecting Kubo's canonical correlation and the retarded Green's function. This relation is used to obtain Eqs. (42) and (44). From the definition of Kubo's canonical correlation, we can transform it as

$$
\begin{aligned}
&(\Pi(t,\mathbf{x}),\Pi(t',\mathbf{x}'))\\
&\equiv \int_0^\beta \frac{d\lambda}{\beta}\mathrm{Tr}[\rho_{\mathrm{eq}}e^{\lambda H}\Pi(t,\mathbf{x})e^{-\lambda H}\Pi(t',\mathbf{x}')]\\
&= \int_0^\beta \frac{d\lambda}{\beta}\langle \Pi(t-i\lambda,\mathbf{x})\Pi(t',\mathbf{x}')\rangle_{\mathrm{eq}}\\
&= -\int_0^\beta \frac{d\lambda}{\beta}\int_0^\infty ds\langle \Pi(t-t'-i\lambda,\mathbf{x})\frac{d}{ds}\Pi(s,\mathbf{x}')\rangle_{\mathrm{eq}}\\
&= i\int_0^\infty ds\int_0^\beta \frac{d\lambda}{\beta}\frac{d}{d\lambda}\langle \Pi(t-t'-i\lambda,\mathbf{x})\Pi(s,\mathbf{x}')\rangle_{\mathrm{eq}}\\
&= i\int_0^\infty \frac{ds}{\beta}\langle[\Pi(s,\mathbf{x}'),\Pi(t-t',\mathbf{x})]\rangle_{\mathrm{eq}}\\
&= i\int_0^\infty \frac{ds}{\beta}\langle[\Pi(s,\mathbf{x}'),\Pi(t-t',\mathbf{x})]\rangle_{\mathrm{eq}}. \qquad (A1)
\end{aligned}
$$

Furthermore, when $t = t'$, we have

$$
\begin{aligned}
(\Pi(0,\mathbf{x}),\Pi(0,\mathbf{x}')) &= i\int_{-\infty}^\infty \frac{ds}{\beta}\theta(s)\langle[\Pi(s,\mathbf{x}'),\Pi(0,\mathbf{x})]\rangle_{\mathrm{eq}}\\
&= -\int_{-\infty}^\infty \frac{ds}{\beta}G^R_\Pi(s,\mathbf{x}'-\mathbf{x})\\
&= -\frac{1}{\beta}\lim_{\omega\to 0}G^R_\Pi(\omega,\mathbf{x}'-\mathbf{x}). \qquad (A2)
\end{aligned}
$$

Then we obtain

$$
\int d^3\mathbf{x}(\Pi(0,\mathbf{x}),\Pi(0,\mathbf{0})) = -\frac{1}{\beta}\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\omega\to 0}G^R_\Pi(\omega,\mathbf{k}). \qquad (A3)
$$

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