

# Quasiparticle theory of shear and bulk viscosities of hadronic matter

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A theoretical framework for the calculation of shear and bulk viscosities of hadronic matter at finite temperature is presented. The framework is based on the quasiparticle picture. It allows for an arbitrary number of hadron species with pointlike interactions, and allows for both elastic and inelastic collisions. Detailed balance is ensured. The particles have temperature-dependent masses arising from mean-field or potential effects, which maintains self-consistency between the equation of state and the transport coefficients. As an example, we calculate the shear and bulk viscosity in the linear  $\sigma$  model. The ratio of shear viscosity to entropy density shows a minimum in the vicinity of a rapid crossover transition, whereas the ratio of bulk viscosity to entropy density shows a maximum.

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## I. INTRODUCTION

One of the amazing *experimental* discoveries of measurements of heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) is the surprising amount of collective flow exhibited by the outgoing hadrons. Collective flow is observed in both the single-particle transverse momentum distribution [1] (radial flow) and in the asymmetric azimuthal distribution around the beam axis [2] (elliptic flow). It is now generally accepted that collective flow is mostly generated early in the nucleus-nucleus collision and is present before partons fragment or coalesce into hadrons [3]. The quark-gluon matter created in these collisions must be strongly interacting, unlike the type of weakly interacting quark-gluon plasma expected to occur at very high temperatures on the basis of asymptotic freedom [4]. Perfect fluid dynamics with zero viscosity reproduces the measurements of radial and elliptic flow quite well up to transverse momenta on the order  $1.5 \text{ GeV}/c$  [5]. These results have been interpreted as strong indicators of early thermalization and collective flow on a time scale of several  $\text{fm}/c$ .

An amazing *theoretical* discovery was made by Kovtun *et al.* [6]. They showed that certain special field theories [anti-deSitter/conformal field theory (AdS/CFT)] that are dual to black branes in higher space-time dimensions [7–9] have the ratio of shear viscosity to entropy density  $\eta/s = 1/4\pi$  (in natural units with  $\hbar = k_B = c = 1$ ). The connection between transport coefficients and gravity arises because both involve commutators of the stress-energy-momentum tensor. They conjectured that all substances have this value as a lower limit, and gave as examples various atomic and molecular systems. In fact, it had been argued much earlier that any substance should have a lower bound on  $\eta/s$  because of the uncertainty principle [10]. Are the RHIC data telling us that the created matter has a very small viscosity, the minimal value of  $\eta/s$ , that it is a *perfect fluid*?

The relatively good agreement between perfect fluid calculations and experimental data for hadrons of low-to-medium transverse momentum at RHIC suggests that the viscosity is small. However, it cannot be zero. Indeed, calculations within AdS/CFT suggest that  $\eta \geq s/4\pi$  [11–17]. Whether or not this is a rigorous lower bound was questioned in

[18–20]. In fact, this bound may be violated when higher order terms involving the Weyl tensor are added to the action [21–23]. A brief summary of these violations may be found in Ref. [24].

What is of even more import is that there are strong theoretical arguments, and evidence from atomic and molecular systems, that  $\eta/s$  should be a minimum in the vicinity of the phase transition or rapid crossover between hadronic matter and quark-gluon plasma<sup>1</sup> [25], and that the ratio of bulk viscosity to entropy density  $\zeta/s$  should be a maximum there [26] (see also Refs. [27–30]).

It ought to be possible to extract numerical values of the viscosities in heavy-ion collisions via scaling violations to perfect fluid flow predictions [31–34]. The program is to solve relativistic viscous fluid equations, with appropriate initial conditions and with a hadron cascade afterburner [35], over a range of beam energies and nuclei and extract  $\eta(T)/s(T)$  and  $\zeta(T)/s(T)$  from comparison with data. Thus, sufficiently precise calculations and measurements should allow for a determination of the ratio  $\eta/s$  as well as the ratio of bulk viscosity to entropy density  $\zeta/s$  as functions of temperature, and that these ratios can pinpoint the location of the phase transition or rapid crossover from hadronic to quark and gluon matter. This is a different method than trying to infer the equation of state of quantum chromodynamics (QCD) in the form of pressure  $P$  as a function of temperature  $T$  or energy density  $\epsilon$ . Because of advances in both theory and computation, vigorous activities are currently underway to determine the dissipative effects in heavy-ion collisions [36–39].

From the theoretical perspective, it should be possible to compute the shear and bulk viscosities directly from QCD at finite temperature. In practice, this is extremely difficult because QCD is generally a strongly interacting theory. Calculations can and have been done at extremely high

<sup>1</sup> According to the theory of dynamical critical phenomena the shear viscosity diverges at the critical point. A typical measurement on  $^3\text{He}$  gives an increase in  $\eta$  of 10% when  $T$  is within 1 part in  $10^4$  of  $T_c$ ; see C. C. Agosta, S. Wang, L. H. Cohen, and H. Meyer, *J. Low Temp. Phys.* **67**, 237 (1987). This divergence is highly unlikely to be observable in high-energy heavy-ion collisions.

temperatures where perturbation theory, applied to quarks and gluons, can be used on account of asymptotic freedom; see Ref. [40] for shear viscosity and Ref. [41] for bulk viscosity. At extremely low temperatures, perturbation theory can again be used because the matter consists only of a very dilute gas of pions, and low-energy pion dynamics is well understood. See Ref. [42] for massive pions; for massless pions, see Ref. [41] for shear viscosity and Ref. [43] for bulk viscosity. There have been a variety of other kinetic theory calculations of the shear and/or bulk viscosities at low-to-moderate temperatures in the literature in recent years [44–48]; these usually include only elastic scattering of one or a few species of hadrons.

In the intermediate region, which may be loosely defined as  $100 < T < 400$  MeV, neither the low nor high temperature approach is accurate. A few lattice QCD simulations have used the Kubo formulas [49] to compute the shear [50,51] and bulk [52] viscosities just above the critical temperature of pure gluon/glueball matter. However, accurate lattice QCD simulations of the properties of hadronic matter are extremely time consuming and the final results are still likely to be far in the future. The reason is that the lattice spacing  $a$  must be small enough to describe the properties of an individual hadron ( $a < 0.05$  fm) whereas the box size  $L$  must be large enough to contain many hadrons forming the dilute gas ( $L > 10$  fm). Hence the number of spatial lattice sites should be at least 200 in each direction. An interesting alternative approach to the intermediate region is a model of classical, nonrelativistic quasiparticles with color charges [53].

Our goal in this article is to provide a theoretical framework in which to calculate the shear and bulk viscosities of hadronic matter. This framework has the following features.

- (i) It is relativistic.
- (ii) It allows for an arbitrary number of hadron species.
- (iii) It allows for both elastic and inelastic collisions.
- (iv) It respects detailed balance.
- (v) It allows for mean fields and temperature-dependent masses.
- (vi) The viscosities and the equation of state are mutually consistent in the sense that the same interactions are used to compute them all.

Obviously some assumptions or approximations must be made for the theory to be applied in practice. The essential assumptions are that quasiparticles are well defined and that the elementary interactions are local. Thus our proposed theoretical framework goes well beyond the classic works of Refs. [54] and [42] which, although relativistic, considered only elastic collisions in dilute gases. The inclusion of not only resonances, but especially inelastic collisions, mean fields, and temperature-dependent masses are essential for an accurate determination of the bulk viscosity [55].

The outline of this article is as follows. In Sec. II we recall the basics of the Boltzmann transport equation. In Sec. III we derive the integral equations for the viscosities by using the Boltzmann equation. In Sec. IV we show how the Landau-Lifshitz condition plays a crucial role for the bulk viscosity. In Sec. V we work out formulas for the viscosities in the relaxation time approximation. In Sec. VI we generalize

the previous results to include mean-field or potential effects and their significance for the bulk viscosity. In Sec. VII we apply the framework to the linear  $\sigma$  model with massive pions. As expected, the ratio  $\eta/s$  has a minimum and the ratio  $\zeta/s$  has a maximum near the rapid crossover transition, which is more pronounced for larger vacuum  $\sigma$  masses. We conclude in Sec. VIII. The reader not interested in mathematical details is referred to Secs. VII and VIII and to the appendix where the main formulas are summarized.

Because the asymmetry between matter and antimatter in high-energy nuclear collisions at RHIC is very small, so are the baryon and electric charge chemical potentials. Therefore, thermal and electrical conductivity are neglected in this article. Their inclusion is straightforward but tedious, and work to include them is in progress.

For ease and clarity of presentation we will, for the most part, display formulas that include only  $2 \rightarrow 2$  reactions, both elastic and inelastic, and formation  $2 \rightarrow 1$  and decay  $1 \rightarrow 2$  of resonances. At certain key points in the article we simply write down formulas for the more general cases. In addition, in practice we generally use classical statistics. The lightest hadron for which this would be the most significant approximation is the pion, but even then the difference between Bose-Einstein and classical statistics have been shown to be inconsequential for the transport coefficients for zero chemical potential [56]. While some of the material here is not original, it is basic to developing the theoretical framework. We present it to make the article self-contained and to define our notation.

## II. BOLTZMANN EQUATION

The rate for  $2 \rightarrow 2$  processes, that is, the number of reactions per unit time per unit volume of the type  $a + b \rightarrow c + d$  is

$$\text{rate} = \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_a}{2E_a(2\pi)^3} \frac{d^3 p_b}{2E_b(2\pi)^3} \frac{d^3 p_c}{2E_c(2\pi)^3} \frac{d^3 p_d}{2E_d(2\pi)^3} \times |\mathcal{M}(a, b|c, d)|^2 (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d) \times f_a f_b [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \quad (1)$$

$$= \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_a}{(2\pi)^3} \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a, b|c, d) \times f_a f_b [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d], \quad (2)$$

whence

$$W(a, b|c, d) = \frac{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}{2E_a 2E_b 2E_c 2E_d} |\mathcal{M}(a, b|c, d)|^2. \quad (3)$$

Here  $s_a$  is the spin of particle  $a$ , etc., and the factor  $1/(1 + \delta_{ab})$  takes into account the possibility that the incoming particles are identical. The amplitude  $\mathcal{M}$  is dimensionless. There are either Bose-enhancement or Pauli-suppression factors in the final state. In the rest frame of the system the single-particle distributions  $f_a$  are normalized such that

$$\int \frac{d^3 p}{(2\pi)^3} f_a(\mathbf{x}, \mathbf{p}, t) = n_a(\mathbf{x}, t) \quad (4)$$

is the spatial density of particles of type  $a$ . In thermal equilibrium,

$$f_a(\mathbf{x}, \mathbf{p}, t) = \frac{1}{e^{(E_a - \mu_a)/T} - (-1)^{2s_a}}, \quad (5)$$

where  $T$  is the temperature and  $\mu_a$  is the chemical potential of the particle. It is obvious that the rate is a Lorentz scalar.

The rate for decay processes, that is, the number of decays per unit time per unit volume of the type  $a \rightarrow c + d$  is

$$\begin{aligned} \text{rate} &= \int \frac{d^3 p_a}{2E_a(2\pi)^3} \frac{d^3 p_c}{2E_c(2\pi)^3} \frac{d^3 p_d}{2E_d(2\pi)^3} |\mathcal{M}(a|c, d)|^2 (2\pi)^4 \\ &\quad \times \delta^4(p_a - p_c - p_d) f_a [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \\ &= \int \frac{d^3 p_a}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) f_a [1 + (-1)^{2s_c} f_c] \\ &\quad \times [1 + (-1)^{2s_d} f_d], \end{aligned} \quad (6)$$

whence

$$W(a|c, d) = \frac{(2\pi)^4 \delta^4(p_a - p_c - p_d)}{2E_a 2E_c 2E_d} |\mathcal{M}(a|c, d)|^2. \quad (7)$$

This  $\mathcal{M}$  has dimension of energy.

How do the  $W$ 's relate to cross sections and decay rates? The relationships for the cross sections are as follows:

$$\begin{aligned} \frac{d\sigma}{d\Omega^*} &= \frac{1}{64\pi^2 s} \frac{p_{\text{final}}^*}{p_{\text{initial}}^*} |\mathcal{M}|^2, \\ \frac{d\sigma}{dt} &= \frac{1}{64\pi s} \frac{1}{(p_{\text{initial}}^*)^2} |\mathcal{M}|^2. \end{aligned} \quad (8)$$

Here  $s$  and  $t$  are the Mandelstam variables (one can tell from the context whether  $t$  represents a Mandelstam variable or time). The differential cross section in the center-of-momentum frame is  $d\sigma/d\Omega^*$  whereas  $d\sigma/dt$  is usually written as a

function of the invariants  $s, t, u$ . Thus,

$$\begin{aligned} W(a, b|c, d) &= \frac{s}{E_a E_b E_c E_d} \frac{p_{\text{initial}}^*}{p_{\text{final}}^*} \frac{d\sigma}{d\Omega^*} \\ &\quad \times (2\pi)^6 \delta^4(p_a + p_b - p_c - p_d). \end{aligned} \quad (9)$$

This agrees with the literature on relativistic Boltzmann equations when only elastic collisions are considered because then  $p_{\text{final}}^* = p_{\text{initial}}^*$ . An alternative form, which may be more useful for cross sections that are not isotropic in the center-of-momentum frame, is

$$W(a, b|c, d) = \frac{2s(p_{\text{initial}}^*)^2}{E_a E_b E_c E_d} \frac{d\sigma}{dt} (2\pi)^5 \delta^4(p_a + p_b - p_c - p_d), \quad (10)$$

where

$$\begin{aligned} 4s(p_{\text{initial}}^*)^2 &= (s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2, \\ 4s(p_{\text{final}}^*)^2 &= (s - m_c^2 - m_d^2)^2 - 4m_c^2 m_d^2. \end{aligned} \quad (11)$$

For the decay  $a \rightarrow c + d$  consider the particle  $a$  at rest. It will decay according to the usual exponential law:

$$\frac{dn_a(t)}{dt} = -\Gamma_{a \rightarrow c+d} n_a(t). \quad (12)$$

One computes that

$$\Gamma_{a \rightarrow c+d} = \frac{p_{\text{final}}^*}{8\pi m_a^2} |\mathcal{M}(a|c, d)|^2, \quad (13)$$

where

$$4m_a^2 (p_{\text{final}}^*)^2 = (m_a^2 - m_c^2 - m_d^2)^2 - 4m_c^2 m_d^2, \quad (14)$$

so that

$$W(a|c, d) = \frac{\pi m_a^2}{E_a E_c E_d p_{\text{final}}^*} \Gamma_{a \rightarrow c+d} (2\pi)^4 \delta^4(p_a - p_c - p_d). \quad (15)$$

Now we consider the Boltzmann equation. Taking into account both gain and loss rates we write it as follows:

$$\begin{aligned} \frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \nabla f_a &= \sum_{bcd} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \left\{ \frac{1}{1 + \delta_{cd}} W(c, d|a, b) f_c f_d [1 + (-1)^{2s_a} f_a] [1 + (-1)^{2s_b} f_b] \right. \\ &\quad \left. - \frac{1}{1 + \delta_{ab}} W(a, b|c, d) f_a f_b [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \right\} \\ &\quad + \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \left\{ \frac{1}{1 + \delta_{cd}} W(c, d|a) f_c f_d [1 + (-1)^{2s_a} f_a] - W(a|c, d) f_a [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \right\} \\ &\quad + \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \left\{ W(c|a, b) f_c [1 + (-1)^{2s_a} f_a] [1 + (-1)^{2s_b} f_b] - \frac{1}{1 + \delta_{ab}} W(a, b|c) f_a f_b [1 + (-1)^{2s_c} f_c] \right\}. \end{aligned} \quad (16)$$

From detailed balance on the microscopic level, energy conservation, and chemical equilibrium as represented by  $\mu_a + \mu_b = \mu_c + \mu_d$  for two-body reactions and by  $\mu_a = \mu_c + \mu_d$  for two-body decays, we find relations between the forward and backward going rates:

$$(1 + \delta_{ab}) W(c, d|a, b) = (1 + \delta_{cd}) W(a, b|c, d), \quad (17)$$

$$W(c, d|a) = (1 + \delta_{cd}) W(a|c, d). \quad (18)$$

The Boltzmann equation then becomes

$$\begin{aligned} \frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \nabla f_a &= \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a, b|c, d) \\ &\times \left\{ f_c f_d [1 + (-1)^{2s_a} f_a] [1 + (-1)^{2s_b} f_b] - f_a f_b [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \right\} \\ &+ \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) \left\{ f_c f_d [1 + (-1)^{2s_a} f_a] - f_a [1 + (-1)^{2s_c} f_c] [1 + (-1)^{2s_d} f_d] \right\} \\ &+ \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} W(c|a, b) \left\{ f_c [1 + (-1)^{2s_a} f_a] [1 + (-1)^{2s_b} f_b] - f_a f_b [1 + (-1)^{2s_c} f_c] \right\}. \quad (19) \end{aligned}$$

For classical statistics, where the Bose and Pauli factors are dropped, the equilibrium phase-space distribution is

$$f_a^{\text{eq}}(\mathbf{x}, \mathbf{p}, t) = e^{-(E_a - \mu_a)/T}. \quad (20)$$

The Boltzmann equation then shortens somewhat:

$$\begin{aligned} \frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \nabla f_a &= \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \\ &\times W(a, b|c, d) \{f_c f_d - f_a f_b\} \\ &+ \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) \{f_c f_d - f_a\} \\ &+ \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} W(c|a, b) \{f_c - f_a f_b\}. \quad (21) \end{aligned}$$

The generalization to arbitrary reactions  $\{i\} \rightarrow \{j\}$  with  $n$  particles in the initial state and  $m$  particles in the final state is now clear. In obvious notation the Boltzmann equation is

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \nabla f_a = \sum_{\{i\}\{j\}} \frac{1}{S} \int' dP_i dP_j W(\{i\}|\{j\}) F[f], \quad (22)$$

where the prime indicates that there is no integration over the momentum of  $a$ . There is a statistical factor for identical particles in the initial state,

$$S = \prod_i n_i!, \quad (23)$$

and products of Bose-Einstein and Fermi-Dirac distributions as appropriate,

$$F[f] = \prod_i \prod_j \{f_j [1 + (-1)^{s_i} f_i] - f_i [1 + (-1)^{s_j} f_j]\}. \quad (24)$$

### III. VISCOSITIES

Now we use the Boltzmann equation to calculate the viscosities. We restrict ourselves to zero chemical potentials. We assume that the system is in approximately local equilibrium, with local temperature  $T(x)$  and flow velocity  $U^\mu(x)$ . In the Landau-Lifshitz approach,  $U^\mu(x)$  is the velocity of energy transport whereas in the Eckart approach,  $U^\mu(x)$  would be

the velocity of baryon number flow [57,58]. However, the net baryon number, electric charge, and all other conserved quantum numbers are taken to be zero. Therefore one cannot use the Eckart approach. Another consequence of all conserved quantum numbers being zero is that thermal conductivity has no meaning.

The symmetric energy-momentum tensor is written as

$$T^{\mu\nu} = -P g^{\mu\nu} + w U^\mu U^\nu + \Delta T^{\mu\nu}, \quad (25)$$

where  $P = P(T)$  is pressure,  $s = dP/dT$  is entropy density,  $\epsilon = -P + Ts$  is energy density, and  $w = Ts = P + \epsilon$  is enthalpy density. These are all measured in a frame in which the fluid is instantaneously at rest. The  $\Delta T^{\mu\nu}$  is the dissipative part. It satisfies the condition,

$$U_\mu \Delta T^{\mu\nu} = 0, \quad (26)$$

on account of the Landau-Lifshitz definition of flow. The entropy current is

$$s^\mu = s U^\mu, \quad (27)$$

and is conserved if dissipative terms are neglected. The most general form of  $\Delta T^{\mu\nu}$  is given by

$$\Delta T^{\mu\nu} = \eta (D^\mu U^\nu + D^\nu U^\mu + \frac{2}{3} \Delta^{\mu\nu} \partial_\rho U^\rho) - \zeta \Delta^{\mu\nu} \partial_\rho U^\rho. \quad (28)$$

Here

$$\Delta^{\mu\nu} = U^\mu U^\nu - g^{\mu\nu} \quad (29)$$

is a projection tensor normal to  $U^\mu$ , and

$$D_\mu = \partial_\mu - U_\mu U^\beta \partial_\beta \quad (30)$$

is a derivative normal to  $U^\mu$ . The  $\eta$  is the shear viscosity and the  $\zeta$  is the bulk viscosity. In the local rest frame of the fluid,

$$\Delta^{0\nu} = 0, \quad \Delta^{ij} = \delta^{ij}, \quad (31)$$

and

$$D_0 = 0, \quad D_i = \partial_i. \quad (32)$$

In this frame,

$$\partial_\mu s^\mu = \frac{\eta}{2T} \left( \partial_i U^j + \partial_j U^i - \frac{2}{3} \delta^{ij} \nabla \cdot \mathbf{U} \right)^2 + \frac{\zeta}{T} (\nabla \cdot \mathbf{U})^2. \quad (33)$$

Nondecrease of entropy requires that both viscosities be non-negative.

We are assuming that the interactions are well localized in space and time. We are assuming that they are, for practical purposes, point or contact interactions. Then the energy-momentum tensor is written as a sum of independent contributions:

$$T^{\mu\nu}(x) = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{p_a^\mu p_a^\nu}{E_a} f_a(x, p). \quad (34)$$

Allow the system to be slightly out of equilibrium. Then we can write

$$f_a(x, p) = f_a^{\text{eq}}(U_\alpha p^\alpha / T) [1 + \phi_a(x, p)], \quad (35)$$

and so

$$\Delta T^{\mu\nu} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{p_a^\mu p_a^\nu}{E_a} f_a^{\text{eq}}(U_\alpha p^\alpha / T) \phi_a(x, p), \quad (36)$$

where  $|\phi_a| \ll 1$ . There is a constraint on  $\phi_a(x, p)$  in that the Landau-Lifshitz condition (26) must be satisfied. It is customary and natural to use the same tensorial decomposition of (28) when expressing  $\phi_a(x, p)$  as a function of space time and momentum.

$$\phi_a = -A_a \partial_\rho U^\rho + C_{\mu\nu}^a (D^\mu U^\nu + D^\nu U^\mu + \frac{2}{3} \Delta^{\mu\nu} \partial_\rho U^\rho). \quad (37)$$

Here  $A_a$  in general will depend on the scalar  $U_\alpha p^\alpha$ . The tensor  $C_{\mu\nu}^a$  could in principle be a linear combination of  $g_{\mu\nu}$  and  $p_\mu p_\nu$ . However, the former gives zero contribution. Therefore we write  $C_{\mu\nu}^a = C_a p_\mu p_\nu$  where  $C_a$  will in general depend on the scalar  $U_\alpha p^\alpha$ .

It is worthwhile emphasizing that the expansion of  $\Delta T^{\mu\nu}$  and  $\phi_a$  in terms of the first order derivatives of the flow velocity is only an approximation. It is referred to as the first order dissipative fluid dynamics. Inclusion of second order derivatives goes under the names of Müller and Israel and Stewart. The second order theory is under intense investigation because of its usefulness in describing high-energy nuclear collisions where space-time gradients are not necessarily small. It is also worth emphasizing that these same quantities are zero for an equilibrated system in uniform flow.

It is now a straightforward matter to equate the two expressions for the dissipative part of the energy-momentum tensor. It is advantageous to work in the local rest frame of the fluid.

$$\zeta = \frac{1}{3} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^2}{E_a} f_a^{\text{eq}}(E_a/T) A_a(E_a), \quad (38)$$

$$\eta = \frac{2}{15} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^4}{E_a} f_a^{\text{eq}}(E_a/T) C_a(E_a). \quad (39)$$

How do we determine the  $A_a$  and  $C_a$ ? The idea is to use the Boltzmann equation where the term  $\partial f_a / \partial t + \mathbf{v}_a \cdot \nabla f_a$  is evaluated using the local equilibrium distribution  $f_a^{\text{eq}}(U_\alpha p^\alpha / T)$ . It will act as a source for the collision term on the other side of the Boltzmann equation. With classical statistics the Boltzmann

equation reads as follows:

$$\begin{aligned} E_a^{-1} p_a^\mu \partial_\mu f_a^{\text{eq}} &= f_a^{\text{eq}} \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \\ &\times f_b^{\text{eq}} W(a, b|c, d) \{\phi_c + \phi_d - \phi_a - \phi_b\} \\ &+ f_a^{\text{eq}} \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) \\ &\times \{\phi_c + \phi_d - \phi_a\} + \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \\ &\times f_c^{\text{eq}} W(c|a, b) \{\phi_c - \phi_a - \phi_b\}. \end{aligned} \quad (40)$$

The first task is to compute the left-hand side of the Boltzmann equation. With  $f_a^{\text{eq}} = \exp(-U_\nu p^\nu / T)$  we have

$$\partial_\mu f_a^{\text{eq}} = -\frac{1}{T} f_a^{\text{eq}} p^\nu \left( \partial_\mu U_\nu - \frac{1}{T} U_\nu \partial_\mu T \right). \quad (41)$$

Using the conservation equations for energy and momentum,  $\partial_\nu T^{\mu\nu} = 0$ , and entropy (since the viscous terms are neglected at this order),  $\partial_\mu s^\mu = 0$ , we may deduce that

$$\Delta^{\mu\nu} \frac{1}{T} \partial_\nu T = -U^\alpha \partial_\alpha U^\mu, \quad (42)$$

which has solution

$$\frac{1}{T} \partial_\mu T = U^\alpha \partial_\alpha U_\mu + \xi U_\mu \partial_\alpha U^\alpha, \quad (43)$$

where  $\xi$  is a function of  $T$  which is undetermined by Eq. (42). It can be determined by substituting the previous expression into the conservation equations. This gives  $\xi = -s/c_V = -v_s^2$ , where  $c_V = d\epsilon/dT = T ds/dT$  is the heat capacity per unit volume and  $v_s^2 = dP/d\epsilon$  is the square of the sound velocity. Thus,

$$\begin{aligned} p^\mu \partial_\mu f_a^{\text{eq}} &= -\frac{1}{T} f_a^{\text{eq}} p^\mu p^\nu [\partial_\mu U_\nu - U_\nu (U^\alpha \partial_\alpha) U_\mu] \\ &= -\frac{1}{2T} f_a^{\text{eq}} p^\mu p^\nu \left[ \left( D_\mu U_\nu + D_\nu U_\mu + \frac{2}{3} \Delta_{\mu\nu} \partial_\rho U^\rho \right) \right. \\ &\quad \left. - \frac{2}{3} \Delta_{\mu\nu} \partial_\rho U^\rho + 2v_s^2 U_\mu U_\nu \partial_\rho U^\rho \right]. \end{aligned} \quad (44)$$

After substituting in the structure of the  $\phi$ 's and grouping terms we get

$$\mathcal{A}^a(\partial_\rho U^\rho) - C_{\mu\nu}^a (D^\mu U^\nu + D^\nu U^\mu + \frac{2}{3} \Delta^{\mu\nu} \partial_\rho U^\rho) = 0, \quad (45)$$

where

$$\begin{aligned} \mathcal{A}^a &= \frac{1}{3E_a T} [(p_a^\alpha U_\alpha)^2 (1 - 3v_s^2) - m_a^2] \\ &+ \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} f_b^{\text{eq}} W(a, b|c, d) \\ &\times \{A_c + A_d - A_a - A_b\} + \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \\ &\times W(a|c, d) \{A_c + A_d - A_a\} + \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \\ &\times f_b^{\text{eq}} W(c|a, b) \{A_c - A_a - A_b\}, \end{aligned} \quad (46)$$

and

$$\begin{aligned}
C_a^{\mu\nu} &= \frac{p_a^\mu p_a^\nu}{2E_a T} + \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) \\
&\times \{C_c p_c^\mu p_c^\nu + C_d p_d^\mu p_d^\nu - C_a p_a^\mu p_a^\nu\} \\
&+ \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} f_b^{\text{eq}} W(c|a, b) \\
&\times \{C_a p_a^\mu p_a^\nu + C_b p_b^\mu p_b^\nu - C_c p_c^\mu p_c^\nu\} \\
&+ \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} f_b^{\text{eq}} W(a, b|c, d) \\
&\times \{C_c p_c^\mu p_c^\nu + C_d p_d^\mu p_d^\nu - C_a p_a^\mu p_a^\nu - C_b p_b^\mu p_b^\nu\}. \quad (47)
\end{aligned}$$

The bulk viscosity is very small or zero in several limits. The first is the conformal limit, which means that the theory has no dimensional parameters, such as mass or intrinsic energy scale. Then  $P \sim T^4$  and  $v_s^2 = 1/3$ , and so the first term on the right-hand side of Eq. (46), the source term, vanishes and so do the  $A_a$ . The second is the nonrelativistic limit of a single species of particle. Then  $P \sim m^{3/2} T^{5/2} \exp(-m/T)$  and  $v_s^2 = T/m$  (plus corrections of higher order in  $T/m$ ). Once again the first term on the right-hand side of Eq. (46) vanishes (to lowest order in  $T/m$ ) and so the bulk viscosity should be very small. These arguments do not apply to the shear viscosity because the source term does not involve the equation of state.

#### IV. LANDAU-LIFSHITZ CONDITION

Equation (46) does not have a unique solution as it stands. For example, consider elastic scattering for just one type of particle. Starting with one solution  $A(E)$  we can generate an infinite number of other solutions by making the shift  $A(E) \rightarrow A'(E) = A(E) - a - bE$ , where  $a$  and  $b$  are arbitrary constants. These constants are associated with particle conservation ( $a$ ) and energy conservation ( $b$ ). This was noted in the literature before. It may be restated in more physical terms. To return a system to kinetic and chemical equilibrium after a change in volume, one might either change the number of particles while keeping the average energy per particle fixed, or one might change the average energy per particle while keeping the total number of particles fixed. Now it is apparent that this ambiguity is associated with the Landau-Lifshitz condition (26), which is also sometimes called the condition of fit when solving Eq. (46).

Consider an arbitrary set of particle species and all possible reactions allowed by the symmetries. Make the shift  $A_a(E_a) \rightarrow A'_a(E_a) = A_a(E_a) - a_a - bE_a$ . The constant  $b$  must be the same for all species of particle. The constants  $a_a$  are just like chemical potentials; they satisfy the same relationships among themselves. Because we are restricting our considerations to systems with zero net quantum numbers, such as electric charge and baryon number, it is obvious that the  $a_a$  are all zero, just as all chemical potentials are zero. The constant  $b$  acts like an inverse temperature and is as yet undetermined.

Suppose that we have a particular solution  $A_a^{\text{par}}$  to Eq. (46); does it satisfy the Landau-Lifshitz condition (26)? The general

solution would be  $A_a(E_a) = A_a^{\text{par}}(E_a) - bE_a$ . Using Eq. (36) the Landau-Lifshitz condition for the  $A$  term is

$$\sum_a \int \frac{d^3 p}{(2\pi)^3} f_a^{\text{eq}}(E_a/T) E_a [A_a^{\text{par}}(E_a) - bE_a] = 0. \quad (48)$$

Here it is useful to know the contributions to the pressure, energy density, entropy density, and heat capacity from a single species of particle.

$$\begin{aligned}
P_a &= T \int \frac{d^3 p}{(2\pi)^3} f_a^{\text{eq}}(E_a/T), \\
\epsilon_a &= \int \frac{d^3 p}{(2\pi)^3} E_a f_a^{\text{eq}}(E_a/T), \\
s_a &= \frac{1}{3T^2} \int \frac{d^3 p}{(2\pi)^3} |\mathbf{p}|^2 f_a^{\text{eq}}(E_a/T), \\
c_{V,a} &= \frac{1}{T^2} \int \frac{d^3 p}{(2\pi)^3} E_a^2 f_a^{\text{eq}}(E_a/T).
\end{aligned} \quad (49)$$

Now the coefficient  $b$  is determined in terms of integrals of the particular solutions:

$$b = \frac{1}{T^2 c_V} \sum_a \int \frac{d^3 p}{(2\pi)^3} f_a^{\text{eq}}(E_a/T) E_a A_a^{\text{par}}(E_a). \quad (50)$$

If the particular solutions already happen to satisfy the Landau-Lifshitz condition, then  $b = 0$ . Substitution of  $A_a(E_a) = A_a^{\text{par}}(E_a) - bE_a$  into Eq. (38), with  $b$  as determined previously, gives an expression for the bulk viscosity:

$$\zeta = \frac{1}{3} \sum_a \int \frac{d^3 p}{(2\pi)^3} E_a f_a^{\text{eq}}(E_a/T) A_a^{\text{par}}(E_a) (|\mathbf{p}|^2 - 3v_s^2 E_a^2). \quad (51)$$

Notice that if the particular solutions happen to satisfy the Landau-Lifshitz condition then Eq. (51) reduces to Eq. (38).

There is no ambiguity with the  $C_a$  in the shear viscosity because of the tensorial structure of the integrand in Eq. (47). Physically the reason has to do with the fact that shear viscosity is associated with the response to changes in shape at fixed volume whereas bulk viscosity is associated with the response to changes in volume at fixed shape.

#### V. RELAXATION TIME APPROXIMATION

Consider the Boltzmann equation [Eq. (21)]. Let us suppose that all species of particles for all values of momentum are in equilibrium except for species  $a$  with momentum  $\mathbf{p}_a$ . Replace all phase-space distributions  $f$  with their equilibrium values  $f^{\text{eq}}$  except for  $f_a$ , which we allow to be out of equilibrium by a small amount. Thus we write  $f_a = f_a^{\text{eq}} + \delta f_a$ . This is the momentum-dependent relaxation time approximation. We approximate the Boltzmann equation by

$$\frac{\partial f_a(\mathbf{x}, t, \mathbf{p}_a)}{\partial t} + \mathbf{v}_a \cdot \nabla f_a(\mathbf{x}, t, \mathbf{p}_a) = -\omega_a(E_a) \delta f_a(\mathbf{x}, t, \mathbf{p}_a), \quad (52)$$

where

$$\begin{aligned} \omega_a(E_a) = & \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a, b|c, d) f_b^{\text{eq}} \\ & + \sum_{cd} \int \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a|c, d) \\ & + \sum_{bc} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} W(c|a, b) f_b^{\text{eq}} \end{aligned} \quad (53)$$

is the frequency of interaction. The equilibration time is defined as

$$\tau_a(E) = \omega_a^{-1}(E). \quad (54)$$

The deviation  $\delta f_a$  is related to the function  $\phi_a$  defined in Eq. (35) by

$$\delta f_a(x, p) = f_a^{\text{eq}}(x, p) \phi(x, p). \quad (55)$$

Therefore we can substitute Eqs. (37) and (44) into Eq. (52) to solve for the functions  $A_a^{\text{par}}$  and  $C_a$ , where  $C_a^{\mu\nu} = C_a p_a^\mu p_a^\nu$ .

$$A_a^{\text{par}}(E_a) = \frac{\tau_a(E_a)}{3T E_a} [(1 - 3v_s^2) E_a^2 - m_a^2], \quad (56)$$

$$C_a(E_a) = \frac{\tau_a(E_a)}{2T E_a}. \quad (57)$$

The viscosities are now readily calculated using these results. As usual, it is advantageous to work in the local rest frame of the fluid:

$$\begin{aligned} \zeta = & \frac{1}{9T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{\tau_a(E_a)}{E_a^2} [(1 - 3v_s^2) E_a^2 - m_a^2]^2 \\ & \times f_a^{\text{eq}}(E_a/T), \end{aligned} \quad (58)$$

$$\eta = \frac{1}{15T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^4}{E_a^2} \tau_a(E_a) f_a^{\text{eq}}(E_a/T). \quad (59)$$

In the relaxation time approximation one must calculate the momentum-dependent relation time or rate from Eq. (53) and then substitute into the previous expressions and perform a one-dimensional integration. These are generalizations of the formulas given in Ref. [59] to an arbitrary number of species of particles with energy-dependent relaxation times.

As a further approximation one may calculate a mean interaction frequency  $\bar{\omega}_a$  and an associated mean relaxation time  $\bar{\tau}_a = \bar{\omega}_a^{-1}$  via

$$\begin{aligned} \bar{\omega}_a = & \frac{1}{n_a} \int \frac{d^3 p}{(2\pi)^3} \omega_a(E_a) f_a^{\text{eq}}(E_a/T), \\ n_a = & \int \frac{d^3 p}{(2\pi)^3} f_a^{\text{eq}}(E_a/T). \end{aligned} \quad (60)$$

However, there is really no need to make this approximation unless one only desires a rough order of magnitude estimate. Depending on the dynamics, the relaxation time may be highly momentum dependent. In either case it is clear that the particles with the longest relaxation time dominate the viscosities, because these particles can transport energy and momentum over greater distances before interacting.

Now comes a subtle point. What if we were to include weak interactions in our considerations? Clearly the relaxation times for weak interactions are orders of magnitude greater than the relaxation times for the strong interactions, so they would dominate the viscosities. The answer is that one must evaluate the actual physical conditions to which the viscous fluid equations are to be applied. For example, in high-energy nuclear collisions the size of the system is on the order of 10 fm whereas the lifetime is of order 10 fm/c. Any electrons, positrons, or neutrinos that might be produced by this system will simply escape and not interact with any of the hadrons. They cannot transport energy and momentum to another part of the system. In addition, due to the weakness of the interaction very few of them will actually be produced. Thus the weak interactions are irrelevant in this situation. The environment in the early universe or supernovae will most likely require inclusion of the effects of the weak interactions because the length and time scales are so much greater.

## VI. MEAN-FIELD OR POTENTIAL EFFECTS

Most hadronic models of hot matter involve temperature-dependent mean fields, such as the Walecka model or the linear sigma model. In the absence of chemical potentials there should be no condensation of vector or tensor fields, only scalar fields. Then the single-particle energies have the form  $E_a = \sqrt{\mathbf{p}^2 + \bar{m}_a^2(T)}$  where  $\bar{m}_a(T)$  is a temperature-dependent effective mass that arises from the mean fields, in other words, potential energy effects. This affects the bulk viscosity, but not the shear viscosity, in several ways.

The Boltzmann equation acquires an extra term on the left side of Eq. (16):

$$\left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_a} \cdot \nabla_x - \nabla_x E_a \cdot \nabla_p \right) f_a(\mathbf{x}, t, \mathbf{p}) = C[f], \quad (61)$$

where  $C[f]$  represents the collision, formation, and decay terms. The extra term involves the force  $\mathbf{F} = d\mathbf{p}/dt = -\nabla_x E$ . To calculate the viscosities the left side is evaluated with the local equilibrium distribution,

$$f_a^{\text{eq}} = \exp[-U_a(x) p_a^\alpha(x)/T(x)]. \quad (62)$$

Now  $p_a^0 = E_a$  depends on  $x$  because  $m$  depends on  $T$  which depends on  $x$ . This approach is not new but was proven or justified many times in the past; see, for example, Refs. [60,61].

A straightforward calculation gives

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_a} \cdot \nabla_x - \nabla_x E_a \cdot \nabla_p \right) f_a^{\text{eq}} \\ & = -\frac{1}{E_a T} f_a^{\text{eq}} \left[ p^\mu p^\nu \left( \partial_\mu U_\nu - \frac{1}{T} U_\nu \partial_\mu T \right) + 2 \frac{d\bar{m}_a^2}{dT} U^\alpha \partial_\alpha T \right]. \end{aligned} \quad (63)$$

The gradient of the temperature was determined previously:

$$\frac{1}{T} \partial_\mu T = U^\alpha \partial_\alpha U_\mu - v_s^2 U_\mu \partial_\alpha U^\alpha. \quad (64)$$

This allows us to write

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_a} \cdot \nabla_x - \nabla_x E_a \cdot \nabla_p \right) f_a^{\text{eq}} \\ &= -\frac{1}{2E_a T} f_a^{\text{eq}} \left[ p^\mu p^\nu \left( D_\mu U_\nu + D_\nu U_\mu + \frac{2}{3} \Delta_{\mu\nu} \partial_\rho U^\rho \right) \right. \\ & \quad \left. - \frac{2}{3} \partial_\rho U^\rho \left( (1 - 3v_s^2) E_a^2 - \bar{m}_a^2 + 3v_s^2 T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right]. \end{aligned} \quad (65)$$

Thus the term  $(1 - 3v_s^2)E_a^2 - \bar{m}_a^2 = \mathbf{p}^2 - 3v_s^2 E_a^2$  in Eq. (46) gets replaced by  $(1 - 3v_s^2)E_a^2 - \bar{m}_a^2 + 3v_s^2 T^2 d\bar{m}_a^2/dT^2 = \mathbf{p}^2 - 3v_s^2(E_a^2 - T^2 d\bar{m}_a^2/dT^2)$ .

Taking into account the mean-field effects modifies the particular solution in the relaxation time approximation:

$$A_a^{\text{par}}(E_a) = \frac{\tau_a(E_a)}{3TE_a} \left[ (1 - 3v_s^2)E_a^2 - \bar{m}_a^2 + 3v_s^2 T^2 \frac{d\bar{m}_a^2}{dT^2} \right]. \quad (66)$$

However, the mean fields also affect the equation of state, the speed of sound, and the Landau-Lifshitz condition, so it is not so straightforward to deduce the bulk viscosity at this point.

### A. Model with no symmetry breaking

Consider a model with  $N$  scalar fields  $\Phi_a$  that has no symmetry breaking, meaning that  $\langle \Phi_a \rangle = 0$  for all  $a$ . It has the effective Lagrangian,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \sum_a (\partial_\mu \Phi_a)^2 - U(\Phi_1, \dots, \Phi_N). \quad (67)$$

Assume that the potential is a polynomial in the fields to arbitrarily high order. It represents localized multiparticle interactions. The quasiparticle approach includes both mean fields and independent thermal fluctuations around the mean fields, all calculated in a thermodynamically self-consistent manner [62–65]. This is sometimes referred to as the Phi-derivable approach, and sometimes as the summation of daisy and superdaisy diagrams. If fermions are present they are integrated out and the effects of their interactions are subsumed in  $U$ . In this approximation only even powers of the fields in the potential play a role. The thermal average may be written as

$$\langle U \rangle = \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N}, \quad (68)$$

where the  $U_{n_1 \dots n_N}$  are constants. The effective masses are obtained from

$$\bar{m}_a^2 = \left\langle \frac{\partial^2 U}{\partial \Phi_a^2} \right\rangle, \quad (69)$$

where it is assumed that the system has been diagonalized in terms of normal modes such that

$$\left\langle \frac{\partial^2 U}{\partial \Phi_a \partial \Phi_b} \right\rangle = \delta_{ab} \bar{m}_a^2. \quad (70)$$

From the combinatorics,

$$\begin{aligned} \left\langle \frac{\partial^2 U}{\partial \Phi_a^2} \right\rangle &= \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} \frac{(2n_a)(2n_a - 1)(2n_a - 3)!!}{(2n_a - 1)} \\ & \quad \times \langle \Phi_a^2 \rangle \dots \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N}, \end{aligned} \quad (71)$$

so that

$$\bar{m}_a^2 = \frac{2}{\langle \Phi_a^2 \rangle} \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} n_a \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N}. \quad (72)$$

The equation of state is given by

$$P = P_0 - V, \quad \epsilon = \epsilon_0 + V, \quad (73)$$

where the subscript 0 refers to the free particle form with effective masses  $\bar{m}_a$ . The energy-momentum tensor is

$$T^{\mu\nu} = T_0^{\mu\nu} + g^{\mu\nu} V. \quad (74)$$

Note that  $Ts = \epsilon + P = \epsilon_0 + P_0 = Ts_0$  where  $s = dP/dT$  and  $s_0 = dP_0/dT$  so that the form of the entropy is unchanged. This is a consequence of the assumption of independent particle motion between collisions in the mean-field approximation. The potential energy density is obtained from

$$\begin{aligned} V &= \langle U \rangle - \frac{1}{2} \sum_a \bar{m}_a^2 \langle \Phi_a \rangle^2 \\ &= \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} [1 - (n_1 + \dots + n_N)] \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N}. \end{aligned} \quad (75)$$

The entropy density is computed from the formulas,

$$\frac{dP}{dT} = \sum_a \frac{\partial P_{0a}}{\partial T} + \sum_a \frac{\partial P_{0a}}{\partial \bar{m}_a^2} \frac{d\bar{m}_a^2}{dT} - \frac{dV}{dT}, \quad (76)$$

$$\frac{\partial P_{0a}}{\partial \bar{m}_a^2} = -\frac{1}{2} \langle \Phi_a^2 \rangle, \quad (77)$$

$$\langle \Phi_a^2 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_a} f_a^{\text{eq}}(E_a/T), \quad (78)$$

whereby it is readily shown that

$$\frac{dV}{dT} = -\frac{1}{2} \sum_a \langle \Phi_a^2 \rangle \frac{d\bar{m}_a^2}{dT}, \quad (79)$$

from which follows

$$\frac{dP}{dT} = \sum_a s_{0a}, \quad (80)$$

so that the model is thermodynamically consistent. There are some additional interesting temperature derivatives that can be derived, such as

$$\frac{dV}{dT} = \sum_a V_a \frac{d\langle \Phi_a^2 \rangle}{dT}, \quad (81)$$

and

$$\frac{d\langle U \rangle}{dT} = \frac{1}{2} \sum_a \bar{m}_a^2 \frac{d\langle \Phi_a^2 \rangle}{dT}. \quad (82)$$



The numbers  $U_{n_1 \dots n_N}$  have the interpretation of representing the interaction of  $2(n_1 + \dots + n_N)$  particles in the initial plus final states. Taking account of the combinatorics, the vertex itself would be

$$\frac{U_{n_1 \dots n_N}}{(2n_1 - 1)!! \dots (2n_N - 1)!!},$$

where it is assumed that all  $n_a \neq 0$ . If an  $n_a$  happened to be zero then the factor  $(2n_a - 1)$  is replaced by 1.

In Fermi liquid theory, functional variation of the energy density with respect to the distribution functions should yield the single-particle energies [66]. That relationship holds here too. The kinetic part of the energy density is  $\epsilon_0$  with the effective masses  $\bar{m}_a$ . Making a variation  $\delta f_a$  also affects the mass. Hence,

$$\begin{aligned} \frac{\delta \epsilon}{\delta f_a} &= E_a + \sum_b \int \frac{d^3 p}{(2\pi)^3} \frac{f_b(E_b)}{2E_b} \frac{\partial \bar{m}_b^2}{\partial \langle \Phi_a^2 \rangle} \frac{\delta \langle \Phi_a^2 \rangle}{\delta f_a} + \frac{\delta V}{\delta f_a} \\ &= E_a + \frac{1}{2} \frac{\delta \langle \Phi_a^2 \rangle}{\delta f_a} \sum_b \frac{\partial \bar{m}_b^2}{\partial \langle \Phi_a^2 \rangle} \langle \Phi_b^2 \rangle + \frac{\delta V}{\delta f_a}. \end{aligned} \quad (83)$$

Variation of the potential part is given by

$$\begin{aligned} \delta V &= \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} [1 - (n_1 + \dots + n_N)] \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} \\ &\times \left[ \frac{n_1}{\langle \Phi_1^2 \rangle} \delta \langle \Phi_1^2 \rangle + \dots + \frac{n_N}{\langle \Phi_N^2 \rangle} \delta \langle \Phi_N^2 \rangle \right], \end{aligned} \quad (84)$$

so that

$$\begin{aligned} \frac{\delta V}{\delta f_a} &= \frac{1}{\langle \Phi_a^2 \rangle} \frac{\delta \langle \Phi_a^2 \rangle}{\delta f_a} \sum_{n_1 \dots n_N} U_{n_1 \dots n_N} [1 - (n_1 + \dots + n_N)] \\ &\times n_a \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N}. \end{aligned} \quad (85)$$

Now it is easy to see from the expression [Eq. (72)] for the mass that  $\partial \bar{m}_a^2 / \partial \langle \Phi_b^2 \rangle$  is a symmetric matrix and that

$$\frac{\delta V}{\delta f_a} = -\frac{1}{2} \frac{\delta \langle \Phi_a^2 \rangle}{\delta f_a} \sum_b \frac{\partial \bar{m}_b^2}{\partial \langle \Phi_a^2 \rangle} \langle \Phi_b^2 \rangle. \quad (86)$$

This cancels the extra term from  $\epsilon_0$  so that

$$\frac{\delta \epsilon}{\delta f_a} = E_a, \quad (87)$$

verifying the internal consistency of the model.

### B. Model with symmetry breaking

Now we allow for one of the fields to condense. For definiteness let it be the  $N$ th one. After making the shift  $\Phi_N \rightarrow \Phi_N + v$ , where  $v$  is the condensate, all fields obey  $\langle \Phi_a \rangle = 0$ . Generalizing the previous analysis we write

$$\langle U \rangle = \sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^l. \quad (88)$$

If there is no condensation, then all  $U_{n_1 \dots n_N; l}$  with  $l > 0$  vanish, and we return to the previous case. Following the same

arguments as before we obtain the effective masses,

$$\bar{m}_a^2 = \frac{2}{\langle \Phi_a^2 \rangle} \sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} n_a \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^l, \quad (89)$$

and the potential energy density,

$$\begin{aligned} V &= \langle U \rangle - \frac{1}{2} \sum_a \bar{m}_a^2 \langle \Phi_a^2 \rangle^2 \\ &= \sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} [1 - (n_1 + \dots + n_N)] \\ &\times \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^l. \end{aligned} \quad (90)$$

The value of the condensate is determined by extremizing the pressure  $P = P_0 - V$  at fixed  $T$ , namely  $(\partial P / \partial v)_T = 0$ . The result is

$$\sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^l = 0. \quad (91)$$

This is just the same condition as  $(\partial U / \partial v) = 0$ .

For the dissipative part of the energy-momentum tensor and the Landau-Lifshitz condition we need

$$\begin{aligned} \Delta V &= \sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} [1 - (n_1 + \dots + n_N)] \\ &\times \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^l \left[ \frac{n_1}{\langle \Phi_1^2 \rangle} \Delta \langle \Phi_1^2 \rangle + \dots + \frac{n_N}{\langle \Phi_N^2 \rangle} \Delta \langle \Phi_N^2 \rangle \right] \\ &- \Delta v \sum_{n_1 \dots n_N} \sum_l U_{n_1 \dots n_N; l} (n_1 + \dots + n_N) l \\ &\times \langle \Phi_1^2 \rangle^{n_1} \dots \langle \Phi_N^2 \rangle^{n_N} v^{l-1}. \end{aligned} \quad (92)$$

The condensate deviates from its equilibrium value because of the corresponding deviation of thermal fluctuations. One may express  $\Delta v$  in terms of the  $\Delta \langle \Phi_a^2 \rangle$  using Eq. (91) but for our purposes there is no need to do so explicitly. For verification of the Fermi liquid result that functional variation of the energy density with respect to the distribution function yields the single-particle energy, it is sufficient to observe that

$$\frac{\delta V}{\delta f_a} = -\frac{1}{2} \frac{\delta \langle \Phi_a^2 \rangle}{\delta f_a} \sum_b \frac{\partial \bar{m}_b^2}{\partial \langle \Phi_a^2 \rangle} \langle \Phi_b^2 \rangle - \frac{1}{2} \frac{\delta v}{\delta f_a} \sum_b \frac{\partial \bar{m}_b^2}{\partial v} \langle \Phi_b^2 \rangle. \quad (93)$$

This exactly cancels the extra term coming from  $\epsilon_0$  so that

$$\frac{\delta \epsilon}{\delta f_a} = E_a, \quad (94)$$

in the presence of a condensate too. From this point on, the expression for  $\Delta T^{\mu\nu}$  and the bulk and shear viscosities is the same as when there is no condensation.

### C. Landau-Lifshitz condition

The single-particle energy at finite temperature is a functional of the distribution functions  $E_a = E_a(\{f\})$ . When there is a small deviation from equilibrium,

$$f_a(x, p) = f_a^{\text{eq}}(E_{a,0}) + \delta f_a(x, p), \quad (95)$$

the single-particle energy changes to

$$E_a = E_{a,0} + \delta E_a. \quad (96)$$

Here the subscript 0 indicates the value the energy would have if there was no departure from equilibrium at all. If  $f_a^{\text{eq}}$  is expressed as a function of the true energy  $E_a$ , then

$$f_a(x, p) = f_a^{\text{eq}}(E_a) + \delta \tilde{f}_a(x, p), \quad (97)$$

where

$$\delta \tilde{f}_a(x, p) = \delta f_a(x, p) - \frac{\partial f_a^{\text{eq}}(E_a)}{\partial E_a} \delta E_a. \quad (98)$$

The structure of the quasiparticle Boltzmann equation is such that it is the function  $\delta \tilde{f}_a$ , which determines the transport coefficients. It is important to realize that energy conservation in the collisions dictates that linearization of the collision term must be performed with respect to the true energy  $E_a$ .

Following the same arguments as earlier, expansion of  $T^{ij}$  from Eq. (74) around local equilibrium, using Eq. (95), leads to

$$\Delta T^{ij} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j}{E_a} \delta \tilde{f}_a. \quad (99)$$

The change in the energy density is given by

$$\begin{aligned} \Delta T^{00} &= \sum_a \int \frac{d^3 p}{(2\pi)^3} E_a \delta f_a \\ &= \sum_a \int \frac{d^3 p}{(2\pi)^3} \left( E_a \delta \tilde{f}_a + E_a \frac{\partial f_a^{\text{eq}}(E_a)}{\partial E_a} \delta E_a \right) \\ &= \sum_a \int \frac{d^3 p}{(2\pi)^3} \left( E_a \delta \tilde{f}_a - \frac{E_a}{T} f_a^{\text{eq}}(E_a) \delta E_a \right) \\ &= \sum_a \int \frac{d^3 p}{(2\pi)^3} \left( E_a \delta \tilde{f}_a - \frac{1}{2T} \frac{d\bar{m}_a^2}{dT} f_a^{\text{eq}}(E_a) \delta T \right). \end{aligned} \quad (100)$$

Recall that

$$\delta f_a = -e^{-E_a/T} \left[ \frac{\delta E_a}{T} - \frac{E_a}{T^2} \delta T \right], \quad (101)$$

and

$$\delta \tilde{f}_a = e^{-E_a/T} \frac{\delta T}{T^2}. \quad (102)$$

Substituting these in Eq. (100) we find that, in the local rest frame,

$$\Delta T^{00} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_a} \left( E_a^2 - \bar{m}_a T \frac{d\bar{m}_a}{dT} \right) \delta \tilde{f}_a. \quad (103)$$

Hence, in a general frame of reference,

$$\Delta T^{\mu\nu} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_a} \left( p_a^\mu p_a^\nu - U^\mu U^\nu T^2 \frac{d\bar{m}_a^2}{dT} \right) \delta \tilde{f}_a. \quad (104)$$

This is the obvious generalization of the result of Ref. [61] to a system with multiple species of particles.

Following the usual arguments we can deduce the viscosities:

$$\zeta = \frac{1}{3} \sum_a \int \frac{d^3 p}{(2\pi)^3 E_a} f_a^{\text{eq}}(E_a/T) A_a(E_a) |\mathbf{p}|^2, \quad (105)$$

$$\eta = \frac{2}{15} \sum_a \int \frac{d^3 p}{(2\pi)^3 E_a} f_a^{\text{eq}}(E_a/T) C_a(E_a) |\mathbf{p}|^4. \quad (106)$$

If one has a particular solution that does not happen to satisfy the Landau-Lifshitz condition, one can make it so by adding a term linear in the energy, just as before:

$$\begin{aligned} &\sum_a \int \frac{d^3 p}{(2\pi)^3 E_a} f_a^{\text{eq}}(E_a/T) \left[ E_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right] \\ &\times [A_a^{\text{par}}(E_a) - b E_a] = 0. \end{aligned} \quad (107)$$

To simplify the resulting formula, it is helpful to use

$$\sum_a \int \frac{d^3 p}{(2\pi)^3} f_a^{\text{eq}}(E_a/T) \left[ |\mathbf{p}|^2 - 3v_s^2 \left( E_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right] = 0, \quad (108)$$

which is a consequence of the identity  $dP/dT = (dP/d\epsilon) d\epsilon/dT = v_s^2 d\epsilon/dT$  in the independent particle models used here. The coefficient  $b$  is thus

$$b = \frac{v_s^2}{T^2 s} \sum_a \int \frac{d^3 p}{(2\pi)^3 E_a} f_a^{\text{eq}}(E_a/T) \left[ E_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right] A_a^{\text{par}}(E_a). \quad (109)$$

Substitution of  $A_a(E_a) = A_a^{\text{par}}(E_a) - b E_a$  into Eq. (105) gives

$$\begin{aligned} \zeta &= \frac{1}{3} \sum_a \int \frac{d^3 p}{(2\pi)^3 E_a} f_a^{\text{eq}}(E_a/T) A_a^{\text{par}}(E_a) \\ &\times \left[ |\mathbf{p}|^2 - 3v_s^2 \left( E_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right]. \end{aligned} \quad (110)$$

Of course, the term proportional to  $v_s^2$  will integrate to zero if the particular solution does satisfy the Landau-Lifshitz condition. The appearance of the factor in square brackets is natural because it matches the source function in Eq. (46).

In the relaxation time approximation this becomes

$$\begin{aligned} \zeta &= \frac{1}{9T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{\tau_a(E_a)}{E_a^2} f_a^{\text{eq}}(E_a/T) \\ &\times \left[ |\mathbf{p}|^2 - 3v_s^2 \left( E_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right]^2. \end{aligned} \quad (111)$$

This expression is obviously positive definite.

## VII. THE LINEAR $\sigma$ MODEL

The linear  $\sigma$  model was long used as a simple renormalizable model of pion dynamics at low energy. Although it cannot claim to be quantitatively accurate, being supplanted by chiral perturbation theory, it is still a much-used model for testing approximations and as a proxy for more involved and detailed low-energy models of QCD. Indeed, we will show that the

model exemplifies the richness of the equation of state and transport coefficients to be expected from QCD.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - U(\sigma, \boldsymbol{\pi}), \quad (112)$$

where

$$U(\sigma, \boldsymbol{\pi}) = \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2 - f^2)^2 - H\sigma. \quad (113)$$

Compare to Eq. (67), where  $\pi_i$  is identified with  $\Phi_i$  for  $i = 1, 2, 3$  and  $\sigma$  is identified with  $\Phi_4$ . The  $SU(2)_R \times SU(2)_L$  chiral symmetry is explicitly broken by the term  $H\sigma$ , which gives the pion a mass. The scalar field has a nonvanishing vacuum expectation value  $v$  determined at the classical level by the equation,

$$\lambda v(v^2 - f^2) = H. \quad (114)$$

The scalar field is thus split into a condensate and a fluctuation,  $\sigma = v + \Delta$ . The three parameters,  $\lambda$ ,  $H$ , and  $f$ , are determined by the vacuum values of the pion decay constant  $f_\pi$  and the pion and sigma masses:

$$\lambda = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2}, \quad (115)$$

$$H = f_\pi^2 m_\pi^2, \quad (116)$$

$$f^2 = \frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2} f_\pi^2. \quad (117)$$

For numerical calculations we take  $f_\pi = 93$  MeV,  $m_\pi = 140$  MeV, and either  $m_\sigma = 600$  or 900 MeV.

### A. Thermodynamics

The equation of state is calculated following the general procedure outlined in Sec. VI. It must be done numerically. For more details the reader is referred to Refs. [62–65].

The temperature dependence of the meson masses and  $\sigma$  condensate  $v$  are shown in Fig. 1 for two illustrative values of the vacuum  $\sigma$  mass. There is no true phase transition, only a crossover from the low-temperature regime where there is a large difference between the two masses and the high-temperature regime where the masses are practically identical. The condensate decreases to very small values at high temperature but never vanishes. The temperature at which the symmetry is approximately restored is around 245 MeV. It should be noted for future reference that the pion and  $\sigma$  masses rise linearly with temperature at high temperature. This is a generic feature of high-temperature field theories.

The entropy density, energy density, and pressure are shown in Fig. 2. These thermodynamic quantities follow a continuous curve with no phase transition. They go to zero exponentially as  $T \rightarrow 0$  because all mesons are massive. At large temperature they show the usual behavior that  $s \sim T^3$  and  $P \sim \epsilon \sim T^4$ . At intermediate temperatures rapid variation of the meson masses result in maxima in  $s/T^3$  and  $\epsilon/T^4$  but not in  $P/T^4$ .

Figure 3 shows the speed of sound squared  $v_s^2$  and heat capacity  $c_V$  for the two choices of the vacuum  $\sigma$  mass. As the vacuum  $\sigma$  mass increases the speed of sound develops a dip and the heat capacity develops a peak around 245 MeV.

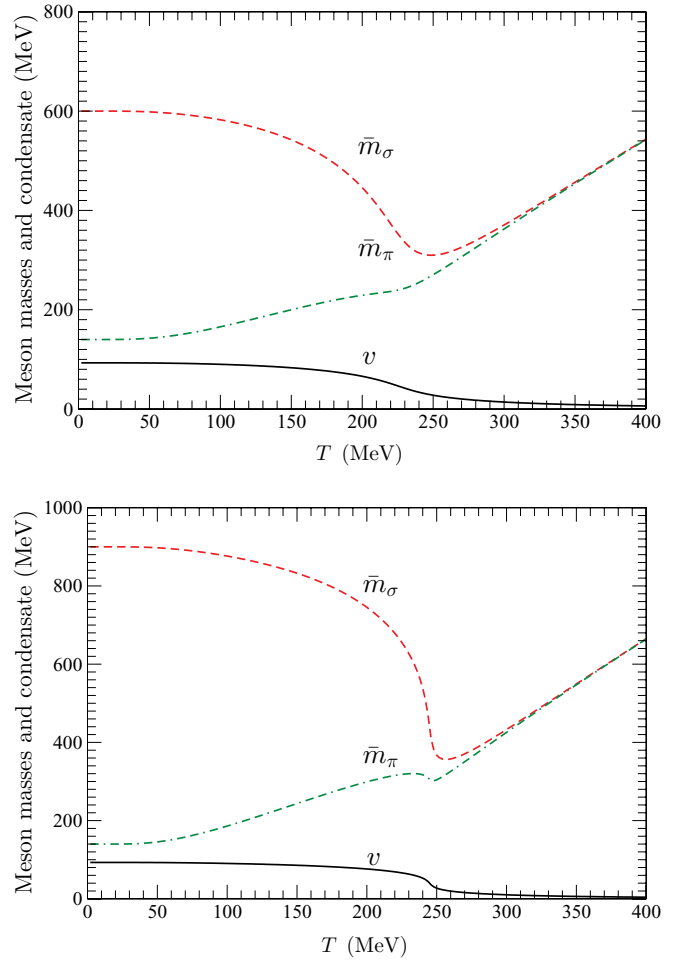


FIG. 1. (Color online) Variation of meson masses and condensate with temperature for a vacuum sigma mass of 600 MeV (top) and 900 MeV (bottom).

This indicates that the system is near a second-order phase transition. At high temperature  $v_s^2 \rightarrow 1/3$  and  $c_V \rightarrow T^3$ , both on account of the fact that the equation of state approaches  $P \sim T^4$ .

### B. Scattering amplitudes

The Feynman rules can easily be determined from the Lagrangian of the linear  $\sigma$  model. At one loop order the various scattering amplitudes are as follows:

$$\begin{aligned} \mathcal{M}_{\pi^a \pi^b; \pi^c \pi^d} &= -2\lambda \left( \frac{s - m_\pi^2}{s - m_\sigma^2} \delta_{ab} \delta_{cd} + \frac{t - m_\pi^2}{t - m_\sigma^2} \delta_{ac} \delta_{bd} + \frac{u - m_\pi^2}{u - m_\sigma^2} \delta_{ad} \delta_{bc} \right), \end{aligned} \quad (118)$$

$$\mathcal{M}_{\pi\sigma; \pi\sigma} = -2\lambda - 4\lambda^2 f_\pi^2 \left( \frac{3}{t - m_\sigma^2} + \frac{1}{u - m_\pi^2} + \frac{1}{s - m_\pi^2} \right), \quad (119)$$

$$\mathcal{M}_{\pi\pi; \sigma\sigma} = -2\lambda - 4\lambda^2 f_\pi^2 \left( \frac{3}{s - m_\sigma^2} + \frac{1}{t - m_\pi^2} + \frac{1}{u - m_\pi^2} \right), \quad (120)$$

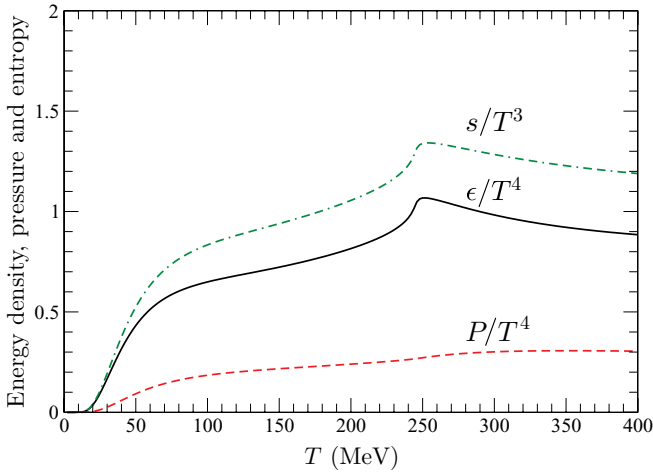
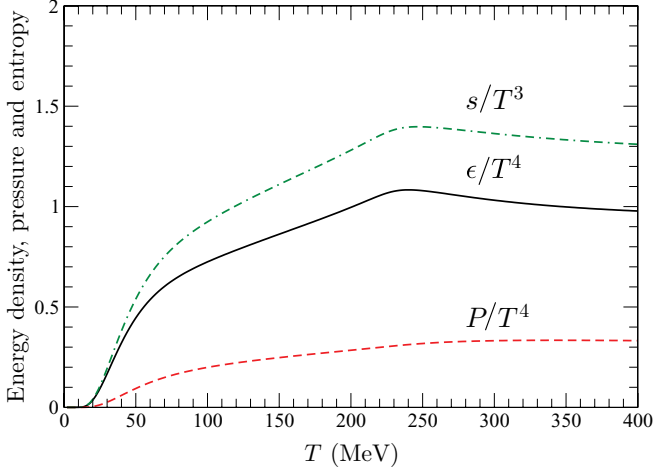


FIG. 2. (Color online) Energy density, pressure, and entropy density as functions of temperature for a vacuum sigma mass of 600 MeV (top) and 900 MeV (bottom).

$$\mathcal{M}_{\sigma\sigma;\sigma\sigma} = -6\lambda - 36\lambda^2 f_\pi^2 \left( \frac{1}{s - m_\sigma^2} + \frac{1}{t - m_\sigma^2} + \frac{1}{u - m_\sigma^2} \right). \quad (121)$$

The poles in the  $s$  and  $u$  channels of  $\pi\pi \rightarrow \pi\pi$  and  $\pi\sigma \rightarrow \pi\sigma$ , respectively, pose trouble as is well known in the literature. We are not aware of any prescription to regularize both of these singularities while satisfying crossing symmetry. It is obvious that the problematic terms are coming from the three-point vertices. Those terms were not included when calculating the equation of state anyway. To be consistent with the equation of state, these terms must be dropped when calculating the viscosities. Equivalently, we approximate the scattering amplitudes by their limits as  $s$ ,  $t$ , and  $u$  all go to infinity. Thus the scattering amplitudes just reduce to constants.

### C. Viscosities

With the transition amplitudes specified, the departure functions  $A_a$  and  $C_a$  for  $a = \pi$  and  $\sigma$  can be determined as solutions to the integral equations  $\mathcal{A}_a = 0$  and  $\mathcal{C}_a^{\mu\nu} = 0$  using

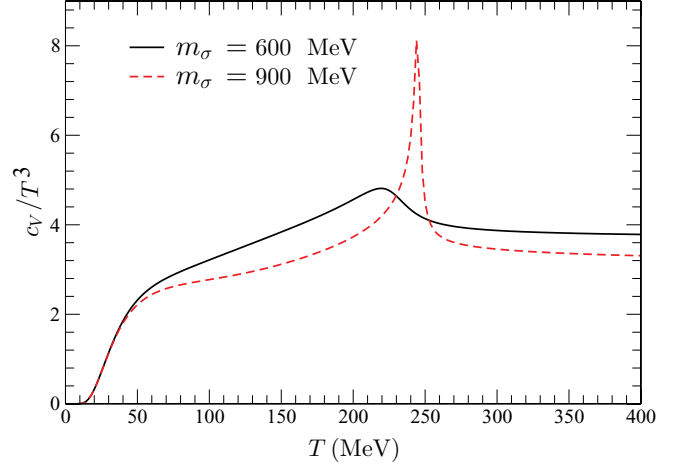
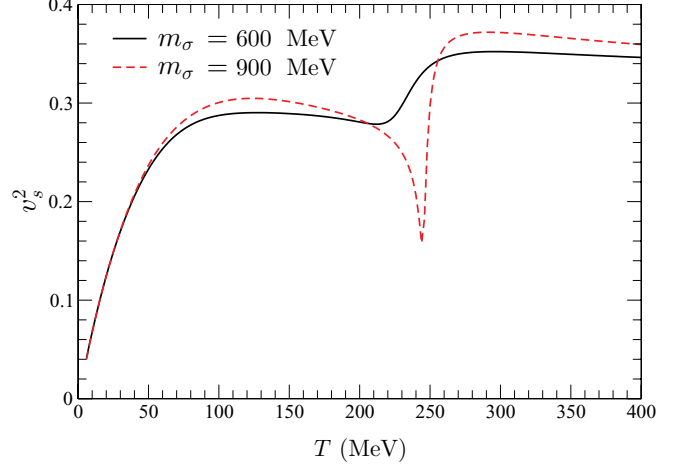


FIG. 3. (Color online) Temperature dependence of the speed of sound (top) and heat capacity (bottom).

Eqs. (46) and (47). When substituted into Eqs. (105) and (106) we will get the bulk and shear viscosities.

The set of integral equations are projected into a suitable vector space and solved by a variational method. The procedure is standard and can be found classic monographs [54,67]. The departure functions are expanded in terms of some chosen basis functions  $g_n$  as

$$C_a = \sum_{n=1}^N c_{a,n} g_n(E_a/T), \quad (122)$$

and the expansion coefficients  $c_{a,n}$  are adjusted to minimize the functional,

$$\mathcal{F} = \sum_a (C_a - \tilde{C}_a, C_a - \tilde{C}_a), \quad (123)$$

where  $\tilde{C}$  is the true solution. We have defined the inner product between two functions  $\chi_1$  and  $\chi_2$  as

$$(\chi_1, \chi_2) = \sum_a \int \frac{d^3 p}{(2\pi)^3} \chi_{1,a}(\mathbf{p}) \mathcal{R}[\chi_{2,a}(\mathbf{p})], \quad (124)$$

where  $\mathcal{R}$  is the collision operator acting on the space of departure functions which, in the notation of Eq. (22), should

be understood as

$$\mathcal{R}[\chi] = \sum_{\{i\}\{j\}} \frac{1}{S} \int' dP_i dP_j W(\{i\}|\{j\}) \prod_{i=1}^n \prod_{j=1}^m f_i \times [1 + (-1)^{s_j} f_j] \left( \sum_i \chi_i - \sum_j \chi_j \right). \quad (125)$$

With the variational technique one hopes to find a good approximation to the departure functions using a modest set of basis functions. However, this is not guaranteed. In this paper we will show numerical results obtained this way for the shear viscosity. Solving the integral equations for the bulk viscosity is more difficult, as is well known, because of the presence of zero modes. Details of the numerical techniques used and their convergence properties will be presented elsewhere.

In Figs. 4 ( $m_\sigma = 600$  MeV) and 5 ( $m_\sigma = 900$  MeV) we show the ratio of shear viscosity to entropy density from a third-order variational calculation of the full Boltzmann equation compared to the relaxation time approximation of the Boltzmann equation. The variational method converges

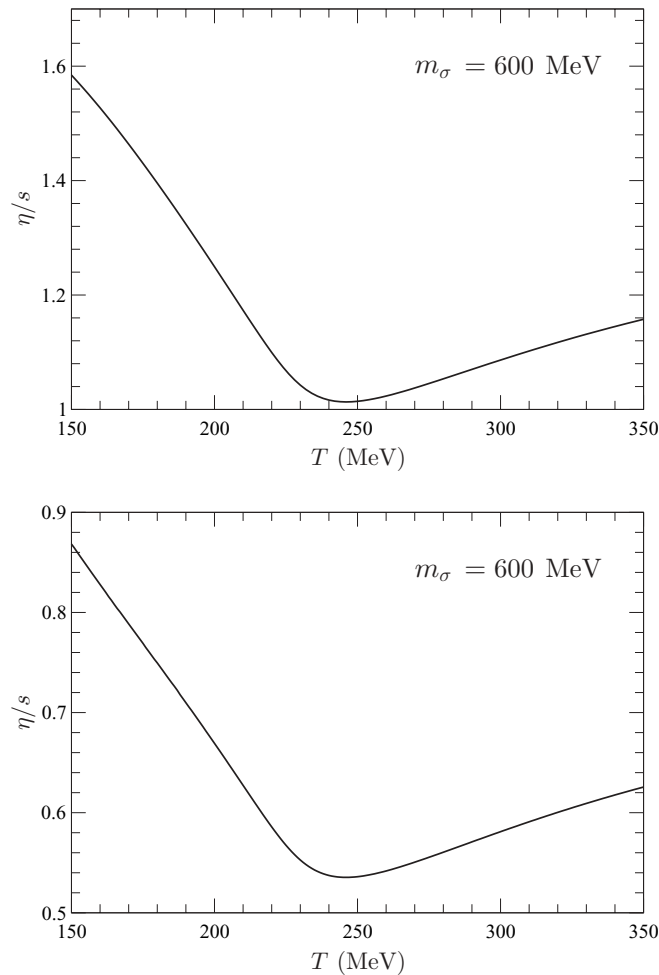


FIG. 4. Ratio of shear viscosity to entropy density from the solution to the integral equation (top) and from the relaxation time approximation (bottom). Recall that the KSS bound is  $1/4\pi \approx 0.08$ . The vacuum sigma mass is 600 MeV.

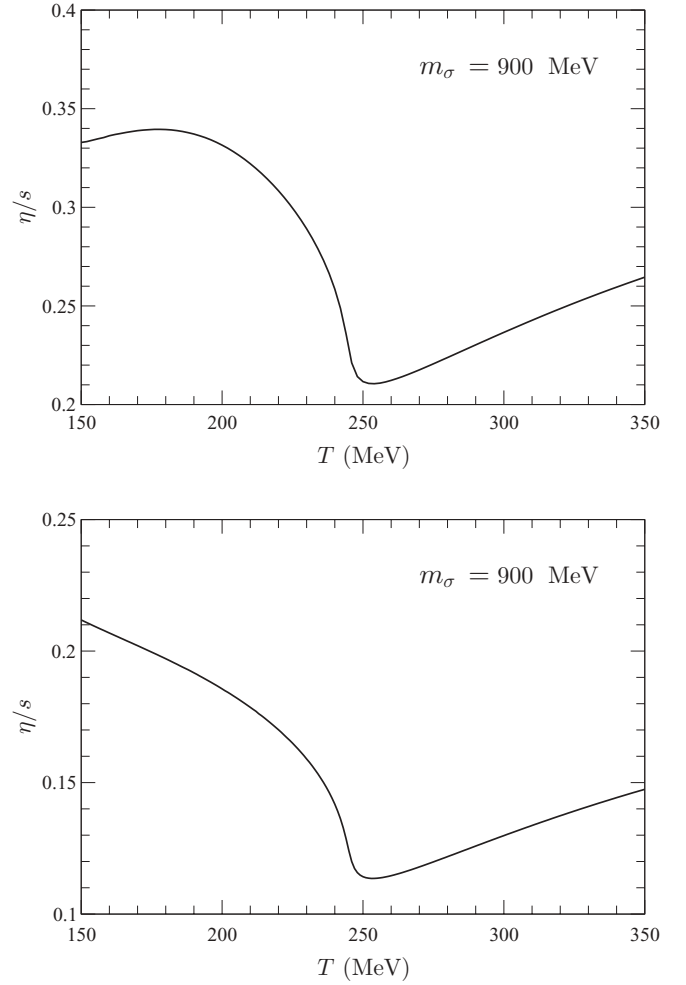


FIG. 5. Ratio of shear viscosity to entropy density from the solution to the integral equation (top) and from the relaxation time approximation (bottom). Recall that the KSS bound is  $1/4\pi \approx 0.08$ . The vacuum sigma mass is 900 MeV.

very quickly. For the present case the first-order solution was already good to within a few percent and it was unnecessary to go beyond the third order. It is satisfying that results from the relaxation time approximation are within a factor of 2 compared to the sophisticated variational method. The difference between the results of the variational method versus the relaxation time approximation is most apparent at low temperature and for  $m_\sigma = 900$  MeV. This can be attributed to how the collision dynamics is approximated when using the relaxation time method. Although the relaxation time used is energy dependent, it is still only an approximation to the full solution of the integral equations.

A minimum near the crossover temperature of 245 MeV is observed in the shear viscosity to entropy density ratio for both  $m_\sigma = 600$  MeV and  $m_\sigma = 900$  MeV. Note that for fixed  $m_\pi$  and  $f_\pi$  the coupling constant  $\lambda$  in this model grows with  $m_\sigma$  [see Eq. (115)]. This leads to increasingly strong interaction, thereby reducing the shear viscosity. The parameters we have chosen are illustrative of QCD phenomenology. Whether or

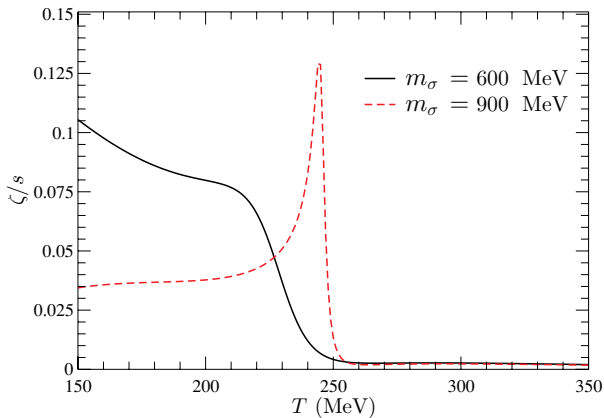


FIG. 6. (Color online) Ratio of bulk viscosity to entropy density in the relaxation time approximation.

not other, more extreme, values of the parameters constitute a possible counterexample of the KSS bound deserves a more careful study and is beyond the scope of this article.

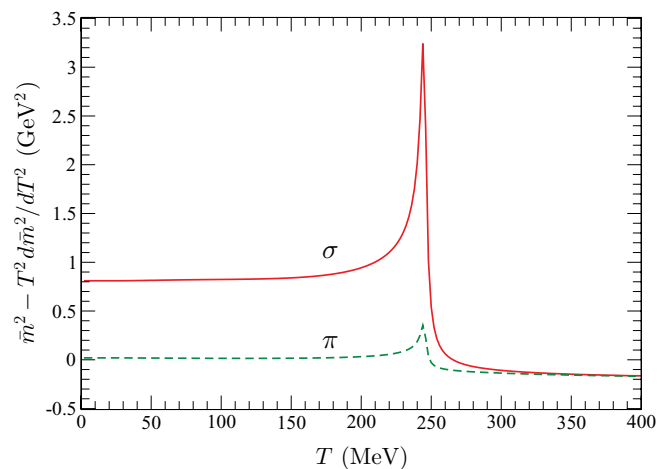
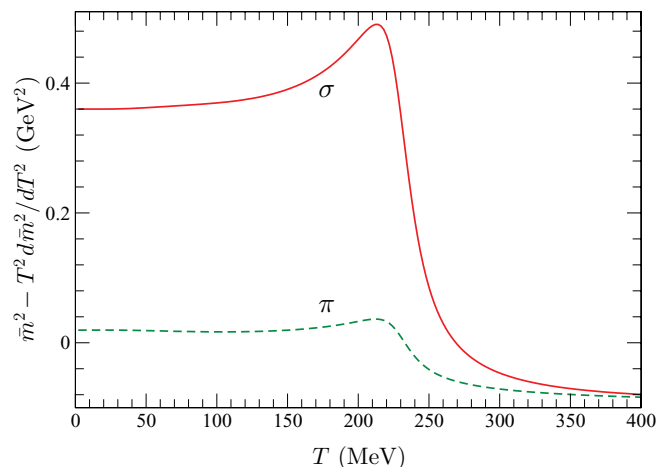


FIG. 7. (Color online) The quantity  $\bar{m}_a^2 - d\bar{m}_a^2/dT^2$  that enters the calculation of the bulk viscosity for a vacuum sigma mass of 600 MeV (top) and 900 MeV (bottom).

The ratio of bulk viscosity to entropy density from the relaxation time approximation,

$$\zeta = \frac{1}{T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{\tau_a(E_a)}{E_a^2} f_a^{\text{eq}}(E_a/T) \times \left[ \left( \frac{1}{3} - v_s^2 \right) |\mathbf{p}|^2 - v_s^2 \left( \bar{m}_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right]^2, \quad (126)$$

is shown in Fig. 6. Although we have not yet managed to solve the integral equations for the bulk viscosity, we do not expect the difference between the two approaches to be significantly different from what we obtained for the shear viscosity. For a vacuum  $\sigma$  mass of 900 MeV, there is a clear maximum in  $\zeta/s$  near the crossover temperature. To ascertain its origin it is only necessary to examine Eq. (126). The ratio is quadratically proportional to the violation of conformality, either in terms of speed of sound,

$$\frac{1}{3} - v_s^2,$$

or in terms of bulk transport mass,

$$\bar{m}_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} = \frac{d}{d\beta^2} (\beta^2 \bar{m}_a^2).$$

A comparison of Figs. 3 and 7 with Fig. 6 clearly shows that  $\zeta/s$  is largest when the violation of conformality is greatest. Note that for the system to be conformal it is only necessary that  $P \sim T^4$  and that  $\bar{m} \sim T$ ; it is not necessary for the effective mass to vanish. Intuitively, the bulk viscosity is enhanced when it is easy to transfer energy between kinetic motion and internal degrees of freedom, such as resonances or heavier mass particle, mean fields, particle production or absorption, or effective masses that vary strongly with temperature. All of these are playing a role here to a greater or lesser extent.

## VIII. CONCLUSION

In this article we constructed a theoretical framework for the calculation of the viscosities of hot hadronic matter. The framework has the benefit that it is relativistic, it allows for an arbitrary number of hadron species, it allows for elastic and inelastic collisions and the formation and decay of resonances, it respects detailed balance, and it allows for temperature-dependent mean fields and temperature-dependent masses. It is a consistent theory in the sense that the same interactions that are used to calculate the equation of state are used to calculate the viscosities. The significance of this is that the bulk viscosity, in particular, depends very dramatically on the equation of state and on the quasiparticle masses. The essential assumption is that quasiparticles are good degrees of freedom. The basic resulting formulas have been assembled in the appendix for ease of reference.

As a nontrivial application we applied this theoretical framework to the linear  $\sigma$  model using the physical vacuum pion mass. As the vacuum  $\sigma$  mass increases, the crossover transition become sharper, the speed of sound develops a dip, and the heat capacity develops a peak. This is well-known physics, but it has significance for the viscosities. As expected, the ratio  $\eta/s$  has a minimum at the crossover temperature

whereas the ratio  $\zeta/s$  develops a maximum there for large enough vacuum  $\sigma$  masses. The minimum value of  $\eta/s$  is greater than the purported bound of  $1/4\pi$ , although we really have nothing new to add here in this direction.

As a byproduct of our theoretical framework we generalized the relaxation time formulas for the shear and bulk viscosities of Gavin [59] to include an arbitrary number of species of hadrons with energy-dependent relation times and temperature-dependent effective masses. These formulas alone can be applied very usefully without much numerical work.

The theoretical framework presented here can be applied to increasingly sophisticated models of hadronic interactions. Work on doing so is in progress. Also in progress is the extension to include chemical potentials. The latter may not be important at RHIC and Large Hadron Collider (LHC) but they will be for heavy-ion collisions at Facility for Antiproton and Ion Research (FAIR).

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### APPENDIX

In this Appendix we gather some of the basic results of the article. For more details see the text.

The Boltzmann equation is

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_a} \cdot \nabla_{\mathbf{x}} - \nabla_{\mathbf{x}} E_a \cdot \nabla_{\mathbf{p}} \right) f_a(\mathbf{x}, t, \mathbf{p}) \\ & = \sum_{\{i\}\{j\}} \frac{1}{S} \int' dP_i dP_j W(\{i\}|\{j\}) F[f], \end{aligned} \quad (\text{A1})$$

where the prime indicates that there is no integration over the momentum of  $a$ . There is a statistical factor for identical particles in the initial state,

$$S = \prod_i n_i!, \quad (\text{A2})$$

and products of Bose-Einstein and Fermi-Dirac distributions as appropriate,

$$F[f] = \prod_i \prod_j \{f_j [1 + (-1)^{s_i} f_i] - f_i [1 + (-1)^{s_j} f_j]\}. \quad (\text{A3})$$

The rate function is

$$W(\{i\}|\{j\}) = \frac{(2\pi)^4 \delta^4(P_i - P_j)}{(\prod_i 2E_i)(\prod_j 2E_j)} |\mathcal{M}(\{i\}|\{j\})|^2. \quad (\text{A4})$$

The departure of the phase-space distributions are expressed in terms of  $\phi_a$  as

$$f_a = f_a^{\text{eq}} (1 + \phi_a), \quad (\text{A5})$$

which furthermore has the tensorial decomposition,

$$\phi_a = -A_a \partial_\rho U^\rho + C_{\mu\nu}^a (D^\mu U^\nu + D^\nu U^\mu + \frac{2}{3} \Delta^{\mu\nu} \partial_\rho U^\rho). \quad (\text{A6})$$

In terms of these the viscosities are

$$\zeta = \frac{1}{3} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^2}{E_a} f_a^{\text{eq}}(E_a/T) A_a(E_a), \quad (\text{A7})$$

$$\eta = \frac{2}{15} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^4}{E_a} f_a^{\text{eq}}(E_a/T) C_a(E_a). \quad (\text{A8})$$

The departure functions satisfy integral equations. For example, for two-body reactions,

$$\begin{aligned} & \frac{1}{E_a T} \left[ \left( \frac{1}{3} - v_s^2 \right) |\mathbf{p}|^2 - v_s^2 \left( \bar{m}_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right] \\ & = \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \\ & \quad \times f_b^{\text{eq}} W(a, b|c, d) \{A_a + A_b - A_c - A_d\}, \end{aligned} \quad (\text{A9})$$

and

$$\begin{aligned} \frac{p_a^\mu p_a^\nu}{2E_a T} & = \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} f_b^{\text{eq}} W(a, b|c, d) \\ & \quad \times \{C_a p_a^\mu p_a^\nu + C_b p_b^\mu p_b^\nu - C_c p_c^\mu p_c^\nu - C_d p_d^\mu p_d^\nu\}. \end{aligned} \quad (\text{A10})$$

For resonance formation and decay and multiparticle reactions see the text. The dissipative part of the energy-momentum tensor can be written as

$$\Delta T^{\mu\nu} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_a} \left( p_a^\mu p_a^\nu - U^\mu U^\nu \frac{d\bar{m}_a^2}{dT^2} \right) \delta \tilde{f}_a, \quad (\text{A11})$$

where  $\delta \tilde{f}_a$  is defined in the text. In the relaxation time approximation the viscosities can be written as

$$\eta = \frac{1}{15T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{p}|^4}{E_a^2} \tau_a(E_a) f_a^{\text{eq}}(E_a/T), \quad (\text{A12})$$

and

$$\begin{aligned} \zeta & = \frac{1}{T} \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{\tau_a(E_a)}{E_a^2} f_a^{\text{eq}}(E_a/T) \\ & \quad \times \left[ \left( \frac{1}{3} - v_s^2 \right) |\mathbf{p}|^2 - v_s^2 \left( \bar{m}_a^2 - T^2 \frac{d\bar{m}_a^2}{dT^2} \right) \right]^2. \end{aligned} \quad (\text{A13})$$

The effective masses are calculated self-consistently using the same potential  $U$  and same approximations as when calculating the equation of state, namely,

$$\bar{m}_a^2 = \left\langle \frac{\partial^2 U}{\partial \Phi_a^2} \right\rangle. \quad (\text{A14})$$

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