Polarization observables in $\pi^0\eta$ photoproduction on the proton

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(Received 21 October 2010; published 25 January 2011)

For $\pi^0\eta$ photoproduction on the nucleon formal expressions are developed for the fivefold differential cross section and the recoil polarization including beam and target polarizations. The polarization observables are described by various beam, target, and beam-target asymmetries for polarized photons and/or polarized nucleons. They are given as bilinear Hermitian forms in the reaction matrix elements divided by the unpolarized cross section. Numerical results for the linear and circular beam asymmetries for $\gamma p \to \pi^0\eta p$ are obtained within an isobar model and are compared with existing data. Predictions are also given for the target asymmetry T_{11}, and the beam-target asymmetries T^c_{10} and T^c_{11} for circularly polarized photons.

DOI: 10.1103/PhysRevC.83.015503 PACS number(s): 13.60.Le, 13.75.−n, 24.70.+s, 25.20.Lj

I. INTRODUCTION

Polarization observables are known to be an essential ingredient in the interpretation of photon-induced meson production reactions, especially if the production process proceeds predominantly via resonance excitations. Their study provides further insight into the details of the underlying reaction mechanisms and possible structure effects. Thus, such observables will serve as an additional critical test for theoretical models.

Today special interest is focused to processes with more than a single pseudoscalar meson in the final state. These reactions constitute a rather new object in particle physics. At present most of the efforts are directed toward an understanding of their general dynamical content. In such a situation, experiments with polarized particles are therefore of special use. Different analyses clearly demonstrate their importance, primarily because the unpolarized data are usually unable to impose sufficient constraints on the model parameters.

Experiments for $\pi\pi$ and $\pi^0\eta$ photoproduction have become a center of attention in recent research programs discussed at the European Laboratory for Structural Assessment (ELSA) and Mainz Microtron (MAMI) facilities [1–5]. A major point of these programs is a study of those resonances for which only a weak evidence exists. It is therefore timely to investigate in detail the polarization structure of double meson photoproduction. Some important steps toward this goal have already been done in Ref. [6], where, in particular, a set of polarization experiments, needed to determine the reaction amplitude, is discussed.

With the present work we want to provide a complete solid basis for the formal expressions of all possible polarization observables that determine the general differential cross section and the proton recoil polarization for $\pi^0\eta$ production on a polarized proton target with polarized photons in a compact and suggestive notation.

Our second goal is to study the properties of those observables for which experimental results already exist or are expected to be measured in the near future. Recently, polarization measurements of different beam asymmetries in $\pi^0\eta$ photoproduction were performed at ELSA [7,8] for the first time. Furthermore, new MAMI results for the target asymmetry T^0_{11} and the beam-target asymmetry T^c_{11} are now expected. Here, we pay some attention to the properties of the circular beam asymmetry measured at MAMI [9]. The analysis of this observable for the similar reaction, $\pi\pi$ photoproduction [10–12], confirms a strong sensitivity of the data to the dynamical content of the amplitude. As will be shown in the present paper, the information contained in the circular asymmetry provides constraints on the contribution of positive parity resonances to $\pi^0\eta$ photoproduction.

The paper is organized as follows. In the next three sections we develop the basic formalism for the differential cross section with inclusion of polarization observables. In Sec. V, the most essential ingredients for the calculation of the T matrix in the isobar model are described. Here we also present and discuss the results on some beam, target, and beam-target asymmetries, which are also compared to the existing data. In several appendixes we describe in detail some ingredients of our formal developments. One should note that throughout this paper π meson always means π^0 meson.

II. KINEMATICS

As a starting point, we first will consider the kinematics of the photoproduction reaction

$$\gamma(k,\vec{\varepsilon}_\mu) + N_i(p_i) \to \pi(q_\pi) + \eta(q_\eta) + N_f(p_f), \qquad (1)$$

defining the notation of the four-momenta of the participating particles

$$k = (\omega_\gamma, \vec{k}), \quad p_i = (E_i, \vec{p}_i), \quad q_\pi = (\omega_\pi, \vec{q}_\pi),$$
$$q_\eta = (\omega_\eta, \vec{q}_\eta), \quad p_f = (E_f, \vec{p}_f). \qquad (2)$$

As a coordinate system we choose a right-handed one with the z axis along the photon momentum $\vec{k}$ and the other axis perpendicular. As is illustrated in Fig. 1 for the laboratory frame, we distinguish three planes:

(i) The reaction plane, spanned by the momenta of incoming photon $\vec{k}$ and $\vec{q}_1$ of particle "1," called the active particle, which usually is detected. This plane intersects the x-z plane along the z axis with an angle ϕ_1.

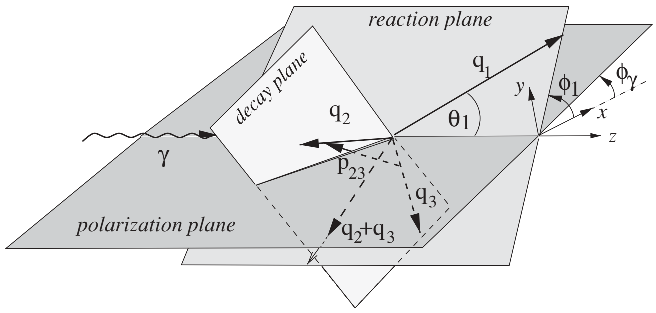

FIG. 1. Kinematics of $\pi\eta$ photoproduction on the nucleon in the laboratory system. The active particle is denoted by "1" and defines the reaction plane. The nonrelativistic relative momentum is denoted by $\vec{p}_{23} = (m_3\vec{q}_2 - m_2\vec{q}_3)/(m_2+m_3)$.

(ii) The polarization or photon plane, spanned by the photon momentum and the direction of maximal linear photon polarization, which intersects the x-z plane along the z axis with an angle ϕ_γ and the reaction plane along the z axis with an angle $|\phi_1 - \phi_\gamma|$.

(iii) The decay plane, spanned by the momenta of the other two outgoing particles "2" and "3," intersecting the reaction plane along the total momentum $\vec{p}_2 + \vec{p}_3$ of the latter two particles. In case that the linear photon polarization vanishes, one can choose $\phi_1 = \phi_\gamma = 0$ and then polarization and reaction planes coincide.

The following formal developments will not depend on whether one chooses as a reference frame the laboratory or the center-of-momentum (c.m.) frame. Furthermore, we will consider the η meson as particle "1" ($\vec{q}_1 := \vec{q}_\eta$) defining the reaction plane, while pion and proton constitute particles "2" and "3," respectively, in the decay plane, i.e., $\vec{q}_2 := \vec{q}_\pi$ and $\vec{q}_3 := \vec{p}_f$. Besides the incoming photon momentum $\vec{k}$, we choose as independent variables for the description of cross section and polarization observables the angle ϕ_γ characterizing the polarization plane, the outgoing η momentum $\vec{q}_\eta = (q_\eta, \theta_\eta, \phi_\eta)$, and the spherical angles $\Omega_{\pi p} = (\theta_{\pi p}, \phi_{\pi p})$ of the relative momentum $\vec{p}_{\pi p}$ of the outgoing pion and nucleon as given by

$$\vec{p}_{\pi p} = (M_p\vec{q}_\pi - m_\pi \vec{p}_f)/(M_p + m_\pi) = (p_{\pi p}, \Omega_{\pi p}). \quad (3)$$

Then the momenta of outgoing pion and nucleon are fixed. For example, the pion momentum reads

$$\vec{q}_\pi = \vec{p}_{\pi p} + \frac{m_\pi}{M_p + m_\pi}(\vec{k} + \vec{p}_i - \vec{q}_\eta). \quad (4)$$

In Sec. V we will also consider configurations where either the outgoing pion or proton is the active particle, i.e., constituting the reaction plane, while the decay plane is spanned by the momenta of the other two particles in the final state, i.e., either eta and proton or pion and eta, respectively.

III. THE T MATRIX

To be specific, we take in this section the outgoing eta as the active particle. The corresponding expressions for the pion as the active particle are obtained simply by the interchange $\eta \leftrightarrow \pi$. For the outgoing proton as the active particle, detailed expressions are listed in Appendix D.

All observables are determined by the T-matrix elements of the electromagnetic $\pi\eta$ production current $\vec{J}_{\gamma\pi\eta}$ between the initial proton and the final outgoing $\pi\eta N_f$ scattering states [indicated by a superscript "$(-)$"]. In a general frame, it is given by

$$T_{m_f \mu m_i} = -^{(-)}\langle \vec{q}_\eta,\, \vec{q}_\pi;\, \vec{p}_f\, m_f \,|\, \vec{\varepsilon}_\mu \cdot \vec{J}_{\gamma\pi\eta}(0)|\vec{p}_i\, m_i\rangle, \quad (5)$$

where m_f denotes the proton spin projection on the relative momentum $\vec{p}_{\pi p}$ of the outgoing pion and proton, and m_i denotes correspondingly the initial proton spin projection on the z axis as a quantization axis. The circular polarization vector of the photon is denoted by $\vec{\varepsilon}_\mu$ with $\mu = \pm 1$. Furthermore, a transverse gauge has been chosen. The knowledge of the specific form of $\vec{J}_{\gamma\pi\eta}$ is not needed for the following formal considerations.

The general form of the T matrix after separation of the overall c.m. motion and insertion of the multipole expansion of the current operator is given in terms of the relative πp momentum and the η momentum by

$$\begin{aligned} T_{m_f\mu m_i}(\vec{p}_{\pi p}, \vec{q}_\eta) &= -^{(-)}\langle \vec{p}_{\pi p}\, m_f;\, \vec{q}_\eta\, |J_{\gamma\pi\eta,\mu}(\vec{k})|m_i\rangle \\ &= \sqrt{2\pi}\sum_L i^L \widehat{L}^{(-)}\langle \vec{p}_{\pi p}\, m_f;\, \vec{q}_\eta\, |O^{\mu L}_\mu|m_i\rangle, \end{aligned} \quad (6)$$

with $\mu = \pm 1$, $\widehat{L} = \sqrt{2L+1}$, and transverse multipoles

$$O^{\mu L}_M = E^L_M + \mu M^L_M. \quad (7)$$

It is convenient to introduce a partial-wave decomposition of the final outgoing scattering state

$$\begin{aligned} ^{(-)}\langle \vec{p}_{\pi p} m_f| = \frac{1}{\sqrt{4\pi}} \sum_{l_{\pi p} j_{\pi p} m_{\pi p}} \widehat{l}_{\pi p} \left(l_{\pi p} 0 \frac{1}{2} m_f | j_{\pi p} m_f \right) \\ \times D^{j_{\pi p}}_{m_f, m_{\pi p}}(\phi_{\pi p}, -\theta_{\pi p}, -\phi_{\pi p}) \\ \times\, ^{(-)}\left\langle p_{\pi p} \left(l_{\pi p} \frac{1}{2} \right) j_{\pi p} m_{\pi p} \right|, \end{aligned} \quad (8)$$

$$^{(-)}\langle \vec{q}_\eta| = \frac{1}{\sqrt{4\pi}} \sum_{l_\eta m_\eta} \widehat{l}_\eta D^{l_\eta}_{0,m_\eta}(\phi_\eta, -\theta_\eta, -\phi_\eta)^{(-)}\langle q_\eta l_\eta m_\eta|, \quad (9)$$

where $m_{\pi p}$ and m_η such as m_i refer to the photon momentum $\vec{k}$ as quantization axis. Here, the rotation matrices $D^j_{m'm}$ are taken in the convention of Rose [13]. Using the Wigner-Eckart theorem, one obtains

$$\begin{aligned} &^{(-)}\left\langle p_{\pi p} \left(l_{\pi p} \frac{1}{2} \right) j_{\pi p} m_{\pi p}; q_\eta l_\eta m_\eta \right| O^{\mu L}_\mu \left| \frac{1}{2} m_i \right\rangle \\ &= \sum_{J M_J} (-1)^{j_{\pi p} - l_\eta + J} \widehat{J} \begin{pmatrix} j_{\pi p} & l_\eta & J \\ m_{\pi p} & m_\eta & -M_J \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ -M_J & \mu & m_i \end{pmatrix} \\ &\quad \times \left\langle p_{\pi p}\, q_\eta; \left[\left(l_{\pi p} \frac{1}{2} \right) j_{\pi p} l_\eta \right] J \,\|\, O^{\mu L} \,\|\, \frac{1}{2} \right\rangle, \end{aligned} \quad (10)$$

with the selection rule $m_{\pi p} + m_\eta = M_J = \mu + m_i$. The angular dependence can be rewritten according to

$$D^{j_{\pi p}}_{m_f, m_{\pi p}}(\phi_{\pi p}, -\theta_{\pi p}, -\phi_{\pi p}) D^{l_\eta}_{0,m_\eta}(\phi_\eta, -\theta_\eta, -\phi_\eta) = d^{j_{\pi p}}_{m_f, m_{\pi p}}(-\theta_{\pi p}) d^{l_\eta}_{0,m_\eta}(-\theta_\eta) e^{i[(m_{\pi p}-m_f)\phi_{\pi p}+m_\eta\phi_\eta]}, \quad (11)$$

where $d^j_{mm'}$ denotes a small rotation matrix [13]. Rearranging

$$(m_{\pi p} - m_f)\phi_{\pi p} + m_\eta \phi_\eta = (m_{\pi p} - m_f)\phi_{pq} + (\mu + m_i - m_f)\phi_\eta \quad (12)$$

with $\phi_{pq} = \phi_{\pi p} - \phi_\eta$, one finds that the dependence on ϕ_η can be separated, i.e.,

$$T_{m_f \mu m_i}(\Omega_{\pi p}, \Omega_\eta) = e^{i(\mu + m_i - m_f)\phi_\eta} t_{m_f\mu m_i}(\theta_{\pi p}, \theta_\eta, \phi_{pq}), \quad (13)$$

where the small t matrix depends only on $\theta_{\pi p}$, θ_η, and the relative azimuthal angle ϕ_{pq}.

Explicitly, one obtains

$$\begin{aligned} t_{m_f\mu m_i}&(\theta_{\pi p}, \theta_\eta, \phi_{pq}) \\ &= \frac{1}{2\sqrt{2\pi}} \sum_{L l_{\pi p} j_{\pi p} m_{\pi p} l_\eta m_\eta J M_J} i^L \widehat{L}\,\widehat{J}\,\widehat{l_\eta}\,\widehat{l_{\pi p}}\,\widehat{j_{\pi p}} \\ &\times (-1)^{J + l_{\pi p} + j_{\pi p} - \frac{1}{2} + m_f - l_\eta} \begin{pmatrix} l_{\pi p} & \frac{1}{2} & j_{\pi p} \\ 0 & m_f & -m_f \end{pmatrix} \\ &\times \begin{pmatrix} j_{\pi p} & l_\eta & J \\ m_{\pi p} & m_\eta & -M_J \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ -M_J & \mu & m_i \end{pmatrix} \\ &\times \left\langle p_{\pi p}\, q_\eta; \left[\left(l_{\pi p}\frac{1}{2}\right) j_{\pi p} l_\eta\right] J \| O^{\mu L} \| \frac{1}{2}\right\rangle \\ &\times d^{j_{\pi p}}_{m_f, m_{\pi p}}(-\theta_{\pi p})\, d^{l_\eta}_{0,m_\eta}(-\theta_\eta)\, e^{i(m_{\pi p} - m_f)\phi_{pq}}. \end{aligned} \quad (14)$$

In the case that parity is conserved, it is quite straightforward to show that the following symmetry relation holds for the inverted spin projections of the small t-matrix elements:

$$t_{-m_f -\mu -m_i}(\theta_{\pi p}, \theta_\eta, \phi_{pq}) = (-1)^{-m_f + \mu + m_i} t_{m_f \mu m_i}(\theta_{\pi p}, \theta_\eta, -\phi_{pq}). \quad (15)$$

Besides the phase factor, one should note the sign change of ϕ_{pq} on the right-hand side. In the derivation of this relation, one has made use of the parity selection rules for the multipole transitions to a final partial wave $|pq[(l_{\pi p}s) j_{\pi p} l_\eta]J\rangle$ with parity $\pi_{J(l_{\pi p}, l_\eta)} = (-1)^{l_{\pi p} + l_\eta}$, which read

$$\begin{aligned} E^L &: \pi_i \pi_{J(l_{\pi p}, l_\eta)} (-1)^L = 1 \to (-1)^{l_{\pi p} + l_\eta + L} = 1, \\ M^L &: \pi_i \pi_{J(l_{\pi p}, l_\eta)} (-1)^L = -1 \to (-1)^{l_{\pi p} + l_\eta + L} = -1. \end{aligned} \quad (16)$$

Therefore, invariance under a parity transformation results in the following property of the reduced matrix element:

$$(-1)^{l_{\pi p} + l_\eta + L} \langle p_{\pi p} q_\eta; \left[\left(l_{\pi p}\tfrac{1}{2}\right) j_{\pi p} l_\eta\right] J \| O^{-\mu L} \| \tfrac{1}{2}\rangle = \langle p_{\pi p} q_\eta; \left[\left(l_{\pi p}\tfrac{1}{2}\right) j_{\pi p} l_\eta\right] J \| O^{\mu L} \| \tfrac{1}{2}\rangle. \quad (17)$$

A corresponding relation for the T-matrix elements follows from the symmetry property (15):

$$T_{-m_f -\mu -m_i}(\theta_{\pi p}, \phi_{\pi p}, \theta_\eta, \phi_\eta) = (-1)^{-m_f + \mu + m_i} T_{m_f \mu m_i}(\theta_{\pi p}, -\phi_{\pi p}, \theta_\eta, -\phi_\eta). \quad (18)$$

These symmetry properties are valid for all three choices of the active particle.

Because the small t-matrix elements are the basic quantities that determine the general differential cross section and the recoil polarization in terms of bilinear hermitean forms in the t-matrix elements, the developments of the next section are independent of which particle is chosen as active.

IV. DIFFERENTIAL CROSS SECTION AND RECOIL POLARIZATION

The starting point for the formal derivation of polarization observables is the evaluation of the following trace with respect to the spin degrees of freedom:

$$A_{I'M'} = c(q_\eta, \theta_\eta, \Omega_{\pi p})\, \mathrm{tr}\big(T^\dagger \tau^{f,[I']}_{M'} T \rho_i\big), \quad (19)$$

for $I' = 0, 1$ and $M' = -I', \ldots, I'$, folded between the density matrix ρ_i for the spin degrees of the initial system and a spin operator $\tau^{f,[I']}_{M'}$ with respect to the final nucleon spin space. The latter is defined by its reduced matrix element

$$\left\langle \tfrac{1}{2} \| \tau^{[I']} \| \tfrac{1}{2} \right\rangle = \sqrt{2}\widehat{I'} \quad \text{for} \quad I' = 0, 1. \quad (20)$$

Note that $\tau^{[1]}$ corresponds to the conventional Pauli spin operator $\vec{\sigma}$. The trace refers to all initial and final state spin degrees of freedom of the incoming photon, target, and recoiling nucleon. The kinematic factor $c(q_\eta, \theta_\eta, \Omega_{\pi p})$ comprises the final-state phase space and the incoming flux. In an arbitrary frame one has

$$\begin{aligned} c(q_\eta, \Omega_q, \Omega_{\pi p}) &= \frac{1}{(2\pi)^5} \frac{M_p^2}{E_i + p_i} \frac{1}{8\omega_\gamma \omega_\eta} \\ &\times \frac{p_{\pi p}^2}{p_{\pi p}(\omega_\pi + E_f) + \frac{(\vec{q}_\pi + \vec{p}_f)\cdot \vec{p}_{\pi p}}{p_{\pi p}(M_p + m_\pi)}(E_f m_\pi - \omega_\pi M_p)}. \end{aligned} \quad (21)$$

The general expression for the differential cross section is given by

$$\frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} = A_{00}, \quad (22)$$

and the final nucleon polarization component P_M with respect to a spherical basis,

$$P_M \frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} = (-1)^M A_{1-M}. \quad (23)$$

With respect to a Cartesian basis, one has as polarization components

$$\begin{aligned} P_x \frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} &= \frac{1}{\sqrt{2}} B_1^-, \quad P_y \frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} = \frac{i}{\sqrt{2}} B_1^+, \\ P_z \frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} &= B_0^+, \end{aligned} \quad (24)$$

where for $M = 0, 1$ we have introduced

$$B_M^\pm = \frac{(-1)^M}{1 + \delta_{M0}} (A_{1M} \pm A_{1-M}). \quad (25)$$

The density matrix ρ_i in (19) is a direct product of the density matrices ρ^γ of the photon and ρ^p of the nucleon

$$\rho_i = \rho^\gamma \otimes \rho^p. \tag{26}$$

For the chosen reference frame, the photon density matrix has the form

$$\rho^\gamma_{\mu\mu'} = \tfrac{1}{2}(\delta_{\mu\mu'} + \vec{P}^\gamma \cdot \vec{\sigma}_{\mu\mu'}) \tag{27}$$

with respect to the circular polarization basis ($\mu = \pm 1$). Here, $|\vec{P}^\gamma|$ describes the total degree of polarization, $P^\gamma_z = P^\gamma_c$ is the difference of right to left circularly polarized photons, i.e., $|P^\gamma_c|$ describes the degree of circular polarization being right or left according to whether $P^\gamma_c > 0$ or < 0, respectively, and $P^\gamma_l = \sqrt{(P^\gamma_x)^2 + (P^\gamma_y)^2}$ describes the degree of linear polarization. By a rotation around the photon momentum by an appropriate angle ϕ_γ, it is possible to have the new x' axis pointing in the direction of maximum linear polarization. Then one has $P^\gamma_{x'} = -P^\gamma_l$ and $P^\gamma_{y'} = 0$ and finds explicitly

$$\rho^\gamma_{\mu\mu'} = \tfrac{1}{2}\big[\big(1 + \mu\, P^\gamma_c\big)\delta_{\mu\mu'} - P^\gamma_l\, \delta_{\mu,-\mu'} e^{-2i\mu\phi_\gamma}\big]. \tag{28}$$

Furthermore, the nucleon density matrix ρ^p can be expressed in terms of irreducible spin operators $\tau^{[I]}$ ($I = 0, 1$) with respect to the initial nucleon spin space, defined in analogy to (20),

$$\rho^p_{m_i m_i'} = \frac{1}{2}\sum_{IM}(-1)^M \left\langle \tfrac{1}{2} m_i \middle| \tau^{[I]}_M \middle| \tfrac{1}{2} m_i' \right\rangle P^p_{I-M}, \tag{29}$$

where $P^p_{00} = 1$, and P^p_{1M} describes the sperical polarization components of the nucleon.

We can assume that the nucleon density matrix is diagonal with respect to an orientation axis $\vec{s}$ having spherical angles (θ_s, ϕ_s) with respect to the chosen coordinate system. Then one has with respect to $\vec{s}$ as a quantization axis,

$$\rho^p_{mm'} = p_m\, \delta_{mm'}, \tag{30}$$

where p_m denotes the probability for finding a nucleon spin projection m on the orientation axis. With respect to this axis, one finds from (29) $P^p_{IM}(\vec{s}) = P^p_I \delta_{M,0}$, where the orientation parameters P^p_I are related to the probabilities $\{p_m\}$ by

$$\begin{aligned} P^p_I &= \sqrt{2}\,\widehat{I}\sum_m (-1)^{\frac{1}{2}-m} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I \\ m & -m & 0 \end{pmatrix} p_m \\ &= \delta_{I0} + (p_{1/2} - p_{-1/2})\,\delta_{I1}. \end{aligned} \tag{31}$$

The polarization components in the chosen laboratory frame are obtained from the P^p_I by a rotation, transforming the quantization axis along the orientation axis into the direction of the photon momentum, i.e.,

$$P^p_{IM}(\vec{z}) = P^p_I\, e^{iM\phi_s} d^I_{M0}(\theta_s). \tag{32}$$

Thus the initial nucleon density matrix becomes finally

$$\rho^p_{m_i m_i'} = \frac{(-1)^{\frac{1}{2}-m_i}}{\sqrt{2}} \sum_{IM} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I \\ m_i' & -m_i & M \end{pmatrix} P^p_I\, e^{-iM\phi_s} d^I_{M0}(\theta_s). \tag{33}$$

This means the nucleon target is characterized by three parameters, namely, the polarization parameter P^p_1 and by the orientation angles θ_s and ϕ_s. If one chooses the c.m. frame as a reference frame, one should note that the nucleon density matrix undergoes no change in the transformation from the laboratory to the c.m. system, because the boost to the c.m. system is collinear with the nucleon quantization axis [14].

The evaluation of the general trace in Eq. (19) can be done analogously to pion photoproduction on the deuteron, as described in detail in Ref. [15]. In fact, one can follow the same steps except for the use of the symmetry relation of Eq. (12) in Ref. [15], which is different in the case of eta-pion production on the nucleon [see Eq. (15)] because of the two pseudoscalar particles in the final state. In terms of the small t-matrix elements, one finds, inserting the density matrices of photon and nucleon, for the general trace,

$$\begin{aligned} A_{I'M'} = \frac{1}{2} \sum_{\mu'\mu IM} P^p_I\, e^{iM\phi_{\eta s}} d^I_{M0}(\theta_s)\, u^{\mu'\mu}_{I'M';IM}\big[\big(1 + \mu\, P^\gamma_c\big)\delta_{\mu\mu'} \\ - P^\gamma_l\, \delta_{\mu,-\mu'} e^{2i\mu\phi_{\eta\gamma}}\big], \end{aligned} \tag{34}$$

with $\phi_{\eta s} = \phi_\eta - \phi_s$ and $\phi_{\eta\gamma} = \phi_\eta - \phi_\gamma$. Furthermore, we have introduced the quantities

$$\begin{aligned} &u^{\mu'\mu}_{I'M';IM}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}) \\ &\quad = c(q_\eta, \Omega_q, \Omega_{\pi p})\widehat{I'}\widehat{I} \sum_{m_f m_f' m_i m_i'} (-1)^{m_f' - m_i} \\ &\quad\times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I' \\ m_f & -m_f' & M' \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I \\ m_i' & -m_i & M \end{pmatrix} \\ &\quad\times t^*_{m_f'\mu' m_i'}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}) t_{m_f \mu m_i}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}). \end{aligned} \tag{35}$$

It is straightforward to prove that they behave under a complex conjugation as

$$\begin{aligned} &\big[u^{\mu'\mu}_{I'M';IM}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq})\big]^* \\ &\quad = (-1)^{M'+M} u^{\mu\mu'}_{I'-M';I-M}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}). \end{aligned} \tag{36}$$

Furthermore, with the help of the symmetry in (15), one finds

$$\begin{aligned} &u^{-\mu'-\mu}_{I'M';IM}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}) \\ &\quad = (-1)^{I'+M'+I+M+\mu'+\mu}\, u^{\mu'\mu}_{I'-M';I-M}(q_\eta, \theta_\eta, \theta_{\pi p}, -\phi_{pq}), \end{aligned} \tag{37}$$

which yields in combination with (36)

$$\begin{aligned} &u^{-\mu'-\mu}_{I'M';IM}(q_\eta, \theta_\eta, \theta_{\pi p}, \phi_{pq}) \\ &\quad = (-1)^{I'+I+\mu'+\mu}\big[u^{\mu\mu'}_{I'M';IM}(q_\eta, \theta_\eta, \theta_{\pi p}, -\phi_{pq})\big]^*. \end{aligned} \tag{38}$$

This relation is quite useful for a further simplification of the semiexclusive differential cross section later on.

Separating the polarization parameters of the photon (P^γ_l and P^γ_c) and the target nucleon (P^p_I), it is then straightforward to show that the trace can be brought into the form

$$\begin{aligned} A_{I'M'} = \frac{1}{2} \sum_{I=0,1} P^p_I \sum_{M=-I}^{I} e^{iM\phi_{\eta s}} d^I_{M0}(\theta_s)\big[v^1_{I'M';IM} + v^{-1}_{I'M';IM} \\ + P^\gamma_c\big(v^1_{I'M';IM} - v^{-1}_{I'M';IM}\big) \\ + P^\gamma_l\big(w^1_{I'M';IM} e^{2i\phi_{\eta\gamma}} + w^{-1}_{I'M';IM} e^{-2i\phi_{\eta\gamma}}\big)\big], \end{aligned} \tag{39}$$

where we have introduced for convenience the quantities

$$v^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = u^{\mu\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad (40)$$

$$w^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = -u^{\mu-\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}). \quad (41)$$

According to Eqs. (36) and (38), they have the following properties under complex conjugation:

$$v^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})^* = (-1)^{M'+M} v^{\mu}_{I'-M';I-M}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad (42)$$

$$w^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})^* = (-1)^{M'+M} w^{-\mu}_{I'-M';I-M}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad (43)$$

$$v^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})^* = (-1)^{I'+I} v^{-\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{pq}), \quad (44)$$

$$w^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})^* = (-1)^{I'+I} w^{\mu}_{I'M';IM}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{pq}). \quad (45)$$

From Eq. (42), it follows in particular that $v^{\mu}_{I'0;I0}$ is real.

A. The differential cross section

For the differential cross section we consider the case $I'=0$ and $M'=0$, i.e., A_{00}, for which we will use the following simplified notation:

$$v^{\mu}_{IM} = v^{\mu}_{00;IM}, \quad (46)$$

$$w^{\mu}_{IM} = w^{\mu}_{00;IM}. \quad (47)$$

The sum over M in Eq. (39) can be rearranged with the help of the relations in Eqs. (42) and (43) and $d^I_{-M0}(\theta_s) = (-1)^M d^I_{M0}(\theta_s)$:

$$\sum_{M=-I}^{I} e^{iM\phi_{\eta s}} d^I_{M0}(\theta_s)\big(v^1_{IM} \pm v^{-1}_{IM}\big) = \sum_{M=0}^{I} \frac{d^I_{M0}(\theta_s)}{1+\delta_{M0}}\big[e^{iM\phi_{\eta s}}\big(v^1_{IM} \pm v^{-1}_{IM}\big) + e^{-iM\phi_{\eta s}}(-1)^M\big(v^1_{I-M} \pm v^{-1}_{I-M}\big)\big] = \sum_{M=0}^{I} \frac{d^I_{M0}(\theta_s)}{1+\delta_{M0}}\big[e^{iM\phi_{\gamma s}}\big(v^1_{IM} \pm v^{-1}_{IM}\big) + \text{c.c.}\big], \quad (48)$$

and furthermore with

$$\psi_M = M\phi_{\eta s} - 2\phi_{\eta\gamma} = (M-2)\phi_\eta - M\phi_s + 2\phi_\gamma, \quad (49)$$

we get

$$\sum_{M=-I}^{I} e^{iM\phi_{\eta s}} d^I_{M0}(\theta_s)\big(w^1_{IM}e^{-2i\phi_{\eta\gamma}} + w^{-1}_{IM}e^{2i\phi_{\eta\gamma}}\big) = \sum_{M=-I}^{I} d^I_{M0}(\theta_s)\big[e^{i\psi_M}w^1_{IM} + e^{-i\psi_M}(-1)^M w^{-1}_{I-M}\big] = \sum_{M=-I}^{I} d^I_{M0}(\theta_s)\big(e^{i\psi_M}w^1_{IM} + \text{c.c.}\big). \quad (50)$$

This then yields for the differential cross section

$$\frac{d\sigma}{d\vec q_\eta d\Omega_{\pi p}} = \sum_{I=0,1} P^p_I \Bigg\{ \sum_{M=0}^{I} \frac{1}{1+\delta_{M0}} d^I_{M0}(\theta_s)\text{Re}\big[e^{iM\phi_{\eta s}}\big(v^+_{IM} + P^\gamma_c v^-_{IM}\big)\big] + P^\gamma_l \sum_{M=-I}^{I} d^I_{M0}(\theta_s)\text{Re}\big(e^{i\psi_M}w^1_{IM}\big)\Bigg\}, \quad (51)$$

where we have defined

$$v^{\pm}_{IM} = v^1_{IM} \pm v^{-1}_{IM}. \quad (52)$$

Now, introducing various beam, target, and beam-target asymmetries by

$$\tau^{0/c}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = \frac{1}{1+\delta_{M0}}\text{Re}\, v^{\pm}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad M \geqslant 0, \quad (53)$$

$$\sigma^{0/c}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = -\text{Im}\, v^{\pm}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad M > 0, \quad (54)$$

$$\tau^{l}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = \text{Re}\, w^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad (55)$$

$$\sigma^{l}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = -\text{Im}\, w^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \quad (56)$$

where we took into account that v^{μ}_{I0} is real.

One obtains as a final expression for the general fivefold differential cross section, including beam and target polarization,

$$\begin{aligned}\frac{d\sigma}{d\vec q_\eta\, d\Omega_{\pi p}} &= \frac{d\sigma_0}{d\vec q_\eta\, d\Omega_{\pi p}}\big[1 + P^\gamma_c T^c_{00} + P^\gamma_l\big(T^l_{00}\cos 2\phi_{\eta\gamma} - S^l_{00}\sin 2\phi_{\eta\gamma}\big) \\ &+ P^p_1\big(\big[T^c_{10} + P^\gamma_c T^c_{10} + P^\gamma_l\big(T^l_{10}\cos 2\phi_{\eta\gamma} - S^l_{10}\sin 2\phi_{\eta\gamma}\big)\big]\cos\theta_s \\ &- \frac{1}{\sqrt{2}}\big\{T^0_{10}\cos\phi_{\eta s} + S^0_{10}\sin\phi_{\eta s} \\ &+ P^\gamma_c\big(T^c_{10}\cos\phi_{\eta s} + S^c_{10}\sin\phi_{\eta s}\big) + P^\gamma_l\big[T^l_{11}\cos(\phi_{\eta s} - 2\phi_{\eta\gamma}) \\ &- T^l_{1-1}\cos(\phi_{\eta s} + 2\phi_{\eta\gamma}) + S^l_{11}\sin(\phi_{\eta s} - 2\phi_{\eta\gamma}) \\ &+ S^l_{1-1}\sin(\phi_{\eta s} + 2\phi_{\eta\gamma})\big]\big\}\sin\theta_s\big)\big], \end{aligned} \quad (57)$$

where the unpolarized differential cross section is given by

$$\frac{d\sigma_0}{d\vec q_\eta\, d\Omega_{\pi p}} = \tau^0_{00}, \quad (58)$$

and the various beam, target, and beam-target asymmetries

$$T^\alpha_{IM} = \frac{\tau^\alpha_{IM}}{\tau^0_{00}}, \quad S^\alpha_{IM} = \frac{\sigma^\alpha_{IM}}{\tau^0_{00}}, \quad \text{for} \quad \alpha \in \{0, c, l\}. \quad (59)$$

The corresponding derivation of the recoil polarization of the outgoing nucleon is presented in Appendix B.

B. The semiexclusive differential cross section $\vec p(\vec\gamma,\eta)\pi p$

We will now turn to semiexclusive reactions where one has to integrate over all variables that are not measured. As an example, we consider the case $\vec{p}(\vec{\gamma},\eta)\pi N$, where only the produced eta is detected. This means integration of the fivefold differential cross section $d\sigma/d\vec{q}_\eta d\Omega_{\pi p}$ over $\Omega_{\pi p}$. The derivation of the resulting cross section is presented in detail in Appendix A. The cross section is governed by the partially integrated asymmetries $\int d\Omega_{\pi p}\tau^\alpha_{IM}$ and $\int d\Omega_{\pi p}\sigma^\alpha_{IM}$ ($\alpha \in \{0, c, l\}$), of which quite a few vanish, either $\int d\Omega_{\pi p}\tau^\alpha_{IM}$ or $\int d\Omega_{\pi p}\sigma^\alpha_{IM}$. The final expression is

$$\frac{d\sigma}{d\vec{q}_\eta} = \frac{d\sigma_0}{d\vec{q}_\eta}\Bigg\{1 + P^\gamma_l \widetilde{\Sigma}^l \cos 2\phi_{\eta\gamma} + P^p_1\Bigg[P^\gamma_l \sum_{M=-1}^{1} \widetilde{T}^l_{1M} \sin(M\phi_{\eta s} - 2\phi_{\eta\gamma}) d^1_{M0}(\theta_s) + \sum_{M=0}^{1}\big(-\widetilde{T}^0_{1M}\sin M\phi_{\eta s} + P^\gamma_c \widetilde{T}^c_{1M}\cos M\phi_{\eta s}\big) d^1_{M0}(\theta_s)\Bigg]\Bigg\}, \tag{60}$$

where the unpolarized cross section and the asymmetries are given by

$$\frac{d\sigma_0}{d\vec{q}_\eta} = \int d\Omega_{\pi p}\, \tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = V_{00}(q_\eta,\theta_\eta), \tag{61}$$

$$\widetilde{\Sigma}^l(q_\eta,\theta_\eta)\frac{d\sigma_0}{d\vec{q}_\eta} = \int d\Omega_{\pi p}\, \tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = W_{00}(q_\eta,\theta_\eta), \tag{62}$$

$$\widetilde{T}^0_{1M}(q_\eta,\theta_\eta)\frac{d\sigma_0}{d\vec{q}_\eta} = \int d\Omega_{\pi p}\, \sigma^0_{1M}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = -(2-\delta_{M0})\mathrm{Im}\,[V_{1M}(q_\eta,\theta_\eta)], \quad \text{for} \quad M = 0, 1, \tag{63}$$

$$\widetilde{T}^c_{1M}(q_\eta,\theta_\eta)\frac{d\sigma_0}{d\vec{q}_\eta} = \int d\Omega_{\pi p}\, \tau^c_{1M}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = (2-\delta_{M0})\mathrm{Re}\,[V_{1M}(q_\eta,\theta_\eta)], \quad \text{for} \quad M = 0, 1, \tag{64}$$

$$\widetilde{T}^l_{1M}(q_\eta,\theta_\eta)\frac{d\sigma_0}{d\vec{q}_\eta} = \int d\Omega_{\pi p}\, \sigma^l_{01M}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = iW_{1M}(q_\eta,\theta_\eta), \quad \text{for} \quad M = 0, \pm 1. \tag{65}$$

Here, the quantities V_{IM} and W_{IM} are related to the small v^1_{IM} and w^1_{IM} by

$$V_{IM}(q_\eta,\,\theta_\eta) = \int d\Omega_{\pi p}\, v^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \tag{66}$$

$$W_{IM}(q_\eta,\theta_\eta) = \int d\Omega_{\pi p}\, w^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}). \tag{67}$$

Because V_{I0} is real according to Eq. (42), the asymmetries $\widetilde{T}^c_{00}$and $\widetilde{T}^0_{10}$ vanish identically. Furthermore, one should note that W_{1M} is purely imaginary. This is shown in Appendix A [see Eq. (A7)]. More explicitly one has

$$\frac{d\sigma}{d\vec{q}_\eta} = \frac{d\sigma_0}{d\vec{q}_\eta}\Bigg[1 + P^\gamma_l \widetilde{\Sigma}^l \cos 2\phi_{\eta\gamma} + P^p_1\Bigg(P^\gamma_l \Bigg\{-\widetilde{T}^l_{10}\cos\theta_s \times \cos 2\phi_{\eta\gamma} - \frac{1}{\sqrt{2}}[(\widetilde{T}^l_{1-1} + \widetilde{T}^l_{11})\sin\phi_{\eta s}\cos 2\phi_{\eta\gamma} + (\widetilde{T}^l_{1-1} - \widetilde{T}^l_{11})\cos\phi_{\eta s}\sin 2\phi_{\eta\gamma}]\sin\theta_s\Bigg\} + \frac{1}{\sqrt{2}}\widetilde{T}^0_{11}\sin\phi_{\eta s}\sin\theta_s - P^\gamma_c\Bigg(\widetilde{T}^c_{11}\cos\phi_{\eta s}\sin\theta_s - \frac{1}{\sqrt{2}}\widetilde{T}^c_{10}\cos\theta_s\Bigg)\Bigg)\Bigg]. \tag{68}$$

We would like to point out that in forward and backward eta emission, i.e., for $\theta_\eta = 0$ and π, the following asymmetries have to vanish:

$$\widetilde{\Sigma}^l = 0, \quad \widetilde{T}^{0,c}_{11} = 0, \quad \text{and} \quad T^l_{1M} = 0, \tag{69}$$

because in that case the differential cross section cannot depend on ϕ_η, because at $\theta_\eta = 0$ or π the azimuthal angle ϕ_η is undefined or arbitrary. This feature can be shown also by straightforward evaluation of V_{IM} and W_{IM}using the explicit representation of the T matrix in Eq. (14). One finds

$$V_{IM}(q_\eta,\theta_\eta = 0/\pi,\theta_{\pi p},\phi_{pq}) = 0 \quad \text{for} \quad M \neq 0,$$
$$W_{IM}(q_\eta,\theta_\eta = 0/\pi,\theta_{\pi p},\phi_{pq}) = 0 \quad \text{for} \quad \text{all } M. \tag{70}$$

The formulas above can be extended readily to the other cases of an active pion or proton through a simple replacement of the appropriate angles with a corresponding redefinition of the various planes in Fig. 1.

C. The total cross section

The general total cross section is obtained from Eq. (60) by integrating over d^3q_η, resulting in

$$\sigma = \sigma_0\big(1 + P^\gamma_c P^p_1 \overline{T}^c_{10}\cos\theta_s\big), \tag{71}$$

where the unpolarized total cross section and the only beam-target asymmetry $\overline{T}^c_{10}$ are given by

$$\sigma_0 = 2\pi\int d\cos\theta_\eta \int_{q^{\min}_\eta}^{q^{\max}_\eta} q^2_\eta\, dq_\eta \frac{d\sigma_0}{d\vec{q}_\eta}, \tag{72}$$

$$\sigma_0\overline{T}^c_{10} = 2\pi\int d\cos\theta_\eta \int_{q_{\min}}^{q_{\max}} q^2\, dq_\eta \frac{d\sigma_0}{d\vec{q}_\eta} T^c_{10}. \tag{73}$$

The integration limits $q^{\min}_\eta$ and $q^{\max}_\eta$ are determined by energy and momentum conservation. There is no dependence on the linear photon polarization as expected.

V. RESULTS AND DISCUSSION

In this section we present our results for those asymmetries of the reaction $\gamma p \to \pi^0\eta p$ for which data already exist or are expected to be measured in the near future. The observables are calculated in the overall γp c.m. frame. The main ingredients of our model are described in detail in Refs. [17,18]. Here we limit ourselves to a brief overview of the model needed for the discussion. The calculation is based on a conventional isobar model as used, for example, for double pion photoproduction in Refs. [19–23]. The model parameters were fitted to the angular distributions of pions in the πp c.m. system, as well as to the distribution over the polar angle of η in the overall c.m. frame. The corresponding data were presented in Refs. [5] and [18] in the region up to a photon laboratory energy $\omega_\gamma = 1.4$ GeV. The present results are obtained in the same energy interval.

The reaction amplitude comprises background and resonance terms

$$t_{m_f\lambda} = t^B_{m_f\lambda} + \sum_{R(J^\pi;T)} t^R_{m_f\lambda}. \tag{74}$$

An individual resonance state $R(J^\pi;T)$ is determined by its spin-parity J^π and isospin T. Instead of the spin projections of the initial particles m_i and μ, respectively, we use their sum $\lambda = m_i + \mu = \pm 1/2,\ \pm 3/2$, which in our coordinate system with the quantization axes along the incident photon momentum corresponds to the initial state helicity.

The resonance sector includes only states with isospin $T = 3/2$. As already noted, analysis of the existing data for $\gamma p \to \pi^0\eta p$ are in agreement with the assumption that in the energy region studied here the reaction is dominated by the D_{33} wave. In the present model, the latter is populated by the $D_{33}(1700)$ and $D_{33}(1940)$ states. The one-star resonance $D_{33}(1940)$ was first introduced into the reaction $\gamma p \to \pi^0\eta p$ in Ref. [1] based on a partial-wave analysis (PWA). In our model the status of this baryon is still not very clear. Primarily we need it in order to maintain the importance of the D_{33} wave at energies above 1.3 GeV, which otherwise would rapidly decrease with increasing energy. Other $T = 3/2$ resonances entering the amplitude are $P_{33}(1600)$, $P_{31}(1750)$, $F_{35}(1905)$, and $P_{33}(1920)$. Their parameters resulting from a fit are listed in Table II of Ref. [18].

As is shown in Refs. [16] and [18], the background contribution is small, so that we can focus our attention on the resonance sector alone. According to the isobar model concept, each resonance term is given by a coherent sum of individual amplitudes corresponding to intermediate transitions to $\eta\Delta(1232)$ and $\pi S_{11}(1535)$ configurations,

$$t^R_{m_f\lambda} = t^{R(\eta\Delta)}_{m_f\lambda} + t^{R(\pi N^*)}_{m_f\lambda}, \tag{75}$$

where the resonances $\Delta(1232)$ and $S_{11}(1535)$ are denoted as Δ and N^*, respectively. The $\pi\eta$ system is assumed not to resonate in our energy interval. The validity of this assumption is confirmed by the results of Ref. [1], where the contribution of the resonance $a_0(980)$ at energies $\omega_\gamma < 1.4$ GeV is shown to be less than 1%.

Each term in Eq. (75) has the form

$$t^{R(\alpha)}_{m_f\lambda}(W, \vec{q}_\pi, \vec{q}_\eta, \vec{p}_f) = c^{(\alpha)}_R A^R_\lambda\, f^{R(\alpha)}_{m_f\lambda}(\vec{q}_\pi, \vec{q}_\eta, \vec{p}_f), \quad \alpha \in \{\eta\Delta, \pi N^*\}, \tag{76}$$

with W being the total c.m. energy. The quantities A^R_λ, which in general depend on W, are helicity functions determining the transition $\gamma p \to R$. The factor $c^{(\alpha)}_R$ absorbs all quantities that are independent of the quantum numbers m_f and λ. Its exact form is irrelevant for the formalism to follow. The angular-dependent part $f^{R(\alpha)}_{m_f\lambda}$ describes the decay of the resonance R into $\pi\eta N$ via intermediate formation of an $\eta\Delta$ or πN^* state.

In the actual calculation we adhere to the nonrelativistic concept of angular momentum so that the angular dependence of the amplitudes (76) is described by means of spherical harmonics

$$f^{R(\eta\Delta)}_{m_f\lambda} \sim \sum_{m,m_\eta,m_\Delta} \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ m & m_f & -m_\Delta \end{pmatrix} \begin{pmatrix} l_\eta & \frac{3}{2} & J \\ m_\eta & m_\Delta & -\lambda \end{pmatrix} \times Y_{1m}(\Omega_{\pi p}) d^{l_\eta}_{m_\eta 0}(\theta_\eta), \tag{77}$$

$$\begin{aligned} f^{R(\pi N^*)}_{m_f\lambda} &\sim \sum_{m_\pi} \begin{pmatrix} l_\pi & \frac{1}{2} & J \\ m_\pi & m_f & -\lambda \end{pmatrix} Y_{l_\pi m_\pi}(\Omega_\pi) \\ &\sim \sum_{m_\pi} \begin{pmatrix} l_\pi & \frac{1}{2} & J \\ m_\pi & m_f & -\lambda \end{pmatrix} \sum_{l=0}^{l_\pi} A^{l_\pi}_l \\ &\times \sum_m \begin{pmatrix} l_\pi - l & l & l_\pi \\ m_\pi - m & m & -m_\pi \end{pmatrix} \\ &\times Y_{lm}(\Omega_{\pi p}) d^{l_\pi - l}_{m_\pi - m 0}(\theta_\eta). \end{aligned} \tag{78}$$

The coefficients $A^{l_\pi}_l$, determined as

$$A^{l_\pi}_l = \left[\frac{m_\pi q_\eta}{(m_\pi + M_p)p_{\pi p}}\right]^l \sqrt{\frac{(2l_\pi - 1)(2l_\pi)!}{(2l-1)(2l_\pi - 2l)!(2l)!}}, \tag{79}$$

stem from the expansion of the function $Y_{l_\pi m_\pi}(\Omega_\pi)$with respect to products of spherical functions depending on $\Omega_{\pi p}$ and Ω_η.

A. The semiexclusive asymmetries for circularly polarized photons and polarized protons

Now we will turn to the case where the active particle (π, η or p) is measured for a fixed invariant mass of the other two final particles, irrespective of the direction θ_α with $\alpha = \pi, \eta$ or p, respectively, for a fixed reaction plane. The resulting semiexclusive differential cross section is obtained by an additional integration over the polar angles θ_α, respectively. It is given by an expression formally analogous to Eq. (60) with the following replacements (for the eta as an active particle as an example):

$$\frac{d\sigma_0}{d\vec{q}_\eta} \to \frac{d\sigma_0}{dM_{\pi p}\, d\phi_\eta} = \int d\cos\theta_\eta \mathcal{K} \frac{d\sigma_0}{d\vec{q}_\eta}, \tag{80}$$

$$\frac{d\sigma_0}{d\vec{q}_\eta}\widetilde{\Sigma}^l(q_\eta, \theta_\eta) \to \frac{d\sigma_0}{dM_{\pi p}\, d\phi_\eta}\widehat{\Sigma}^l(M_{\pi p}) = \int d\cos\theta_\eta \mathcal{K} \frac{d\sigma_0}{d\vec{q}_\eta}\widetilde{\Sigma}^l(q_\eta, \theta_\eta), \tag{81}$$

$$\frac{d\sigma_0}{d\vec{q}_\eta}\widetilde{T}^{\alpha}_{IM}(q_\eta,\theta_\eta) \to \frac{d\sigma_0}{dM_{\pi p}\,d\phi_\eta}\widehat{T}^{\alpha}_{IM}(M_{\pi p}) = \int d\cos\theta_\eta \mathcal{K}\frac{d\sigma_0}{d\vec{q}_\eta}\widetilde{T}^{\alpha}_{IM}(q_\eta,\theta_\eta), \quad \alpha \in \{0,l,c\}. \tag{82}$$

The factor $\mathcal{K}$ takes into account the transformation of the differential, i.e., $q_\eta^2\,dq_\eta = \mathcal{K}\,dM_{\pi p}$. In the γp c.m. frame, this factor is independent of θ_η and reads

$$\mathcal{K} = \frac{q_\eta \omega_\eta M_{\pi p}}{W}. \tag{83}$$

For the case of an active pion or proton, one simply has to make the following replacements: $\eta \to \pi$ or p and $\pi p \to \eta p$ or $\pi\eta$, respectively.

We now consider circularly polarized photons and allow for polarized protons, i.e., $P_c^\gamma \neq 0$, $P_1^p \neq 0$, and $P_l^\gamma = 0$. Furthermore, we set the azimuthal η angle to $\phi_\eta = 0$. Then one obtains explicitly

$$\left.\frac{d\sigma}{dM_{\pi p}\,d\phi_\eta}\right|_{\phi_\eta=0} = \left.\frac{d\sigma_0}{dM_{\pi p}\,d\phi_\eta}\right|_{\phi_\eta=0}\left\{1 + P_1^p\left[\frac{1}{\sqrt{2}}\widehat{T}^0_{11}\sin\phi_s\sin\theta_s + P_c^\gamma\left(\widehat{T}^c_{10}\cos\theta_s - \frac{1}{\sqrt{2}}\widehat{T}^c_{11}\cos\phi_s\sin\theta_s\right)\right]\right\}. \tag{84}$$

As a side remark, angular distributions irrespective of the energy of the active particle may be obtained in a similar manner via appropriate integration of the cross section in (60) over the energy of the active particle.

Figure 2 demonstrates our predictions for the semiinclusive target asymmetry $\widehat{T}^0_{11}$ as well as for the double-polarization observables $\widehat{T}^c_{11}$ and $\widehat{T}^c_{10}$. In the single D_{33} resonance model, including only $D_{33}(1700)$ and $D_{33}(1940)$, both asymmetries $\widehat{T}^0_{11}$ and $\widehat{T}^c_{11}$ should vanish completely. The corresponding angular distributions (in Fig. 3 we show the dependence of $\widehat{T}^c_{11}$ on $\cos\theta_\eta$ and $\cos\theta_\pi$) are odd functions of $\cos\theta_{\eta/\pi}$, so that they vanish after integration over the polar angle. The full model, in which also positive parity resonances are included, gives an even component in both asymmetries, thus leading to a rather intricate energy dependence, as is shown in Fig. 2.

It is also worth noting that for the active pion, the dependence of $\widehat{T}^c_{11}$ on θ_π is rather similar to that observed for single π^0 photoproduction in the Δ region. This may be owing to the dominance of the s wave in the $\eta\Delta$ channel and to the relatively large η mass, so that the Δ decay is not contaminated by the presence of an η meson.

Of special interest is the observable $\widehat{T}^c_{10}$. In the single D_{33} model, its value is almost independent of $M_{\pi p}$ (or $M_{\eta p}$). For example, if only the $D_{33}(1700)$ resonance is retained in the amplitude, it is approximately equal to

$$\widehat{T}^c_{10} \approx \frac{1-a^2}{2(1+a^2)}, \quad \text{with} \quad a = \frac{A_{3/2}}{A_{1/2}}, \tag{85}$$

where A_λ is a helicity function corresponding to the transition $\gamma N \to D_{33}(1700)$ [see our ansatz (76) for the resonance

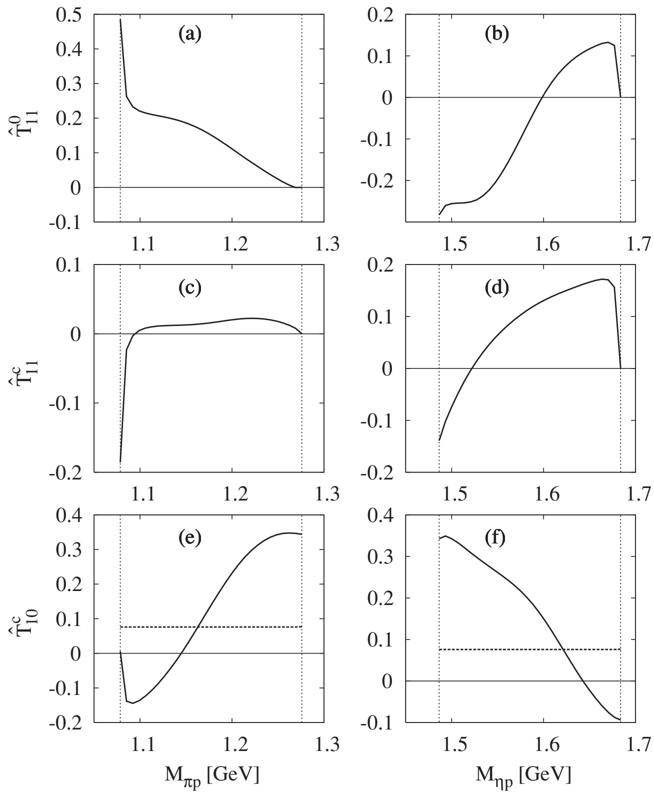

FIG. 2. The target asymmetry $\widehat{T}^0_{11}$ [(a) and (b)], and the beam-target asymmetries for circularly polarized photons $\widehat{T}^c_{11}$ [(c) and (d)] and $\widehat{T}^c_{10}$ [(e) and (f)] for the η as the active particle (left-hand panels) as a function of the πp invariant mass spectrum $M_{\pi p}$, and for the pion as the active particle (right-hand panels) as a function of $M_{\eta p}$, calculated at a laboratory photon energy of 1.3 GeV. The solid line presents the full calculation. The dashed line is obtained including the $D_{33}(1700)$ resonance only. The asymmetries $\widehat{T}^0_{11}$ and $\widehat{T}^c_{11}$ vanish in the single D_{33} model. The vertical dotted lines mark the boundaries of the available kinematical region.

amplitudes]. Taking $a = 1.1$ from the analysis of Ref. [5] (see Fig. 6 of Ref. [5] at $E_\gamma = 1.3$ GeV), we will have, according to Eq. (85), $\widehat{T}^c_{10} = -0.05$, in general agreement with the result shown by the dashed line in Fig. 2. If both resonances $D_{33}(1700)$and $D_{33}(1940)$ are included, $\widehat{T}^c_{10}$ remains constant, but its value is no longer determined by a simple relation analogous to (85). As we can see, inclusion of other resonances, resulting in a strong interference with the leading partial wave, changes crucially the shape of $\widehat{T}^c_{10}$.

It is also interesting to note that, in contrast to single pseudoscalar meson photoproduction, $\widehat{T}^c_{10}$ does not approach unity at very forward and backward η angles [see panel (c) in Fig. 3]. The reason for this behavior lies in the spin 3/2 of the Δ resonance, so that angular momentum conservation does not require $\lambda = 1/2$ at $\theta_\eta = 0(\pi)$, as in the case of a single meson.

B. The semiexclusive asymmetries for linearly polarized photons and polarized protons

For only linearly polarized photons the semiexclusive cross section is again obtained from Eq. (60) for $P_c^\gamma = 0$ and $\phi_\eta = 0$

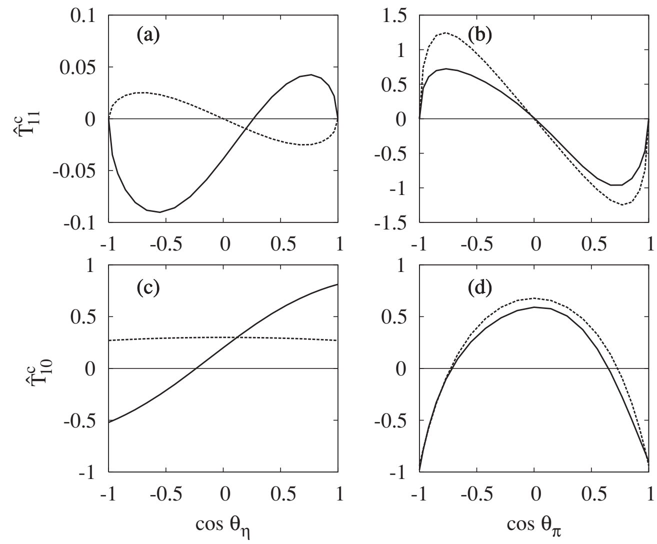

FIG. 3. The beam-target asymmetries for circularly polarized photons $\widehat{T}^c_{11}$ and $\widehat{T}^c_{10}$ at a laboratory photon energy of 1.3 GeV as function of the polar angles of active eta [left-hand panels (a) and (c)] and active pion [right-hand panels (b) and (d)] in the γp c.m. frame. The solid curve is the full model calculation. The dashed curve includes only the $D_{33}(1700)$ and $D_{33}(1940)$ resonances.

with the replacements of Eqs. (80)–(82):

$$\left.\frac{d\sigma}{dM_{\pi p}d\phi_\eta}\right|_{\phi_\eta=0} = \left.\frac{d\sigma_0}{dM_{\pi p}d\phi_\eta}\right|_{\phi_\eta=0}\Bigg[1 + P^\gamma_l\Bigg(\widehat{\Sigma}^l\cos 2\phi_\gamma\, P^p_1 + \Bigg\{-\widehat{T}^l_{10}\cos\theta_s\cos 2\phi_\gamma + \frac{1}{\sqrt{2}}\big[\big(\widehat{T}^l_{1-1}+\widehat{T}^l_{11}\big)\sin\phi_s \times \cos 2\phi_\gamma + \big(\widehat{T}^l_{1-1}-\widehat{T}^l_{11}\big)\cos\phi_s\sin 2\phi_\gamma\big]\sin\theta_s\Bigg\}\Bigg)\Bigg], \tag{86}$$

where ϕ_γ measures the angle between the reaction and the photon plane. The gross features of the beam asymmetry for linearly polarized photons $\widehat{\Sigma}^l$ as a function of the πN or ηN invariant energies were already discussed in detail in Ref. [18]. Therefore, we show here only the additional beam-target asymmetries $\widehat{T}^l_{1M}$ in Fig. 4.

Furthermore, we present results for the asymmetries called I^c and I^s, which were recently measured at ELSA [8]. In this experiment, the direction of the eta meson was detected in the reaction plane in coincidence with the pion proton pair for a fixed orientation of the decay plane integrated over the direction within this plane of $\vec{p}_{\pi p}$ as function of the angle between the reaction plane and the decay plane. The initial proton was unpolarized. For the comparison of our results with the data, we have adjusted the calculation to the experimental kinematic conditions of these measurements. First of all, we changed the coordinate system as defined in Fig. 1 for the x-z plane coinciding with the reaction plane, i.e., $\phi_\eta = 0$ (z axis parallel to $\vec{k}$ and y axis parallel to $\vec{k}\times\vec{q}_\eta$) by rotating

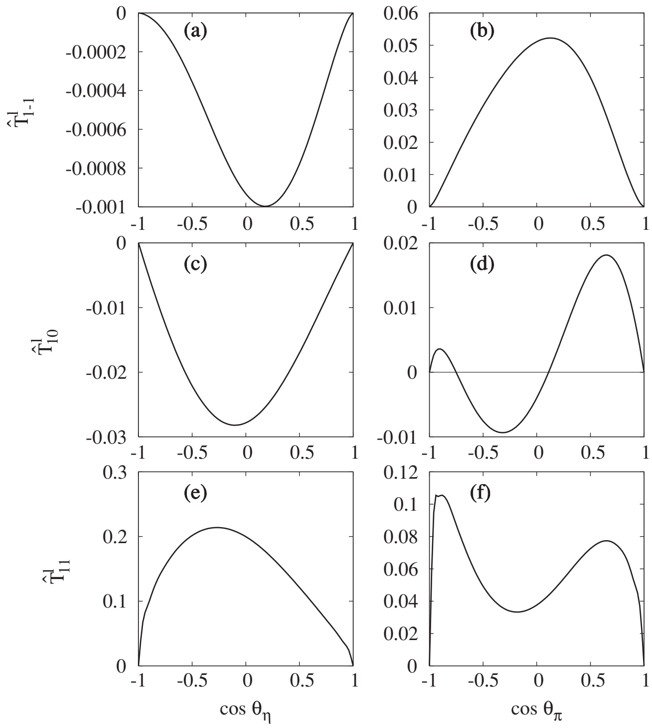

FIG. 4. The beam-target asymmetries for linearly polarized photons $\widehat{T}^l_{1-1}$, $\widehat{T}^l_{10}$, and $\widehat{T}^l_{11}$ at a laboratory photon energy of 1.3 GeV as function of the polar angles of an active eta [left-hand panels (a), (c), and (e)] and an active pion [right-hand panels (b), (d), and (f)] in the γp c.m. frame.

it around the y axis such that the new z^* axis is aligned along the vector $\vec{q}_\pi + \vec{p}_f$. With respect to the rotated coordinate system, the relative momentum $\vec{p}_{\pi p}$ has the spherical angles $\Omega^*_{\pi p} = (\theta^*_{\pi p}, \phi^*_{\pi p})$, and the decay plane intersects the reaction plane with the azimuthal angle $\phi^*_{\pi p}$. This is illustrated in Fig. 5 for the c.m. system. In the rotated γp c.m. system, the corresponding expressions for the amplitudes $f^{R(\alpha)}_{m_f\lambda}$ can be obtained easily from Eqs. (77) and (78) via a positive rotation of $Y_{lm}(\Omega_p)$ by an angle $\theta_R = \theta_\eta + \pi$ around the y axis. With respect to the new variables, one obtains a set of new structure functions $\tau/\sigma^{(*)\alpha}_{IM}(q_\eta, \theta_\eta, \theta^*_{\pi p}, \phi^*_{pq})$, which are related to the old

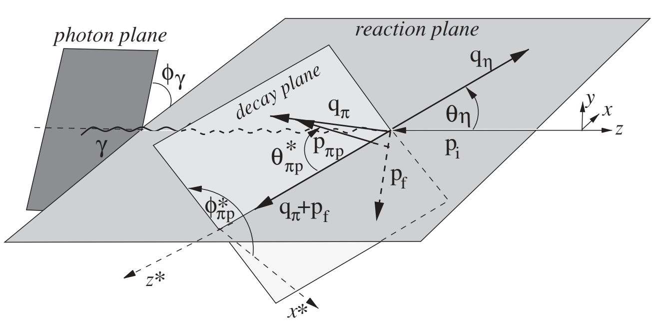

FIG. 5. Kinematics of $\pi\eta$ photoproduction on the nucleon for an active eta in the c.m. system with a rotated coordinate system.

ones by the Jacobian

$$J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p}) = \left|\frac{\partial(\cos\theta_{\pi p},\phi_{\pi p})}{\partial(\cos\theta^*_{\pi p},\phi^*_{\pi p})}\right| \quad (87)$$

according to

$$\tau/\sigma^{(*)\alpha}_{IM}(q_\eta,\theta_\eta,\theta^*_{\pi p},\phi^*_{pq}) = \tau/\sigma^{\alpha}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) \times |J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p})| \quad \text{for} \quad \alpha\in\{0,l,c\}. \quad (88)$$

From the relations between $(\theta_{\pi p},\phi_{\pi p})$ and $(\theta^*_{\pi p},\phi^*_{\pi p})$,

$$\cos\theta_{\pi p} = \cos\theta^*_{\pi p}\cos\theta_R - \sin\theta^*_{\pi p}\cos\phi_{\pi p}\sin\theta_R, \quad (89)$$

$$\cot\phi_{\pi p} = \cot\phi^*_{\pi p}\cos\theta_R + \frac{\cot\theta^*_{\pi p}}{\sin\phi^*_{\pi p}}\sin\theta_R, \quad (90)$$

with θ_R denoting the rotation angle, one can see that $\theta_{\pi p}$ and $\phi_{\pi p}$ are, respectively, even and odd functions of $\phi^*_{\pi p}$ (what may also be obvious from the geometric considerations). Explicitly, one finds for the Jacobian

$$J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p}) = \frac{\sin^2\phi_{\pi p}}{\sin^2\theta^*_{\pi p}\sin^2\phi^*_{\pi p}}[(\sin\theta^*_{\pi p}\cos\theta_R + \cos\phi^*_{\pi p}\cos\theta^*_{\pi p}\sin\theta_R)^2 + \sin^2\phi^*_{\pi p}\sin^2\theta_R]. \quad (91)$$

The above found symmetry of the angle transformation is reflected by the invariance of the Jacobian under a simultaneous sign change of $\phi_{\pi p}$ and $\phi^*_{\pi p}$, i.e.,

$$J(\cos\theta_{\pi p},-\phi_{\pi p};\cos\theta^*_{\pi p},-\phi^*_{\pi p}) = J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p}), \quad (92)$$

which will be used later on.

As mentioned in the formal part, we also consider partitions $\pi+(\eta N)$ and $p+(\pi\eta)$ in which the decay plane is spanned in the former case by the vectors $\vec q_\eta$ and $\vec p_f$, formally replacing $\Omega^*_{\pi p}$ by $\Omega^*_{\eta p}=(\theta^*_{\eta p},\phi^*_{\eta p})$, and in the latter case by $\vec q_\pi$ and $\vec q_\eta$ with $\Omega^*_{\pi\eta}=(\theta^*_{\pi\eta},\phi^*_{\pi\eta})$.

In order to evaluate the corresponding semiexclusive observables, one has to integrate over dq_η and $d\cos\theta^*_{\pi p}$ the general expression for the differential cross section, which reads for $\phi_\eta=0$, $P^\gamma_c=0$, and $P^p_1=0$,

$$\frac{d\sigma}{d\vec q_\eta\, d\Omega^*_{\pi p}} = \frac{d\sigma_0}{d\vec q_\eta\, d\Omega^*_{\pi p}}\left[1+P^\gamma_l\left(T^{(*)l}_{00}\cos 2\phi_\gamma + S^{(*)l}_{00}\sin 2\phi_\gamma\right)\right]. \quad (93)$$

This then yields in the notation of Ref. [6] (one should note that ϕ' in Ref. [8] is related to ϕ_γ by $\phi_\gamma=\phi'-\pi/2$)

$$\frac{d\sigma}{d\phi_\eta\, d\phi^*_{\pi p}} = \frac{d\sigma_0}{d\phi_\eta\, d\phi^*_{\pi p}}\left\{1-P^\gamma_l\left[I^c(\phi^*_{\pi p})\cos 2\phi_\gamma + I^s(\phi^*_{\pi p})\sin 2\phi_\gamma\right]\right\}, \quad (94)$$

where the linear beam asymmetries I^c and I^s are determined by the coefficients $S^{(*)l}_{00}$ and $T^{(*)l}_{00}$ in Eq. (93):

$$I^c(\phi^*_{\pi p})\frac{d\sigma}{d\phi_\eta\, d\phi^*_{\pi p}} = -\int d\cos\theta^*_{\pi p}\int_{q^{\min}_\eta}^{q^{\max}_\eta} q^2_\eta\, dq_\eta \frac{d\phi_\eta\, d\sigma_0}{d\phi^*_{\pi p}} \times T^{(*)l}_{00}(q_\eta,\theta_\eta;\theta^*_{\pi p},\phi^*_{\pi p}) = -\int d\cos\theta^*_{\pi p}\int_{q^{\min}_\eta}^{q^{\max}_\eta} q^2_\eta\, dq_\eta \times \tau^{(*)l}_{00}(q_\eta,\theta_\eta;\theta^*_{\pi p},\phi^*_{\pi p}), \quad (95)$$

$$I^s(\phi^*_{\pi p})\frac{d\sigma}{d\phi_\eta\, d\phi^*_{\pi p}} = -\int d\cos\theta^*_{\pi p}\int_{q^{\min}_\eta}^{q^{\max}_\eta} q^2_\eta\, dq_\eta \frac{d\phi_\eta\, d\sigma_0}{d\phi^*_{\pi p}} \times S^{(*)l}_{00}(q_\eta,\theta_\eta;\theta^*_{\pi p},\phi^*_{\pi p}) = -\int d\cos\theta^*_{\pi p}\int_{q^{\min}_\eta}^{q^{\max}_\eta} q^2_\eta\, dq_\eta \times \sigma^{(*)l}_{00}(q_\eta,\theta_\eta;\theta^*_{\pi p},\phi^*_{\pi p}). \quad (96)$$

Using Eqs. (45), (55), and (56), one can easily show that $I^c(\phi^*_{\pi p})$ and $I^s(\phi^*_{\pi p})$ are, respectively, even and odd functions of the angle $\phi^*_{\pi p}$, i.e.,

$$I^c(-\phi^*_{\pi p}) = I^c(\phi^*_{\pi p}), \qquad I^s(-\phi^*_{\pi p}) = -I^s(\phi^*_{\pi p}). \quad (97)$$

Indeed, from the symmetry relation (45) with $\phi_\eta=0$ and thus $\phi_{pq}=\phi_{\pi p}$ and the definitions (55) and (56) follows

$$\tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{\pi p}) = \tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{\pi p}), \quad (98)$$

$$\sigma^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{\pi p}) = -\sigma^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{\pi p}). \quad (99)$$

Furthermore, from Eq. (88) and the invariance in Eq. (92) of the Jacobian, one finds

$$\begin{aligned}\tau^{(*)l}_{00}&(q_\eta,\theta_\eta,\theta^*_{\pi p},-\phi^*_{\pi p})\\ &= \tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{\pi p})|J(\cos\theta_{\pi p},-\phi_{\pi p};\cos\theta^*_{\pi p},-\phi^*_{\pi p})|\\ &= \tau^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{\pi p})|J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p})|\\ &= \tau^{(*)l}_{00}(q_\eta,\theta_\eta,\theta^*_{\pi p},\phi^*_{\pi p}),\end{aligned} \quad (100)$$

$$\begin{aligned}\sigma^{(*)l}_{00}&(q_\eta,\theta_\eta,\theta^*_{\pi p},-\phi^*_{\pi p})\\ &= \sigma^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{\pi p})|J(\cos\theta_{\pi p},-\phi_{\pi p};\cos\theta^*_{\pi p},-\phi^*_{\pi p})|\\ &= -\sigma^l_{00}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{\pi p})|J(\cos\theta_{\pi p},\phi_{\pi p};\cos\theta^*_{\pi p},\phi^*_{\pi p})|\\ &= -\sigma^{(*)l}_{00}(q_\eta,\theta_\eta,\theta^*_{\pi p},\phi^*_{\pi p}).\end{aligned} \quad (101)$$

From these relations the noted symmetries of Eq. (97) follow directly with the help of the definitions in Eqs. (95) and (96).

In Figs. 6 and 7 we compare our results with the data. In view of the fact that the data were not included in the fit of the model parameters, the agreement is reasonable. Already the single D_{33} model [including only $D_{33}(1700)$ and $D_{33}(1940)$] reproduces the experimentally observed shape and magnitude of the observables, so that admixtures of other terms leads to relatively small corrections. Our results are in general agreement with those obtained in Ref. [24], except, maybe,

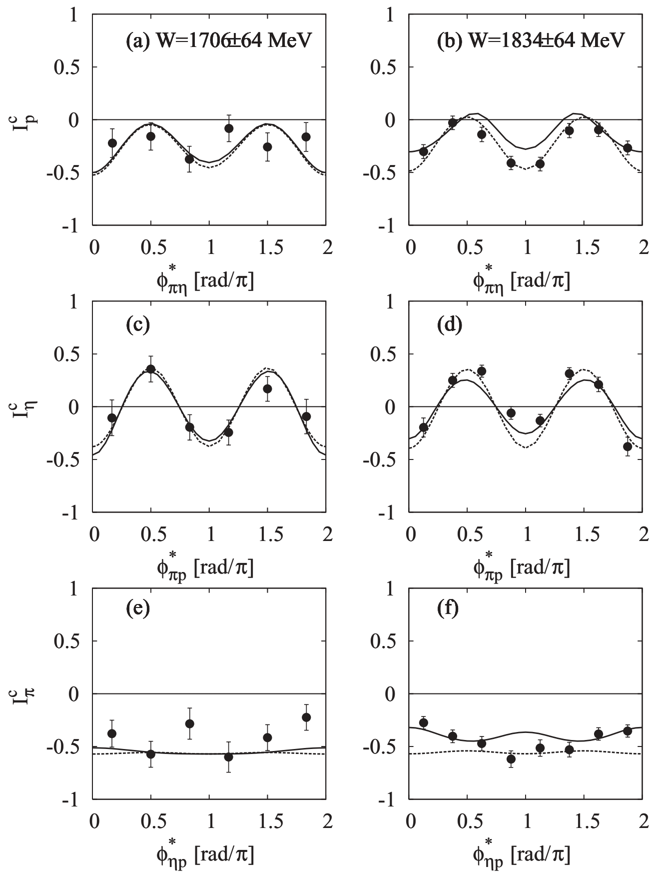

FIG. 6. The beam asymmetry I^c calculated for two total c.m. energies W. The data are from Ref. [8] (only the statistical errors are shown). The upper two panels (a) and (b) refer to an active proton, the two middle panels (c) and (d) refer to an active eta, and the two lower panels (e) and (f) refer to an active pion as function of the angle between the corresponding reaction plane and decay planes as counted from the reaction plane (see Fig. 5). Notation of the curves as in Fig. 3.

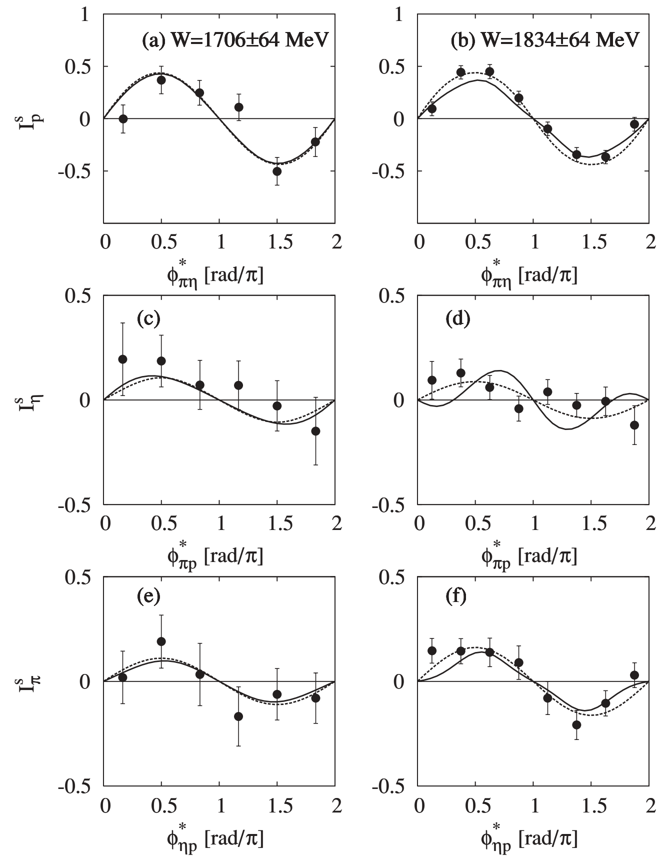

FIG. 7. Same as in Fig. 6 for the beam asymmetry I^s.

I^s_π, for which the model [24] predicts vanishingly small values (see Fig. 4 of the cited paper).

At the end of this section we will briefly return to circularly polarized photons. Without target polarization, one has as a semiexclusive cross section for the same experimental conditions as above,

$$\frac{d\sigma}{d\phi_\eta\, d\phi^*_{\pi p}} = \frac{d\sigma_0}{d\phi_\eta\, d\phi^*_{\pi p}}\left(1 + P^\gamma_c\, T^{(*)c}_{00}\right), \qquad (102)$$

where only one beam asymmetry appears. In Ref. [6] this circular photon asymmetry was introduced with the notation $I^\odot$, i.e.,

$$\frac{d\sigma_0}{d\phi_\eta\, d\phi^*_{\pi p}}\, T^{(*)c}_{00}(\phi^*_{\pi p}) = \frac{d\sigma_0}{d\phi_\eta\, d\phi^*_{\pi p}}\, I^\odot(\phi^*_{\pi p}) = \frac{1}{2}\frac{d\sigma^+ - d\sigma^-}{d\phi_\eta\, d\phi^*_{\pi p}} = \int d\cos\theta^*_{\pi p} \int_{q_\eta^{\min}}^{q_\eta^{\max}} q_\eta^2\, dq_\eta\, \tau^{(*)c}_{00}(q_\eta, \theta_\eta; \theta^*_{\pi p}, \phi^*_{\pi p}), \quad (103)$$

where $d\sigma^\pm$ denotes the cross section corresponding to the photon beam with a helicity $P^\gamma_c = \lambda_\gamma = \pm 1$, respectively. Furthermore, as in Ref. [9], instead of $I^\odot$, we will consider an observable whose definition slightly differs from Eq. (103), namely,

$$W^c(\phi^*_{\pi p}) = \frac{2\pi}{\sigma}\frac{d\sigma}{d\phi_\eta\, d\phi^*_{\pi p}} T^{(*)c}_{00} = \frac{\pi}{\sigma}\frac{d\sigma^+ - d\sigma^-}{d\phi_\eta\, d\phi^*_{\pi p}}, \qquad (104)$$

with σ being the unpolarized total cross section. According to the definitions in Eqs. (52) and (53), and the symmetry property in Eq. (44), $W^c$is an odd function of the argument $\phi^*_{\pi p}$ and therefore may be expanded into a sine series

$$W^c(\phi^*_{\pi p}) = \sum_n A_n \sin n\phi^*_{\pi p}. \qquad (105)$$

For further analyses it is convenient to have an analytic expression for $W^c(\phi^*)$ of Eq. (104), and we neglect for simplicity the small background. Furthermore, as already noted, in our energy region the reaction seems to be dominated by the D_{33} wave accompanied by relatively small admixtures of resonance states in other waves, in our case, P_{33}, P_{31}, and F_{35}. The latter contribute mainly as long as the corresponding amplitudes can interfere with that coming from the D_{33} excitation. In this connection, we will retain in further relations only those terms that are linear in the "weak" amplitudes. Then the integrand in Eq. (85), calculated up to the first order in $t^{P_{31}}$,

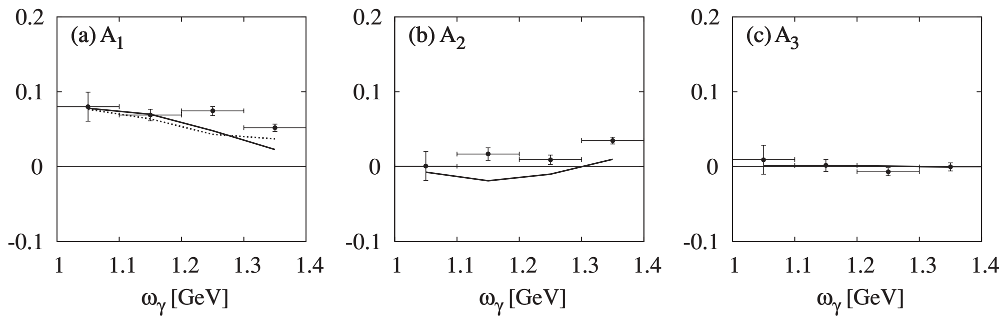

FIG. 8. Coefficients A_n $(n = 1, 2, 3)$ of the sine expansion (105). Notations as in Fig. 3. The data are from Ref. [9].

$t^{P_{33}}$, and $t^{F_{35}}$, reads

$$\tau_{00}^{(*)c} \simeq \left|t_{m_f\lambda}^{D_{33}}\right|^2 + 2\,\mathrm{Re}\left\{t_{m_f\lambda}^{*D_{33}} t_{m_f\lambda}^{P_{31}} + t_{m_f\lambda}^{*D_{33}} t_{m_f\lambda}^{P_{33}} + t_{m_f\lambda}^{*D_{33}} t_{m_f\lambda}^{F_{35}}\right\} - (\lambda \to -\lambda). \quad (106)$$

Using Eqs. (76)–(78) in (106), one obtains for the asymmetry in Eq. (104)

$$W^c(\phi_{\pi p}^*) = A_1 \sin\phi_{\pi p}^* + A_2 \sin 2\phi_{\pi p}^*, \quad (107)$$

where the coefficients A_1 and A_2 are expressed in terms of resonance parameters and are given in Appendix C. Of key importance is the fact that the first term in (107) is almost exclusively determined by the D_{33} wave. The contributions of other waves into A_1 are quadratic in the corresponding amplitudes and may therefore be neglected. As a result, the "weak" resonances enter only into the second term of Eq. (107), which is owing to an interference of the amplitudes $t^{P_{31}}$, $t^{P_{33}}$, and $t^{F_{35}}$ with the dominant $t^{D_{33}}$. In this respect, the $\sin 2\phi_{\pi p}^*$ admixture in the asymmetry $W^c(\phi_{\pi p}^*)$ may be viewed as a signature of positive parity states in $\pi^0\eta$ photoproduction.

In Fig. 8 we compare our calculation for A_n, $n = 1, 2, 3$, with the results obtained from the measurements of Ref. [9]. As one can see, the single D_{33} resonance model reproduces rather well the coefficient A_1 in the whole energy interval. As expected, addition of other resonances does not visibly change its value, because, as already noted, the corresponding contributions are of second order in the "small" amplitudes. For A_2 the agreement is worse. In particular, the model gives a wrong sign of this coefficient. It is also worth noting that A_2 has a rather small value at $\omega_\gamma \leqslant 1.3$ GeV. Unfortunately, the data do not allow us to find the reason for this fact, whether it is a consequence of a general smallness of individual contributions, or whether it is caused by an accidental cancelation between different terms. The last coefficient A_3 is comparable with zero, which is in line with our discussion above as well as with the model predictions. In the general case, the term with $\sin 3\phi_{\pi p}^*$ would be owing to an interference of D_{33} with negative parity resonances such as S_{31}, D_{35}, etc. In this respect, its smallness may be considered as an indication of an insignificant role of these states in this reaction.

VI. CONCLUSION

In this work we have derived formal expressions for the differential cross section and the recoil polarization of $\pi\eta$ photoproduction on the nucleon, including various polarization asymmetries with respect to polarized photons and nucleons.

A general analysis allowing the determination of the moduli and relative phases of the four independent photoproduction amplitudes requires a complete set of polarization experiments, which for photoproduction of two pseudoscalar mesons is discussed, e.g., in Ref. [6]. However, in the $\pi\eta$ case, owing to the assumed dominance of the D_{33} wave, the information on bilinear combinations of the amplitudes may require much smaller parameters. The situation is similar to that existing in η photoproduction, which is known to be dominated by the S_{11} wave in a wide energy region. Making use of this fact has allowed, e.g., an almost model-independent extraction of the parameters of the resonance $D_{13}(1520)$ in a much cleaner way, than in π photoproduction, where it overlaps with a multitude of other resonance states.

As noted above, according to the analyses of Refs. [1] and [18], in the energy region below $\omega_\gamma = 1.4$ GeV, the main contribution beyond the D_{33} resonance should come from the positive parity states P_{33}, P_{31}, and F_{35}, which reveal themselves through their interference with the dominant D_{33} amplitude. Our results show that the corresponding "small" amplitudes may be identified, e.g., through their contribution to the second Fourier coefficient A_2 in the sine series for $W^c(\phi_{\pi p}^*)$ in Eq. (105).

It is also important to note that the D_{33} resonance decays predominantly into an s-wave $\eta\Delta$ state. As a result, in the single D_{33} model (only the D_{33} wave is included into the amplitude) most of the polarization observables vanish. Therefore, the results of polarization measurements are expected to be sensitive to even small admixtures of "weak" resonances.

A comprehensive program for single- and double-polarization measurements of the reaction $\gamma p \to \pi^0\eta p$ is planned for the near future at MAMI and ELSA. The information obtained by these new experiments will provide stringent constraints on the quantum numbers of the resonance states entering the reaction amplitude.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (SFB 443, SFB/TR16), and the RF Federal program "Kadry" (contract P691). A.F. would like to thank the Institut für Kernphysik of the Johannes Gutenberg-Universität Mainz for the kind hospitality.

APPENDIX A: SEMIEXCLUSIVE DIFFERENTIAL CROSS SECTION $\vec{p}(\vec{\gamma},\eta)\pi p$

To derive the general expression for the semiexclusive cross section, we first introduce the quantities

$$W_{IM}(q_\eta,\theta_\eta) = \int d\Omega_{\pi p}\, w^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = -\frac{\widehat{I}}{\sqrt{2}}\int d\Omega_{\pi p}\, c(q_\eta,\theta_\eta,\Omega_{\pi p})\sum_{m_i m_i'}(-1)^{\frac{1}{2}-m_i} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I \\ m_i' & -m_i & M\end{pmatrix}\sum_{m_f} t^*_{m_f 1 m_i'}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})\, t_{m_f -1 m_i}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}), \tag{A1}$$

$$V^{\pm}_{IM}(q_\eta,\theta_\eta) = V^1_{IM}(q_\eta,\theta_\eta) \pm V^{-1}_{IM}(q_\eta,\theta_\eta), \tag{A2}$$

with

$$V^{\mu}_{IM}(q_\eta,\theta_\eta) = \int d\Omega_{\pi p}\, v^{\mu}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) = \frac{\widehat{I}}{\sqrt{2}}\int d\Omega_{\pi p}\, c(q_\eta,\theta_\eta,\Omega_{\pi p}) \times \sum_{m_i m_i'}(-1)^{\frac{1}{2}-m_i}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & I \\ m_i' & -m_i & M\end{pmatrix}\sum_{m_f} t^*_{m_f \mu m_i'}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq})\, t_{m_f \mu m_i}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}). \tag{A3}$$

Using now the property (44), one finds with the help of

$$\int_0^{2\pi} d\phi_{\pi p}\, f(-\phi_{pq}) = \int_0^{2\pi} d\phi_{\pi p}\, f(\phi_{pq}) \tag{A4}$$

for a periodic function $f(\phi_{pq}+2\pi) = f(\phi_{pq})$ (please note $\phi_{pq} = \phi_{\pi p} - \phi_\eta$), the relation

$$\begin{aligned} V^{-1}_{IM}(q_\eta,\theta_\eta) &= \int d\Omega_{\pi p}\, v^{-1}_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},\phi_{pq}) \\ &= (-1)^I \int d\Omega_{\pi p}\, v^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{pq})^* \\ &= (-1)^I V^1_{IM}(q_\eta,\theta_\eta)^*, \end{aligned} \tag{A5}$$

and thus

$$V^{\pm}_{IM}(q_\eta,\theta_\eta) = V^1_{IM}(q_\eta,\theta_\eta) \pm (-1)^I V^1_{IM}(q_\eta,\theta_\eta)^*. \tag{A6}$$

Correspondingly, using (45), one obtains

$$\begin{aligned} W_{IM}(q_\eta,\theta_\eta)^* &= (-1)^I \int d\Omega_{\pi p}\, w^1_{IM}(q_\eta,\theta_\eta,\theta_{\pi p},-\phi_{pq}) \\ &= (-1)^I W_{IM}(q_\eta,\theta_\eta). \end{aligned} \tag{A7}$$

From the two foregoing equations, we can conclude that V^+_{IM} and W_{IM} are real for $I=0$ and imaginary for $I=1$, whereas V^-_{IM} is imaginary for $I=0$ and real for $I=1$. Therefore, according to (53)–(56), the following integrated asymmetries vanish:

$$\int d\Omega_{\pi p}\, \tau^{\alpha}_{IM} = 0 \quad \text{for} \begin{Bmatrix} \alpha\in\{0,l\} & \text{and} & I=1, \\ \alpha\in\{c\} & \text{and} & I=0 \end{Bmatrix}, \tag{A8}$$

$$\int d\Omega_{\pi p}\, \sigma^{\alpha}_{IM} = 0 \quad \text{for} \begin{Bmatrix} \alpha\in\{0,l\} & \text{and} & I=0, \\ \alpha\in\{c\} & \text{and} & I=1 \end{Bmatrix}. \tag{A9}$$

Instead of using these results for deriving from (57) the threefold semiexclusive differential cross section, we prefer to start from the expression in (51), and obtain

$$\frac{d^3\sigma}{dq_\eta\, d\Omega_\eta} = \sum_{I=0,1} P^p_I \Bigg\{ \sum_{M=0}^{I} \frac{1}{1+\delta_{M0}} d^I_{M0}(\theta_s) \times \mathrm{Re}\,[e^{iM\phi_{\eta s}}(V^+_{IM} + P^\gamma_c V^-_{IM})] + P^\gamma_l \sum_{M=-I}^{I} d^I_{M0}(\theta_s)\mathrm{Re}\,(e^{i\psi_M} W_{IM})\Bigg\}. \tag{A10}$$

This expression can be simplified using the fact that $i^{\delta_{I1}} W_{IM}$, $i^{\delta_{I1}} V^+_{IM}$, and $i^{1-\delta_{I1}} V^-_{IM}$ are real according to (A6) and (A7). The latter two quantities can be written as

$$i^{\delta_{I1}} V^+_{IM} = 2\,\mathrm{Re}\,(i^{\delta_{I1}} V^1_{IM}), \tag{A11}$$

$$i^{1-\delta_{I1}} V^-_{IM} = 2\,\mathrm{Re}\,(i^{1-\delta_{I1}} V^1_{IM}) = -2\,\mathrm{Im}\,(i^{-\delta_{I1}} V^1_{IM}). \tag{A12}$$

Now using

$$\begin{aligned} \mathrm{Re}\,(e^{iM\phi_{\eta s}} V^+_{IM}) &= \mathrm{Re}\,[e^{i(M\phi_{\eta s}-\delta_{I1}\pi/2)} i^{\delta_{I1}} V^+_{IM}] \\ &= 2\,\mathrm{Re}\,(i^{\delta_{I1}} V^1_{IM})\cos(M\phi_{\eta s}-\delta_{I1}\pi/2), \end{aligned} \tag{A13}$$

$$\begin{aligned} \mathrm{Re}\,(e^{iM\phi_{\eta s}} V^-_{IM}) &= \mathrm{Re}\left[\frac{1}{i} e^{i(M\phi_{\eta s}+\delta_{I1}\pi/2)} i^{1-\delta_{I1}} V^-_{IM}\right] \\ &= -2\,\mathrm{Im}\,(i^{-\delta_{I1}} V^1_{IM})\sin(M\phi_{\eta s}+\delta_{I1}\pi/2), \end{aligned} \tag{A14}$$

$$\begin{aligned} \mathrm{Re}\,(e^{i\psi_M} W_{IM}) &= \mathrm{Re}\,[e^{i(\psi_M-\delta_{I1}\pi/2)} i^{\delta_{I1}} W_{IM}] \\ &= i^{\delta_{I1}} W_{IM}\cos(\psi_M-\delta_{I1}\pi/2), \end{aligned} \tag{A15}$$

we find as the final form for the threefold semiexclusive differential cross section (60).

APPENDIX B: THE RECOIL POLARIZATION

For the recoil polarization, we have to evaluate according to (24) the quantities $B^{\pm}_{M'}$ of (25). From (39) we obtain for $M' = 0, 1$,

$$B^{\pm}_{M'} = \frac{(-1)^{M'}}{2(1+\delta_{M'0})} \sum_{I=0,1} P^p_I \sum_{M=-I}^{I} e^{iM\phi_{\eta s}} d^I_{M0}(\theta_s) \times \left[\widetilde{v}^{1\pm}_{M';IM} + \widetilde{v}^{-1\pm}_{M';IM} + P^\gamma_c\left(\widetilde{v}^{1\pm}_{M';IM} - \widetilde{v}^{-1\pm}_{M';IM}\right) + P^\gamma_l\left(\widetilde{w}^{1\pm}_{M';IM}e^{-2i\phi_{\eta\gamma}} + \widetilde{w}^{-1\pm}_{M';IM}e^{2i\phi_{\eta\gamma}}\right)\right], \tag{B1}$$

where for convenience we have defined

$$\widetilde{v}/\widetilde{w}^{\mu;\pm}_{M';IM} = v/w^{\mu}_{1M';IM} \pm v/w^{\mu}_{1-M';IM}. \tag{B2}$$

One should note that

$$\widetilde{v}/\widetilde{w}^{\mu;+}_{0;IM} = 2\, v/w^{\mu}_{10;IM} \quad \text{and} \quad \widetilde{v}/\widetilde{w}^{\mu;-}_{0;IM} = 0. \tag{B3}$$

These quantities obey the obvious property

$$\widetilde{v}/\widetilde{w}^{\mu;\pm}_{-M';IM} = \pm\widetilde{v}/\widetilde{w}^{\mu;\pm}_{M';IM}. \tag{B4}$$

Then one obtains as a final expression for the Cartesian nucleon recoil polarization components as defined in (24), including beam and target polarization contributions,

$$P_{x_i}\frac{d\sigma}{d\vec{q}_\eta\, d\Omega_{\pi p}} = \sum_{I=0,1} P^p_I \Bigg(\sum_{M=0}^{I} d^I_{M0}(\theta_s)\big\{\tau^0_{x_i;IM}\cos(M\phi_{\eta s}) + \sigma^0_{x_i;IM}\sin(M\phi_{\eta s}) + P^\gamma_c\big[\tau^c_{x_i;IM}\cos(M\phi_{\eta s}) + \sigma^c_{x_i;IM}\sin(M\phi_{\eta s})\big]\big\} + P^\gamma_l \sum_{M=-I}^{I} d^I_{M0}(\theta_s)\big(\tau^l_{x_i;IM}\cos\psi_M + \sigma^l_{x_i;IM}\sin\psi_M\big)\Bigg), \tag{B5}$$

where the various beam, target, and beam-target asymmetries are given by

$$\tau/\sigma^0_{x/y;IM} = \mp\frac{1}{\sqrt{2}(1+\delta_{M0})}\text{Re/Im}\left(\widetilde{v}^{1;-}_{1;IM} + \widetilde{v}^{-1;-}_{1;IM}\right), \tag{B6}$$

$$\tau/\sigma^c_{x/y;IM} = \mp\frac{1}{\sqrt{2}}\text{Re/Im}\left(\widetilde{v}^{1;-}_{1;IM} - \widetilde{v}^{-1;-}_{1;IM}\right), \tag{B7}$$

$$\tau/\sigma^l_{x/y;IM} = \mp\frac{1}{\sqrt{2}}\text{Re/Im}\left(\widetilde{w}^{1;-}_{1;IM}\right). \tag{B8}$$

$$\tau/\sigma^0_{z;IM} = \frac{1}{2(1+\delta_{M0})}\text{Re/Im}\left(\widetilde{v}^{1;+}_{0;IM} + \widetilde{v}^{-1;+}_{0;IM}\right) = \frac{1}{1+\delta_{M0}}\text{Re/Im}\left(v^1_{10;IM} + v^{-1}_{10;IM}\right), \tag{B9}$$

$$\tau/\sigma^c_{z;IM} = \tfrac{1}{2}\text{Re/Im}\left(\widetilde{v}^{1;+}_{0;IM} - \widetilde{v}^{-1;+}_{0;IM}\right) = \text{Re/Im}\left(v^1_{10;IM} - v^{-1}_{10;IM}\right), \tag{B10}$$

$$\tau/\sigma^l_{z;IM} = \tfrac{1}{2}\text{Re/Im}\left(\widetilde{w}^{1;+}_{0;IM}\right) = \text{Re/Im}\left(w^1_{10;IM}\right), \tag{B11}$$

where we have used (B3) for P_z.

APPENDIX C: THE EXPANSION COEFFICIENTS

The first two coefficients in the Fourier expansion of W^c in Eq. (105) may be derived from the general expressions in Eqs. (76), (77), and (78). Using the actual resonance quantum numbers, one obtains after straightforward manipulations

$$A_1 = \frac{\pi}{\sigma}\left[\left(A^{D_{33}}_{3/2}\right)^2 + \frac{1}{3}\left(A^{D_{33}}_{1/2}\right)^2\right]\int \text{Im}\left(c^{*(1)}_{D_{33}}c^{(2)}_{D_{33}}\right) \times \sin^2\theta^*_{\pi p}\, d\theta^*_{\pi p}\, d\omega_\eta, \tag{C1}$$

$$A_2 = -\frac{\pi}{\sigma}\left(F_{D_{33}P_{31}} + F_{D_{33}P_{33}} + F_{D_{33}F_{35}}\right). \tag{C2}$$

The individual terms on the right-hand side of Eq. (C2) read

$$F_{D_{33}P_{31}} = -\frac{4}{3\sqrt{3}}A^{D_{33}}_{1/2}A^{P_{31}}_{1/2}\int \text{Im}\left(c^{*(1)}_{D_{33}}c^{(\eta\Delta)}_{P_{31}}\right) \times \sin^2\theta^*_{\pi p}\, d\theta^*_{\pi p}\, d\omega_\eta, \tag{C3}$$

$$F_{D_{33}P_{33}} = \frac{8}{3\sqrt{15}}\left(A^{D_{33}}_{3/2}A^{P_{33}}_{3/2} - A^{D_{33}}_{1/2}A^{P_{33}}_{1/2}\right)\int \text{Im}\Bigg[c^{*(1)}_{D_{33}}c^{(\eta\Delta)}_{P_{33}} + \sqrt{\frac{2}{3}}\left(c^{*(2)}_{D_{33}}c^{(\pi N^*)}_{P_{33}}p^2_{\pi p} - c^{*(1)}_{D_{33}}c^{(\pi N^*)}_{P_{33}}X_\pi q_\eta p_{\pi p}\right)\Bigg] \times \sin^2\theta^*_{\pi p}\, d\theta^*_{\pi p}\, d\omega_\eta, \tag{C4}$$

$$F_{D_{33}F_{35}} = -\frac{1}{\sqrt{15}}\left(\sqrt{6}A^{D_{33}}_{3/2}A^{F_{35}}_{3/2} + A^{D_{33}}_{1/2}A^{F_{35}}_{1/2}\right) \times \int \text{Im}\Bigg[c^{*(1)}_{D_{33}}c^{(\eta\Delta)}_{F_{35}} - 2\Big(c^{*(2)}_{D_{33}}c^{(\pi N^*)}_{F_{35}}p^2_{\pi p} - c^{*(1)}_{D_{33}}c^{(\pi N^*)}_{F_{35}}X_\pi q_\eta p_{\pi p}\Big)\Bigg]\sin^2\theta^*_{\pi p}\, d\theta^*_{\pi p}\, d\omega_\eta, \tag{C5}$$

where $X_\pi = m_\pi/(M_p + m_\pi)$. In the expressions above, $\vec{p}_{\pi p}$ is, as previously, the relative πp momentum. The factors $c^{(\alpha)}_R$ $\alpha \in \{\eta\Delta, \pi N^*\}$ appear in the general ansatz for the resonance amplitudes in Eq. (76). For convenience we have introduced in Eqs. (C3)–(C5) the following notations for the combinations of the coefficients $c^{(\alpha)}_{D_{33}}$:

$$c^{(1)}_{D_{33}} = c^{(\eta\Delta)}_{D_{33}} + \frac{p}{q_\pi}c^{(\pi N^*)}_{D_{33}}, \quad c^{(2)}_{D_{33}} = -\frac{q_\eta}{q_\pi}X_\pi c^{(\pi N^*)}_{D_{33}}. \tag{C6}$$

APPENDIX D: THE *T*-MATRIX FOR AN ACTIVE PROTON

For an active proton, the partial-wave decomposition of the final state reads

$${}^{(-)}\langle\vec{q}_{\pi\eta}| = \frac{1}{\sqrt{4\pi}}\sum_{l_{\pi\eta}m_{\pi\eta}}\widehat{l}_{\pi\eta}\, D^{l_{\pi\eta}}_{0,m_{\pi\eta}}(\phi_{\pi\eta}, -\theta_{\pi\eta}, -\phi_{\pi\eta}) \times {}^{(-)}\langle q_{\pi\eta}l_{\pi\eta}m_{\pi\eta}|, \tag{D1}$$

$${}^{(-)}\langle\vec{p}_p m_f| = \frac{1}{\sqrt{4\pi}}\sum_{l_p j_p m_p}\widehat{l}_p\left(l_p 0\frac{1}{2}m_f|j_p m_f\right) \times D^{j_p}_{m_f,m_p}(\phi_p, -\theta_p, -\phi_p)\,{}^{(-)}\left\langle p_p\left(l_p\frac{1}{2}\right)j_p m_p\right|, \tag{D2}$$

where again $m_{\pi\eta}$ and m_p refer to the photon momentum $\vec{k}$ as the quantization axis. Then we follow the same steps as in Eqs. (10)–(14). With the help of the multipole decomposition and the Wigner-Eckart theorem, one obtains

$$^{(-)}\left\langle q_{\pi\eta}\, l_{\pi\eta} m_{\pi\eta};\, p_p \left(l_p \frac{1}{2}\right) j_p m_p \right| O_\mu^{\mu L} \left| \frac{1}{2} m_i \right\rangle = \sum_{JM_J} (-1)^{l_{\pi\eta}-j_p+J}\, \widehat{J} \begin{pmatrix} l_{\pi\eta} & j_p & J \\ m_{\pi\eta} & m_p & -M_J \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ -M_J & \mu & m_i \end{pmatrix} \times \left\langle q_{\pi\eta}\, p_p; [l_{\pi\eta}(l_p s) j_p] J \| O^{\mu L} \| \frac{1}{2} \right\rangle, \tag{D3}$$

with the selection rule $m_p + m_{\pi\eta} = M_J = \mu + m_i$. Rewriting the angular dependence,

$$D^{j_p}_{m_f,m_p}(\phi_p, -\theta_p, -\phi_p) D^{l_{\pi\eta}}_{0,m_{\pi\eta}}(\phi_{\pi\eta}, -\theta_{\pi\eta}, -\phi_{\pi\eta}) = d^{j_p}_{m_f,m_p}(-\theta_p) d^{l_{\pi\eta}}_{0,m_{\pi\eta}}(-\theta_{\pi\eta}) e^{i[(m_p-m_f)\phi_p + m_{\pi\eta}\phi_{\pi\eta}]}, \tag{D4}$$

and rearranging

$$(m_p - m_f)\phi_p + m_{\pi\eta}\phi_{\pi\eta} = m_{\pi\eta}\phi_{pq} + (\mu + m_i - m_f)\,\phi_p \tag{D5}$$

with $\phi_{pq} = \phi_{\pi\eta} - \phi_p$, one finds that the dependence on ϕ_p can be separated, i.e.,

$$T_{m_f \mu m_i}(\Omega_p, \Omega_{\pi\eta}) = e^{i(\mu+m_i-m_f)\phi_p}\, t_{m_f \mu m_i}(\theta_p, \theta_{\pi\eta}, \phi_{pq}), \tag{D6}$$

where the small t matrix depends only on θ_p, $\theta_{\pi\eta}$, and the relative azimuthal angle ϕ_{pq}.

The explicit form for the t matrix in the case of an active proton then reads

$$t_{m_f \mu m_i}(\theta_{\pi\eta}, \theta_p, \phi_{pq}) = \frac{1}{2\sqrt{2\pi}} \sum_{L l_{\pi\eta} m_{\pi\eta} l_p j_p m_p J J M_J} i^L \widehat{L}\widehat{J}\,\widehat{l_p}\,\widehat{j_p}\,\widehat{l_{\pi\eta}} \times (-1)^{J+l_{\pi\eta}-\frac{1}{2}+m_f-l_p-j_p} \begin{pmatrix} l_p & \frac{1}{2} & j_p \\ 0 & m_f & -m_f \end{pmatrix} \times \begin{pmatrix} l_{\pi\eta} & j_p & J \\ m_{\pi\eta} & m_p & -M_J \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ -M_J & \mu & m_i \end{pmatrix} \times \left\langle p_{\pi\eta} p_p; \left[l_{\pi\eta} \left(l_p \frac{1}{2}\right) j_p \right] J \| O^{\mu L} \| \frac{1}{2} \right\rangle \times d^{l_{\pi\eta}}_{0,m_{\pi\eta}}(-\theta_{\pi\eta}) d^{j_p}_{m_f,m_p}(-\theta_p) e^{i m_{\pi\eta}\phi_{pq}}. \tag{D7}$$

Parity transformation leads to the following property of the reduced matrix element:

$$(-1)^{l_{\pi\eta}+l_p+L} \left\langle p_{\pi\eta} p_p; \left[l_{\pi\eta} \left(l_{\pi p} \tfrac{1}{2}\right) j_{\pi p} \right] J \| \mathcal{O}^{-\mu L} \| \tfrac{1}{2} \right\rangle = \left\langle p_{\pi p} p_p; \left[l_{\pi\eta} \left(l_{\pi p} \tfrac{1}{2}\right) j_{\pi p} \right] J \| \mathcal{O}^{\mu L} \| \tfrac{1}{2} \right\rangle, \tag{D8}$$

which in turn gives the symmetry property of Eq. (15).

[1] I. Horn *et al.*, Eur. Phys. J. A **38**, 173 (2008).
[2] Ch. Weinheimer *et al.*, Nucl. Phys. A **721**, 781c (2003).
[3] T. Nakabayashi *et al.*, Phys. Rev. C **74**, 035202 (2006).
[4] J. Ajaka *et al.*, Phys. Rev. Lett. **100**, 052003 (2008).
[5] V. L. Kashevarov *et al.*, Eur. Phys. J. A **42**, 141 (2009).
[6] W. Roberts and T. Oed, Phys. Rev. C **71**, 055201 (2005).
[7] E. Gutz *et al.*, Eur. Phys. J. A **35**, 291 (2008).
[8] E. Gutz *et al.*, Phys. Lett. B **687**, 11 (2010).
[9] V. L. Kashevarov *et al.*, Phys. Lett. B **693**, 551 (2010).
[10] S. Strauch *et al.*, Phys. Rev. Lett. **95**, 162003 (2005).
[11] D. Krambrich *et al.*, Phys. Rev. Lett. **103**, 052002 (2009).
[12] L. Roca, Nucl. Phys. A **748**, 192 (2005).
[13] E. M. Rose, *Elementary Theory of Angular Momentum* (Wiley New York, 1957).
[14] B. A. Robson, *The Theory of Polarization Phenomena* (Clarendon, Oxford, 1974).
[15] H. Arenhövel and A. Fix, Phys. Rev. C **72**, 064004 (2005).
[16] M. Döring, E. Oset, and D. Strottman, Phys. Rev. C **73**, 045209 (2006).
[17] A. Fix, M. Ostrick, and L. Tiator, Eur. Phys. J. A **36**, 61 (2008).
[18] A. Fix, V. L. Kashevarov, A. Lee, and M. Ostrick, Phys. Rev. C **82**, 035207 (2010).
[19] J. A. Gomez Tejedor and E. Oset, Nucl. Phys. A **600**, 413 (1996).
[20] L. Y. Murphy and J. M. Laget, Report DAPHNIA/SPhN, 96-10 (1996).
[21] K. Ochi, M. Hirata, and T. Takaki, Phys. Rev. C **56**, 1472 (1997).
[22] M. Ripani *et al.*, Nucl. Phys. A **672**, 220 (2000).
[23] A. Fix and H. Arenhövel, Eur. Phys. J. A **25**, 115 (2005).
[24] M. Doring, E. Oset, and U. G. Meissner, Eur. Phys. J. A **46**, 315 (2010).