Bohr Hamiltonian with Davidson potential for triaxial nuclei

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A solution of the Bohr Hamiltonian appropriate for triaxial shapes, involving a Davidson potential in β and a steep harmonic oscillator in γ, centered around $\gamma = \pi/6$, is developed. Analytical expressions for spectra and $B(E2)$ transition rates ranging from a triaxial vibrator to the rigid triaxial rotator are obtained and compared to experimental results. Using a variational procedure, we point out that the $Z(5)$ solution, in which an infinite square well potential in β is used, corresponds to the critical point of the shape phase transition from a triaxial vibrator to the rigid triaxial rotator.

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I. INTRODUCTION

The advent of critical point symmetries [1–4], manifested experimentally [5–7] in nuclei on or near the point of shape phase transitions, revived interest in special solutions [8,9] of the Bohr Hamiltonian [10]. While shape phase transitions in nuclei originally were found [4,11] in the framework of the interacting boson model (IBM) [12], the first examples of critical point symmetries, the $E(5)$ symmetry [1] [corresponding to the second-order phase transition between spherical and γ-unstable (soft with respect to axial asymmetry) nuclei] and the $X(5)$ symmetry [2] (appropriate for the first-order phase transition between spherical and prolate deformed nuclei), have been developed as special solutions of the Bohr Hamiltonian, using an infinite square well potential in the β degree of freedom (related to the magnitude of the deformation). In $E(5)$, the potential is independent of the γ degree of freedom, related to the shape of the nucleus, whereas in $X(5)$ the potential can be separated into two terms, $u(\beta) + v(\gamma)$, the latter being a steep harmonic oscillator centered around $\gamma = 0$, corresponding to prolate deformed nuclei. The $Z(5)$ solution [13], developed later, formally resembles the $X(5)$ case in using a separable potential and an infinite square well potential in β, but it differs drastically in using a steep harmonic oscillator potential in the γ degree of freedom centered around $\gamma = \pi/6$, corresponding to triaxial shapes.

Triaxial shapes in nuclei have been considered for a long time, since the introduction of the rigid triaxial rotor [14,15], despite the fact that very few candidates have been found experimentally [16,17]. In the framework of the IBM, triaxial shapes can occur in three different cases:

(i) In the IBM-1 framework, in which no distinction between protons and neutrons is made, the inclusion of higher order (three-body) terms is needed [18,19].
(ii) In the IBM-2 framework, in which protons and neutrons are used as distinct entities, the inclusion of one-body and two-body terms suffices [20–22].
(iii) In the sdg-IBM framework (using s, d, and g bosons), the presence of the g boson also suffices [23].

Shape phase transitions involving rigid triaxial shapes have been studied recently in the IBM-2 framework [20–22], while in the sdg-IBM framework no transitions toward stable triaxial shapes have been found so far [23].

In the present work, the Z(5) solution is modified by replacing the infinite square well potential in β by a Davidson potential [24],

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{1}$$

where β_0 corresponds to the position of the minimum of the potential. This solution is going to be called $Z(5)$-D. Similar studies already exist in the literature for both the $E(5)$ [8,25] and $X(5)$ [26,27] cases. In addition, other potentials, such as β^{2n} potentials [28–30] and the Morse [31] and Kratzer [8] potentials, have been used in the $E(5)$ (γ-unstable) [32] and $X(5)$ ($\gamma \approx 0$, prolate deformed) [33] frameworks.

In addition to providing results that are easily comparable to experiment analytical solutions for the spectra and $B(E2)$ transition rates, the present study leads to an important by-product. Using a variational procedure applied earlier in the $E(5)$ and $X(5)$ frameworks [26,27], one can see that the $Z(5)$ solution can be interpreted as corresponding to the critical point of a shape phase transition between a triaxial vibrator and a rigid triaxial rotator.

In Secs. II and III, the β part and the γ part of the spectrum are considered, and $B(E2)$ transition rates are calculated in Sec. IV. Numerical results, including results of the previously mentioned variational procedure, are shown in Sec. V, and in Sec. VI a brief comparison to experimental results is attempted. Finally, conclusions and plans for further work are found in Sec. VII.

II. THE β PART OF THE SPECTRUM

The original Bohr Hamiltonian [10] is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \tag{2}$$

where β and γ are the usual collective coordinates, Q_k ($k = 1, 2, 3$) are the components of angular momentum in the intrinsic frame, and B is the mass parameter.

In the case in which the potential has a minimum around $\gamma = \pi/6$, the term involving the components of the angular momentum can be written [13] in the form $4(Q_1^2 + Q_2^2 + Q_3^2) - 3Q_1^2$. By using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (2), introducing [2] reduced energies $\epsilon = 2BE/\hbar^2$ and reduced potentials $u = 2BV/\hbar^2$, and assuming [2] that the reduced potential can be separated into two terms, one depending on β and the other depending on γ [i.e., $u(\beta, \gamma) = u(\beta) + v(\gamma)$], the Schrödinger equation can be approximately separated into two equations,

$$\left\{-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{4\beta^2}[4L(L+1) - 3\alpha^2] + u(\beta)\right\}\xi_{L,\alpha}(\beta) = \epsilon_\beta\xi_{L,\alpha}(\beta), \tag{3}$$

$$\left[-\frac{1}{\langle\beta^2\rangle\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + v(\gamma)\right]\eta(\gamma) = \epsilon_\gamma\eta(\gamma), \tag{4}$$

where L is the angular momentum quantum number, α is the projection of the angular momentum on the body-fixed $\hat{x}'$ axis (α has to be an even integer [34]), $\langle\beta^2\rangle$ is the average of β^2 over $\xi(\beta)$, and $\epsilon = \epsilon_\beta + \epsilon_\gamma$. It should be noticed that the separation of variables is approximate, since in Eq. (4) the quantity $\langle\beta^2\rangle$ appears, which depends on the quantum numbers L and α, appearing in Eq. (3). Therefore, an approximate separation of variables is achieved in the adiabatic limit, as in Ref. [35]. As a consequence, the energy relation $\epsilon = \epsilon_\beta + \epsilon_\gamma$ is also approximate.

The total wave function should have the form $\Psi(\beta, \gamma, \theta_i) = \xi_{L,\alpha}(\beta)\eta(\gamma)\mathcal{D}^L_{M,\alpha}(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, and M and α are the eigenvalues of the projections of angular momentum on the laboratory fixed $\hat{z}$ axis and the body-fixed $\hat{x}'$ axis respectively.

Instead of the projection α of the angular momentum on the $\hat{x}'$ axis, it is customary to introduce the wobbling quantum number [34,36] $n_w = L - \alpha$, which labels a series of bands with $L = n_w, n_w + 2, n_w + 4, \ldots$ (with $n_w > 0$) next to the ground-state band (with $n_w = 0$) [34].

Equation (3) has been solved in the case in which $u(\beta)$ is an infinite well potential, and the corresponding solution is called $Z(5)$ [13]. The spectrum is given by roots of Bessel functions, for which the notation has been kept the same as in Ref. [2], namely $E_{s,n_w,L}$, with the ground-state band corresponding to $s = 1, n_w = 0$.

Equation (3) is exactly solvable also in the case in which the potential has the form of a Davidson potential [24],

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{5}$$

where β_0 is the position of the minimum of the potential. When plugging the Davidson potential into Eq. (3), the β_0^4/β^2 term of the potential is combined with the $[4L(L+1) - 3\alpha^2]/4\beta^2$ term appearing there, and the equation is solved exactly [25,37] the eigenfunctions being Laguerre polynomials of the form

$$\xi_{n,n_w,L}(\beta) = \xi_{n,\alpha,L}(\beta) = \left[\frac{2n!}{\Gamma\left(n + a + \frac{5}{2}\right)}\right]^{1/2}\beta^a L_n^{a+\frac{3}{2}}(\beta^2)e^{-\beta^2/2}, \tag{6}$$

where $\Gamma(z)$ stands for the Γ function, n is the usual oscillator quantum number (which should be distinguished from the wobbling quantum number n_w), $L_n^a(z)$ denotes the Laguerre polynomials [38], and

$$a = -\frac{3}{2} + \sqrt{\frac{4L(L+1) - 3\alpha^2 + 9}{4} + \beta_0^4} = -\frac{3}{2} + \sqrt{\frac{L(L+4) + 3n_w(2L - n_w) + 9}{4} + \beta_0^4}. \tag{7}$$

The energy eigenvalues are then (in $\hbar\omega = 1$ units)

$$E_{n,L}^{(n_w)} = 2n + a + \frac{5}{2} = 2n + 1 + \sqrt{\frac{L(L+4) + 3n_w(2L - n_w) + 9}{4} + \beta_0^4}, \tag{8}$$

where $n = 0, 1, 2, \ldots$. One can see that a formal correspondence between the energy levels of the $Z(5)$ model and the present model, to which we refer as the $Z(5)$-D model, can be established through the relation $n = s - 1$, which expresses a formal one-to-one correspondence between the states in the two spectra, while the origins of the two quantum numbers are different, s labeling the order of a zero of a Bessel function and n labeling the number of zeros of a Laguerre polynomial. For the energy states, the notation $E_{s,n_w,L} = E_{n+1,n_w,L}$ is used, as in Refs. [2,13]. Therefore, the ground-state band corresponds to $s = 1$ ($n = 0$) and $n_w = 0$.

In the limit $\beta_0 \to \infty$, one can expand the square root in Eq. (8) and keep only the lowest order term, thus obtaining

$$E_L^{(n_w)} = A[L(L+4) + 3n_w(2L - n_w)], \tag{9}$$

where A is a constant, which is the spectrum of the triaxial rotator obtained in Ref. [34].

In the special case $\beta_0 = 0$, that is, in the case where a harmonic oscillator is used, one obtains a parameter-free (up to overall scale factors), exactly soluble model to which we refer as the $Z(5)$-β^2 model, in analogy to the $E(5)$-β^{2n} [28,29] and $X(5)$-β^{2n} [30] models. This model represents a triaxial vibrator.

III. THE γ PART OF THE SPECTRUM

The γ part of the spectrum is obtained from Eq. (4), as described in Ref. [13], by putting in it a harmonic oscillator potential having a minimum at $\gamma = \pi/6$, that is,

$$v(\gamma) = \frac{1}{2}c\left(\gamma - \frac{\pi}{6}\right)^2 = \frac{1}{2}c\tilde{\gamma}^2, \qquad \tilde{\gamma} = \gamma - \frac{\pi}{6}. \tag{10}$$

In the case of $\gamma \approx \pi/6$, a simple harmonic oscillator equation in the variable $\tilde{\gamma}$ occurs. Similar potentials and solutions in the γ variable have been considered in [10,39].

The total energy in the case of the $Z(5)$-D model is then

$$E(n, n_w, L, n_{\tilde{\gamma}}, \beta_0) = E_0 + A\left[2n + 1 + \sqrt{\frac{L(L+4) + 3n_w(2L - n_w) + 9}{4} + \beta_0^4}\right] + Bn_{\tilde{\gamma}}, \tag{11}$$

where $n_{\tilde{\gamma}}$ is the number of oscillator quanta in the $\tilde{\gamma}$ degree of freedom, and E_0, A, and B are free parameters.

It should be noticed that in Eq. (4) there is a latent dependence on s, L, and n_w "hidden" in the $\langle\beta^2\rangle$ term. The approximate separation of the β and γ variables is achieved by considering an adiabatic limit, as in the $X(5)$ case [2,35].

IV. B(E2) TRANSITION RATES

The quadrupole operator is given by

$$T^{(\mathrm{E2})}_{\mu} = t\beta\left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i)\cos\left(\gamma - \frac{2\pi}{3}\right) + \frac{1}{\sqrt{2}}[\mathcal{D}^{(2)}_{\mu,2}(\theta_i) + \mathcal{D}^{(2)}_{\mu,-2}(\theta_i)]\sin\left(\gamma - \frac{2\pi}{3}\right)\right], \tag{12}$$

where t is a scale factor, while in the Wigner functions, $\mathcal{D}^{(2)}$, the quantum number α appears next to μ, and the quantity $\gamma - 2\pi/3$ in the trigonometric functions is obtained from $\gamma - 2\pi k/3$ for $k = 1$, since in the present case the projection α along the body-fixed $\hat{x}'$ axis is used.

The symmetrized wave function for $Z(5)$-D reads

$$\Psi(\beta, \gamma, \theta_i) = \xi_{n,\alpha,L}(\beta)\eta_{n_{\tilde{\gamma}}}(\tilde{\gamma})\sqrt{\frac{2L+1}{16\pi^2(1+\delta_{\alpha,0})}} \times \left[\mathcal{D}^{(L)}_{\mu,\alpha} + (-1)^L\mathcal{D}^{(L)}_{\mu,-\alpha}\right], \tag{13}$$

where the normalization factor occurs from the standard integrals involving two Wigner functions [40] and is the same as in [34]. The α has to be an even integer [34], while for $\alpha = 0$ it is clear that only even values of L are allowed, since the symmetrized wave function is vanishing otherwise.

The calculation of $B(E2)$'s proceeds as in Ref. [13] and need not be repeated here. In the calculation of matrix elements, the integral over $\tilde{\gamma}$ leads to unity [because of the normalization of $\eta(\tilde{\gamma})$, taking into account that γ in Eq. (12) is fixed to the $\pi/6$ value, because of the steep potential used in γ], while the integral over β takes the form

$$I_{\beta}(n_i, L_i, \alpha_i, n_f, L_f, \alpha_f) = \int \beta\xi_{n_i,\alpha_i,L_i}(\beta)\xi_{n_f,\alpha_f,L_f}(\beta)\beta^4 d\beta, \tag{14}$$

where the β factor comes from Eq. (12) and the β^4 factor comes from the volume element [10]. It is worth remembering, though, that a $\Delta\alpha = \pm 2$ selection rule occurs, which results in vanishing quadrupole moments.

V. NUMERICAL RESULTS

A. Spectra

The lowest bands for the $Z(5)$-D model are shown in Table I for the limiting parameter values $\beta_0 = 0$ [the $Z(5)$-β^2 model] and $\beta \to \infty$ (the triaxial rotor model [34,39]), as well as for the intermediate value $\beta_0 = 2$ (for illustrative purposes). The levels of $Z(5)$ [13] are also shown for comparison. The bands shown are as follows:

(i) The ground-state band (gsb), with ($s = 1, n_w = 0$).
(ii) The quasi-γ_1 band, composed of the even L levels with ($s = 1, n_w = 2$) and the odd L levels with ($s = 1, n_w = 1$).
(iii) The quasi-γ_2 band, composed of the even L levels with ($s = 1, n_w = 4$) and the odd L levels with ($s = 1, n_w = 3$).
(iv) The quasi-β_1 band, with ($s = 2, n_w = 0$).
(v) The quasi-β_2 band, with ($s = 3, n_w = 0$).

Since the last two bands go to infinity for $\beta_0 \to \infty$, the energy levels for $\beta_0 = 3$ have been shown instead.

In all cases, $B = 0$ has been used in Eq. (11), that is, the term involving $n_{\tilde{\gamma}}$ has been ignored.

For all bands, a uniform raising of the energies from the triaxial vibrator ($\beta_0 = 0$) values to the triaxial rigid rotator ($\beta_0 \to \infty$) values is observed.

A quantity that is very sensitive to structural changes (since it is a discrete derivative of energies) is the odd-even staggering in γ bands, described by quantity [16]

$$S(J) = \frac{E(J^+_\gamma) + E((J-2)^+_\gamma) - 2E((J-1)^+_\gamma)}{E(2^+_1)}, \tag{15}$$

which measures the displacement of the $(J-1)^+_\gamma$ level relative to the average of its neighbors, J^+_γ and $(J-2)^+_\gamma$, normalized to the energy of the first excited state of the ground-state band, 2^+_1.

It is known [17] that γ-soft shapes exhibit staggering with negative values at even L and positive values at odd L, while triaxial γ-rigid shapes exhibit the opposite behavior, that is, positive values at even L and negative values at odd L. In Table I, it is clear that the present models exhibit strong staggering of the triaxial type, with the even-L levels growing much faster with L than the odd-L levels.

B. Variational procedure

A variational procedure appropriate for locating the behavior of various physical quantities at a critical point has been introduced [26,27] and applied for recovering the E(5) [1] and X(5) [2] ground-state bands from Davidson potentials in the relevant frameworks. The method is applicable in cases in which one has a one-parameter potential spanning the region between two limiting symmetries. The method is based

TABLE I. Energy spectra of the Z(5)-D model (for different values of the parameter β_0), and for the Z(5) model [13]. $\beta_0 = 0$ corresponds to the Z(5)-β^2 model (a triaxial vibrator), while $\beta \to \infty$ is the rigid triaxial rotator [14,15]. The notation L_{s,n_w} is used. All levels are measured from the ground state, $0_{1,0}$, and are normalized to the first excited state, $2_{1,0}$. See Subsec. V A for further discussion. In addition, the energy levels resulting from the variational procedure of Subsec. V B are reported (labelled by "Var"), along with the parameter values $\beta_{0,m}$ at which they are obtained. See Subsec. V B for further discussion.

β_0	0	2	∞				β_0	0	2	∞			
L_{s,n_w}				$\beta_{0,m}$	Var	$Z(5)$	L_{s,n_w}				$\beta_{0,m}$	Var	$Z(5)$
$0_{1,0}$	0.000	0.000	0.000			0.000							
$2_{1,0}$	1.000	1.000	1.000			1.000							
$4_{1,0}$	2.150	2.521	2.667	1.375	2.341	2.350							
$6_{1,0}$	3.353	4.424	5.000	1.474	3.956	3.984							
$8_{1,0}$	4.579	6.596	8.000	1.562	5.819	5.877							
$10_{1,0}$	5.817	8.957	11.667	1.640	7.915	8.019							
$12_{1,0}$	7.063	11.450	16.000	1.713	10.237	10.403							
$14_{1,0}$	8.313	14.039	21.000	1.780	12.781	13.024							
$16_{1,0}$	9.566	16.698	26.667	1.843	15.544	15.878							
$18_{1,0}$	10.821	19.410	33.000	1.902	18.523	18.964							
$20_{1,0}$	12.077	22.163	40.000	1.960	21.719	22.279							
							$2_{1,2}$	1.734	1.932	2.000	1.336	1.833	1.837
$3_{1,1}$	2.343	2.807	3.000	1.392	2.586	2.597	$4_{1,2}$	3.649	4.930	5.667	1.496	4.386	4.420
$5_{1,1}$	3.791	5.177	6.000	1.507	4.597	4.634	$6_{1,2}$	5.281	7.917	10.000	1.607	6.981	7.063
$7_{1,1}$	5.169	7.703	9.667	1.600	6.790	6.869	$8_{1,2}$	6.791	10.898	15.000	1.697	9.713	9.864
$9_{1,1}$	6.511	10.333	14.000	1.681	9.182	9.318	$10_{1,2}$	8.234	13.874	20.667	1.776	12.615	12.852
$11_{1,1}$	7.832	13.035	19.000	1.754	11.778	11.989	$12_{1,2}$	9.635	16.847	27.000	1.846	15.703	16.043
$13_{1,1}$	9.140	15.788	24.667	1.822	14.581	14.882	$14_{1,2}$	11.008	19.818	34.000	1.911	18.986	19.443
$15_{1,1}$	10.438	18.579	31.000	1.885	17.593	18.000	$16_{1,2}$	12.360	22.787	41.667	1.972	22.468	23.056
$17_{1,1}$	11.730	21.399	38.000	1.944	20.815	21.341	$18_{1,2}$	13.698	25.755	50.000	2.030	26.154	26.884
$19_{1,1}$	13.017	24.241	45.667	2.001	24.248	24.905	$20_{1,2}$	15.024	28.721	59.000	2.085	30.046	30.928
							$4_{1,4}$	4.066	5.663	6.667	1.526	5.012	5.056
$5_{1,3}$	4.939	7.268	9.000	1.585	6.406	6.476	$6_{1,4}$	6.221	9.753	13.000	1.665	8.644	8.767
$7_{1,3}$	6.699	10.711	14.667	1.692	9.537	9.683	$8_{1,4}$	8.075	13.541	20.000	1.767	12.282	12.508
$9_{1,3}$	8.313	14.039	21.000	1.780	12.781	13.024	$10_{1,4}$	9.773	17.143	27.667	1.853	16.021	16.372
$11_{1,3}$	9.841	17.289	28.000	1.856	16.180	16.536	$12_{1,4}$	11.374	20.618	36.000	1.928	19.904	20.396
$13_{1,3}$	11.314	20.486	35.667	1.925	19.752	20.237	$14_{1,4}$	12.910	24.003	45.000	1.996	23.953	24.598
$15_{1,3}$	12.747	23.642	44.000	1.989	23.509	24.137	$16_{1,4}$	14.399	27.320	54.667	2.059	28.182	28.991
$17_{1,3}$	14.152	26.768	53.000	2.049	27.460	28.241	$18_{1,4}$	15.853	30.586	65.000	2.118	32.601	33.581
$19_{1,3}$	15.536	29.871	62.667	2.105	31.611	32.553	$20_{1,4}$	17.281	33.811	76.000	2.174	37.217	38.373

β_0	0	2	3				β_0	0	2	3			
L_{s,n_w}				$\beta_{0,m}$	Var	$Z(5)$	L_{s,n_w}				$\beta_{0,m}$	Var	$Z(5)$
$0_{2,0}$	2.528	5.921	12.274			3.913	$0_{3,0}$	5.055	11.842	24.548			9.782
$2_{2,0}$	3.528	6.921	13.274			5.697	$2_{3,0}$	6.055	12.842	25.548			12.343
$4_{2,0}$	4.678	8.442	14.903			7.962	$4_{3,0}$	7.205	14.363	27.177			15.506
$6_{2,0}$	5.881	10.345	17.110			10.567	$6_{3,0}$	8.408	16.266	29.384			19.059
$8_{2,0}$	7.107	12.517	19.835			13.469	$8_{3,0}$	9.634	18.439	32.109			22.933
$10_{2,0}$	8.345	14.878	23.015			16.646	$10_{3,0}$	10.873	20.799	35.289			27.103
$12_{2,0}$	9.590	17.371	26.588			20.088	$12_{3,0}$	12.118	23.292	38.862			31.552
$14_{2,0}$	10.840	19.960	30.497			23.788	$14_{3,0}$	13.368	25.881	42.771			36.272
$16_{2,0}$	12.093	22.619	34.692			27.740	$16_{3,0}$	14.621	28.540	46.966			41.258
$18_{2,0}$	13.348	25.331	39.129			31.942	$18_{3,0}$	15.876	31.253	51.403			46.504
$20_{2,0}$	14.605	28.084	43.772			36.390	$20_{3,0}$	17.132	34.005	56.046			52.007

on the fact that if a shape phase transition occurs between these two symmetries, the rate of change of various physical quantities should become maximum at the critical point [41]. The parameter value corresponding to the maximum, $\beta_{0,m}$, is determined for each value of angular momentum separately.

The variational procedure used here resembles the standard Ritz variational procedure of quantum mechanics [42], in which a trial wave function containing a free parameter is used, whereas here a potential containing a free parameter is used, the difference being that in the Ritz approach the

TABLE II. Intraband and interband B(E2) transition rates, normalized to the one between the two lowest states, B(E2;$2_{1,0} \to 0_{1,0}$), are given for Z(5)-D model (for different values of the parameter β_0), for the Z(5) model [13], and for the O(6) limit of the IBM [12]. $\beta_0 = 0$ corresponds to the Z(5)-β^2 model (a triaxial vibrator), while $\beta \to \infty$ is the rigid triaxial rotator [14,15]. The notation L_{s,n_w} is used, while the initial state is labeled by (i) and the final state is labeled by (f).

β_0		0	2	∞			β_0		0	2	∞		
$L^{(i)}_{s,n_w}$	$L^{(f)}_{s,n_w}$				$O(6)$	$Z(5)$	$L^{(i)}_{s,n_w}$	$L^{(f)}_{s,n_w}$				$O(6)$	$Z(5)$
$2_{1,0}$	$0_{1,0}$	1.000	1.000	1.000	1.000	1.000	$2_{2,0}$	$0_{2,0}$	1.480	1.284	1.000		0.774
$4_{1,0}$	$2_{1,0}$	1.834	1.493	1.389	1.429	1.590	$4_{2,0}$	$2_{2,0}$	2.440	1.865	1.389		1.192
$6_{1,0}$	$4_{1,0}$	2.919	2.041	1.731	1.667	2.203	$6_{2,0}$	$4_{2,0}$	3.647	2.477	1.731		1.643
$8_{1,0}$	$6_{1,0}$	3.955	2.497	1.912	1.818	2.635	$8_{2,0}$	$6_{2,0}$	4.746	2.955	1.912		1.975
$10_{1,0}$	$8_{1,0}$	4.976	2.934	2.024	1.923	2.967	$10_{2,0}$	$8_{2,0}$	5.804	3.398	2.024		2.242
$12_{1,0}$	$10_{1,0}$	5.989	3.370	2.100	2.000	3.234	$12_{2,0}$	$10_{2,0}$	6.844	3.836	2.100		2.466
$14_{1,0}$	$12_{1,0}$	6.999	3.810	2.155	2.059	3.455	$14_{2,0}$	$12_{2,0}$	7.874	4.277	2.155		2.660
$16_{1,0}$	$14_{1,0}$	8.007	4.257	2.197	2.105	3.642	$16_{2,0}$	$14_{2,0}$	8.897	4.723	2.197		2.829
$18_{1,0}$	$16_{1,0}$	9.014	4.708	2.230	2.143	3.803	$18_{2,0}$	$16_{2,0}$	9.915	5.175	2.230		2.980
$20_{1,0}$	$18_{1,0}$	10.020	5.164	2.256	2.174	3.944	$20_{2,0}$	$18_{2,0}$	10.931	5.630	2.256		3.115
$4_{1,2}$	$2_{1,2}$	0.912	0.682	0.595	0.873	0.736	$3_{1,1}$	$2_{1,2}$	2.731	2.006	1.786	1.190	2.171
$6_{1,2}$	$4_{1,2}$	1.583	0.986	0.734	1.240	1.031	$5_{1,1}$	$4_{1,2}$	1.991	1.244	0.955	0.434	1.313
$8_{1,2}$	$6_{1,2}$	2.816	1.617	1.051	1.462	1.590	$7_{1,1}$	$6_{1,2}$	2.183	1.262	0.851	0.231	1.260
$10_{1,2}$	$8_{1,2}$	4.038	2.215	1.278	1.614	2.035	$9_{1,1}$	$8_{1,2}$	2.247	1.241	0.746	0.145	1.164
$12_{1,2}$	$10_{1,2}$	5.231	2.786	1.446	1.726	2.394	$11_{1,1}$	$10_{1,2}$	2.265	1.214	0.657	0.100	1.069
$14_{1,2}$	$12_{1,2}$	6.395	3.339	1.574	1.813	2.690	$13_{1,1}$	$12_{1,2}$	2.265	1.189	0.585	0.073	0.984
$16_{1,2}$	$14_{1,2}$	7.535	3.877	1.674	1.882	2.938	$15_{1,1}$	$14_{1,2}$	2.258	1.168	0.526	0.056	0.910
$18_{1,2}$	$16_{1,2}$	8.655	4.406	1.755	1.938	3.151	$17_{1,1}$	$16_{1,2}$	2.247	1.149	0.478	0.044	0.846
$20_{1,2}$	$18_{1,2}$	9.759	4.927	1.822	1.985	3.335	$19_{1,1}$	$18_{1,2}$	2.236	1.133	0.437	0.036	0.790
$2_{1,2}$	$2_{1,0}$	1.865	1.520	1.429	1.429	1.620	$3_{1,1}$	$4_{1,0}$	1.618	1.147	1.000	0.476	1.243
$4_{1,2}$	$4_{1,0}$	0.459	0.323	0.273	0.794	0.348	$5_{1,1}$	$6_{1,0}$	1.449	0.917	0.714	0.430	0.972
$6_{1,2}$	$6_{1,0}$	0.292	0.187	0.143	0.579	0.198	$7_{1,1}$	$8_{1,0}$	1.351	0.796	0.556	0.374	0.808
$8_{1,2}$	$8_{1,0}$	0.211	0.127	0.088	0.462	0.129	$9_{1,1}$	$10_{1,0}$	1.287	0.724	0.455	0.327	0.696
$10_{1,2}$	$10_{1,0}$	0.165	0.094	0.059	0.386	0.092	$11_{1,1}$	$12_{1,0}$	1.243	0.676	0.385	0.291	0.614
$12_{1,2}$	$12_{1,0}$	0.135	0.074	0.043	0.332	0.069	$13_{1,1}$	$14_{1,0}$	1.211	0.643	0.333	0.261	0.551
$14_{1,2}$	$14_{1,0}$	0.114	0.061	0.032	0.292	0.054	$15_{1,1}$	$16_{1,0}$	1.186	0.619	0.294	0.237	0.507
$16_{1,2}$	$16_{1,0}$	0.099	0.052	0.025	0.261	0.043	$17_{1,1}$	$18_{1,0}$	1.167	0.601	0.263	0.216	0.459
$18_{1,2}$	$18_{1,0}$	0.087	0.045	0.020	0.236	0.035	$19_{1,1}$	$20_{1,0}$	1.151	0.587	0.238	0.199	0.425
$20_{1,2}$	$20_{1,0}$	0.078	0.040	0.017	0.215	0.030							
$5_{1,1}$	$3_{1,1}$	1.667	1.147	0.955	0.955	1.235							
$7_{1,1}$	$5_{1,1}$	2.891	1.778	1.310	1.319	1.851							
$9_{1,1}$	$7_{1,1}$	4.061	2.338	1.535	1.528	2.308							
$11_{1,1}$	$9_{1,1}$	5.191	2.865	1.690	1.668	2.665							
$13_{1,1}$	$11_{1,1}$	6.292	3.374	1.802	1.771	2.952							
$15_{1,1}$	$13_{1,1}$	7.373	3.873	1.887	1.850	3.190							
$17_{1,1}$	$15_{1,1}$	8.440	4.366	1.954	1.913	3.392							
$19_{1,1}$	$17_{1,1}$	9.496	4.856	2.007	1.965	3.566							

parameter is determined by minimizing the energy, whereas here the parameter is found to maximize the rate of change of the relevant physical quantity. L-dependent potentials, like the ones occurring here, have been used in nuclear physics in optical model potentials [43–45], as well as in the study of quasimolecular resonances [46]. The method is also analogous to the variable moment of inertia model (VMI) [47], in which the energy is minimized with respect to the moment of inertia (which depends on the angular momentum) separately for each value of the angular momentm L.

In the present case, as seen in Subsec. II A, the Davidson potentials of Eq. (1) lead to a triaxial vibrator Z(5)-β^2 for $\beta_0 = 0$, while they give the rigid triaxial rotator [14,15,34] for $\beta_0 \to \infty$. Applying the variational procedure to the energy ratios E(L)/E(2) of the ground-state band ($s = 1$, $n_w = 0$) of the Z(5)-D model, where β_0 is the free parameter serving to span the region between the two limiting cases, we are led to the results shown in Table I, where for each value of the angular momentum L the location of the maximum, $\beta_{0,m}$, and the corresponding energy (normalized to the energy of the first excited state) are given. It is clear that the band determined through the variational procedure agrees very well with the ground-state band of the $Z(5)$ model. The agreement remains equally good for the $s = 1, n_w = 1, 2, 3, 4$ bands, also shown in

Table I, thus indicating that the $Z(5)$ model is possibly related to a shape phase transition from a triaxial vibrator to the rigid triaxial rotator.

C. B(E2) transition rates

Both intraband and interband $B(E2)$ transition rates for the same models are reported in Table II. In addition, results for the O(6) limit of the IBM [12] are shown for comparison, derived from the expressions given in Ref. [12], the final results reading

$$R_{g,g}(L+2\to L) = \frac{B[E2;(L+2)_g\to L_g]}{B(E2;2_g\to 0_g)} = \frac{5}{2}\frac{(L+2)}{(L+5)}\frac{(2N-L)(2N+L+8)}{4N(N+4)}, \quad (16)$$

$$R_{\gamma_{\rm even},g}(L\to L) = \frac{B(E2;L_\gamma\to L_g)}{B(E2;2_g\to 0_g)} = \frac{10(L+1)}{(L+5)(2L-1)}\frac{(2N-L)(2N+L+8)}{4N(N+4)}, \quad (17)$$

$$R_{\gamma_{\rm odd},g}(L\to L+1) = \frac{B[E2;L_\gamma\to (L+1)_g]}{B(E2;2_g\to 0_g)} = \frac{5(L-1)(2L+3)}{L(L+6)(2L+1)}\frac{(2N-L-1)(2N+L+9)}{4N(N+4)}, \quad (18)$$

$$R_{\gamma_{\rm even}\to\gamma_{\rm even}}(L+2\to L) = \frac{B[E2;(L+2)_\gamma\to L_\gamma]}{B(E2;2_g\to 0_g)} = \frac{5L(2L+7)}{2(L+7)(2L+3)}\frac{(2N-L-2)(2N+L+10)}{4N(N+4)}, \quad (19)$$

$$R_{\gamma_{\rm odd}\to\gamma_{\rm odd}}(L+2\to L) = \frac{B[E2;(L+2)_\gamma\to L_\gamma]}{B(E2;2_g\to 0_g)} = \frac{5(L-1)(L+3)(L+4)}{2(L+1)(L+2)(L+8)}\frac{(2N-L-3)(2N+L+11)}{4N(N+4)}, \quad (20)$$

$$R_{\gamma_{\rm odd}\to\gamma_{\rm even}}(L\to L-1) = \frac{B[E2;L_\gamma\to (L-1)_\gamma]}{B(E2;2_g\to 0_g)} = \frac{30(L+2)}{(L-1)(L+6)(2L+1)}\frac{(2N-L-1)(2N+L+9)}{4N(N+4)}. \quad (21)$$

In all of the previous equations, N stands for the boson number. Numerical results for $N\to\infty$ are reported in Table II. We remark that the $O(6)$ predictions for $N\to\infty$ are very similar to the ones of the rigid triaxial rotator [14,15], that is, to those of the $Z(5)$-D model for $\beta_0\to\infty$.

TABLE III. Comparison of theoretical predictions of the $Z(5)$-D model, labeled by the relevant β_0 value, to experimental spectra of ^{128}Xe [48], ^{130}Xe [49], and ^{132}Xe [50]. In each column, all energies are normalized to the energy of the relevant 2_1^+ state. The quality measure σ of Eq. (22) is used. See Sec. VI for further discussion.

L_{s,n_w}	^{128}Xe exp	^{128}Xe $\beta_0=1.32$	^{130}Xe exp	^{130}Xe $\beta_0=1.11$	^{132}Xe exp	^{132}Xe $\beta_0=0$
$4_{1,0}$	2.333	2.323	2.247	2.255	2.157	2.150
$6_{1,0}$	3.922	3.805	3.627	3.621	3.163	3.353
$8_{1,0}$	5.674	5.372	5.031	5.040		
$10_{1,0}$	7.597	6.986	6.457	6.489		
$12_{1,0}$			7.867	7.956		
$14_{1,0}$			9.458	9.434		
$2_{1,2}$	2.189	1.830	2.093	1.793	1.944	1.734
$4_{1,2}$	3.620	4.180	3.373	3.961	2.940	3.649
$6_{1,2}$	5.150	6.284				
$3_{1,1}$	3.228	2.555	3.045	2.471	2.701	2.343
$5_{1,1}$	4.508	4.360	4.051	4.125	3.246	3.791
$7_{1,1}$	6.165	6.138				
$0_{2,0}$	3.574	3.452	3.346	3.028	2.771	2.528
$2_{2,0}$	4.515	4.452				
σ		0.495		0.297		0.422

VI. COMPARISON TO EXPERIMENT

As seen from Table I, one should look for nuclei having ground-state bands characterized by $R_{4/2}=E(4)/E(2)$ ratios between 2.150 and 2.667, while the γ_1 bandhead (normalized to the 2_1^+ state) should be between 1.734 and 2.000 and the β_1 bandhead (normalized in the same way) should be above 2.528. The Xe isotopes $^{128-132}$Xe, lying below the $N=82$ shell closure, nearly fulfill these conditions. Results of one-parameter (β_0) rms fits are shown in Table III, with σ being the quality measure

$$\sigma=\sqrt{\frac{\sum_{i=1}^{n}[E_i(\exp)-E_i(\text{th})]^2}{(n-1)E(2_1^+)^2}}. \quad (22)$$

TABLE IV. Comparison of theoretical predictions of the $Z(5)$-D model, labeled by the relevant β_0 value, to experimental $B(E2)$ values of ^{128}Xe [48] and ^{132}Xe [50]. In each column, all $B(E2)$'s are normalized to the $2_1^+\to 0_1^+$ transition. See Sec. VI for further discussion.

$L^{(i)}_{s,n_w}$	$L^{(f)}_{s,n_w}$	^{128}Xe exp	^{128}Xe $\beta_0=1.32$	^{132}Xe exp	^{132}Xe $\beta_0=0$
$4_{1,0}$	$2_{1,0}$	1.468 ± 0.201	1.648	1.238 ± 0.180	1.834
$6_{1,0}$	$4_{1,0}$	1.940 ± 0.275	2.464		
$8_{1,0}$	$6_{1,0}$	2.388 ± 0.398	3.228		
$2_{1,2}$	$2_{1,0}$	1.194 ± 0.187	1.673	1.775 ± 0.288	1.865
$2_{1,2}$	$0_{1,0}$	0.016 ± 0.002	0.000	0.003 ± 0.001	0.000

The overall agreement is good, with the notable exception of the even-L levels of the quasi-γ_1 band, which grow too fast, as already remarked at the end of Subsec. V A. As a result, the theoretical predictions exhibit strong triaxial odd-even staggering, which is not seen experimentally. Indeed, the Xe isotopes are known [17] to exhibit staggering of the γ-soft type, in contrast to the strong triaxial γ-rigid staggering shown here by the theoretical values. The only nuclei found in the extended recent search of Ref. [17] to possess γ_1 bands with triaxial shapes are ^{112}Ru, ^{170}Er, ^{192}Os, ^{192}Pt, and ^{232}Th, all of them located in the nuclear chart far from the Xe isotopes considered here.

In Table IV the existing $B(E2)$ transition rates of the same nuclei are compared to the $Z(5)$-D model predictions for the parameter values obtained from fitting the spectra. No fitting of the $B(E2)$ values has been performed. The theoretical predictions are in general higher than the experimental values, but in most cases lie within or near the experimental error bars.

VII. CONCLUSIONS

$Z(5)$ [13] is a solution of the Bohr Hamiltonian similar to the $X(5)$ [2] solution, with the notable difference that it regards triaxial shapes ($\gamma \approx \pi/6$) instead of prolate deformed shapes ($\gamma \approx 0$). Predictions for spectra and $B(E2)$ transition rates are parameter independent (up to overall scale factors).

In the present $Z(5)$-D solution, the infinite square well $u(\beta)$ potential, used in $Z(5)$, is replaced by the Davidson potential [24], involving a free parameter, β_0. As a result, $Z(5)$-D can cover the region between a triaxial vibrator and the rigid triaxial rotator [14,15]. In addition to providing easily comparable to experiment analytical solutions for spectra and $B(E2)$ values within this wide region, the present solution has an interesting by-product. Using a variational procedure [26,27], we point out that the $Z(5)$ solution corresponds to the critical point of the shape phase transition from a triaxial vibrator to the rigid triaxial rotator. However, the $Z(5)$ solution is *not* a special case of $Z(5)$-D, obtained for a specific parameter value, or a limiting case of $Z(5)$-D. By using the Davidson potential, one can cover the whole way from triaxial vibrator to triaxial rotator, but one cannot get the critical point as a special case. This is due to the shape of the Davidson potential, which is not flat, as the potential is expected to be at the critical point. The same situation has occurred in the ESD model [51], in which the Davidson potential is used in order to interpolate between a vibrator and the prolate axial rotator with $\gamma \approx 0$. By using the Exactly Separable Davidson (ESD) model, one can obtain very good fits of many nuclei from the prolate rotator limit down to close to the critical point, but one cannot describe the nuclei very close to the critical point [51].

Concerning the separation of variables, which allowed for analytical solutions, a potential of the form $u(\beta) + v(\gamma)$ has been used, bringing in the approximations used in $X(5)$ [2]. These approximations can be avoided in two ways:

(i) Using potentials of the form $u(\beta) + v(\gamma)/\beta^2$, which are known [8] to allow for exact separation of variables without any approximations.

(ii) Using the powerful techniques of the algebraic collective model [52–54], which allow for the exact numerical diagonalization of any Bohr Hamiltonian.

The first path has been used for a detailed study of the Davidson potential plugged in the Bohr Hamiltonian for $\gamma \approx 0$ [51]. The main advantage of this solution is that all bands are treated on equal footing with respect to the influence of the $v(\gamma)$ potential, while in the present solution only the quasi-γ bands are affected. A similar study for $\gamma \approx \pi/6$ case would be interesting. The analytical solution and a brief comparison to experiment in the Os region has already been given in Ref. [55].

The second path has been recently used for the description of a triaxial symmetry top [54], as well as for the study the onset of rigid triaxial deformation [56]. Further investigations of triaxial shapes using this powerful tool should also be revealing.

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