Comparative tests of isospin-symmetry-breaking corrections to superallowed $0^+ \rightarrow 0^+$ nuclear β decay

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We present a test with which to evaluate the calculated isospin-symmetry-breaking corrections to superallowed $0^+ \rightarrow 0^+$ nuclear β decay. The test is based on the corrected experimental $\mathcal{F}t$ values being required to satisfy conservation of the vector current (CVC). When applied to six sets of published calculations, the test demonstrates quantitatively that only one set, the one based on the shell model with Saxon-Woods radial wave functions, provides satisfactory agreement with CVC. This test can easily be applied to any sets of calculated correction terms that are produced in future.

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I. INTRODUCTION

Superallowed $0^+ \rightarrow 0^+$ β decay between $T = 1$ nuclear analog states has been a subject of continuous and often intense study for six decades. The ft values for such transitions are nearly independent of nuclear-structure ambiguities and depend uniquely on the vector part of the weak interaction. Their measurement gives us access to clean tests of some of the fundamental precepts of weak-interaction theory, and, over the years, this strong motivation has led to very high precision being achieved both in the experiments and in the theory used to interpret them.

The most recent survey of world data [1] finds ten of these superallowed transitions with measured ft values known to 0.1% precision or better and three more that have a precision between 0.1% and 0.3%. An analysis of the ft values [1] demonstrated that the vector coupling constant G_V has the same value for all 13 transitions to within $\pm 0.013\%$, thus confirming a key part of the conserved vector current (CVC) hypothesis, and it set an upper limit on a possible scalar current at 0.2% of the vector current. With both these outcomes established, the results could then be used to extract a value for $V_{\rm ud}$, the up-down element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, with which the top-row unitarity test of the CKM matrix yielded the result [2] 0.9999(6). This is in remarkable agreement with the standard model, and the tight uncertainty significantly limits the scope for any new physics beyond the model. Further tightening of the uncertainty would increase the impact of this result even more.

Although the role played by nuclear structure is relatively small, the precision currently reached by experiment is such that the theoretical uncertainties introduced by correction terms required in the analysis of the ft-value data now predominate over the experimental uncertainties. Two of these correction terms depend on nuclear structure, and together they are the second largest contributor to the overall uncertainty in $V_{\rm ud}$. The largest contributor is the nucleus-independent component of the radiative correction but, at present, there seems little opportunity for further improvement there.

Thus, it is the nuclear-structure-dependent terms that have attracted the greatest attention, particularly recently. The most widely used of these latter correction terms are those calculated by the present authors, which have been tabulated for all the superallowed transitions of interest in Ref. [3]. However, there are a growing number of alternative choices [4–9] available for one of the two correction terms—the one that accounts for isospin symmetry breaking—including a set we offer ourselves [1]. There has also been a claim, albeit unsupported by any detailed computations, that our calculations neglect a radial excitation term, which is purported to be important [10]. To counterbalance that, however, there are two recent papers that confirm our result: one [11] does so based on a semiempirical analysis of the data, while the other [12] quotes the average results from a Skyrme-density-functional-theory calculation in which simultaneous isospin and angular-momentum projections have been incorporated.

Clearly, it would be valuable if the various sets of calculated isospin-symmetry-breaking correction terms could be tested against the data and their relative merits quantitatively evaluated, since this must surely be a first step in any attempt to reduce the uncertainty attributed to these corrections. In this paper, we address ourselves to devising and then applying such a test.

We begin by describing how information on the fundamental weak-interaction parameters is extracted from the experimental ft-value data. We will overview the role played by all the theoretical corrections but will focus, in particular, on the isospin-symmetry-breaking term. This will lead naturally to the test we propose as a means of evaluating the efficacy of any calculated set of these terms available now or in the future. We will then outline the methods currently used to calculate the isospin-symmetry-breaking term and proceed to apply our test to each of them. Finally, we will evaluate the results of the test and present our conclusions.

II. THE ANALYSIS OF SUPERALLOWED BETA TRANSITIONS

Superallowed Fermi beta decay between 0^+ states depends uniquely on the vector part of the hadronic weak interaction.

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According to CVC, when the decay occurs between isospin $T = 1$ analog states, the measured ft values should be the same irrespective of the nucleus, viz.,

$$ft = \frac{K}{G_V^2 |M_F|^2} = \text{const}, \quad (1)$$

where $K/(\hbar c)^6 = 2\pi^3 \hbar \ln 2/(m_e c^2)^5 = (8120.2787 \pm 0.0011) \times 10^{-10}$ GeV^{-4} s, G_V is the vector coupling constant for semileptonic weak interactions, and M_F is the Fermi matrix element. The CVC hypothesis asserts that the vector coupling constant G_V is a true constant and not renormalized to another value in the nuclear medium.

In practice, Eq. (1) has to be amended slightly. First, there are radiative corrections because, for example, the emitted electron may emit a bremsstrahlung photon that goes undetected in the experiment. Second, isospin is not an exact symmetry in nuclei, so the nuclear matrix element M_F is not the same for all superallowed transitions but is slightly reduced from its ideal value by a different amount in each case. This leads us to write

$$|M_F|^2 = |M_0|^2 (1 - \delta_C), \quad (2)$$

where M_0 is the exact-symmetry value, which for $T = 1$ states is $M_0 = \sqrt{2}$, and δ_C is the isospin-symmetry-breaking correction, which takes on a different (small) value for each transition. Thus, we define a "corrected" $\mathcal{F}t$ value as

$$\mathcal{F}t \equiv ft(1 + \delta'_R)(1 + \delta_{\rm NS} - \delta_C) = \frac{K}{2G_V^2 (1 + \Delta_R^V)}, \quad (3)$$

where, in addition to the terms already defined, Δ_R^V is the transition-independent part of the radiative correction and the terms δ'_R and $\delta_{\rm NS}$ comprise the transition-dependent part of the radiative correction, the former being a function only of the electron's energy and the Z of the daughter nucleus, while the latter, like δ_C, depends in its evaluation on the details of nuclear structure.

From Eq. (3), it can be seen that a single measured transition establishes a value for $\mathcal{F}t$ and, hence, G_V. This result could, in principle, then be used to determine $V_{\rm ud}$ via the relationship $V_{\rm ud} = G_V/G_F$, where G_F is the well-known weak-interaction constant for muon decay [13]. However, a value for $V_{\rm ud}$ derived from a single superallowed transition would be reliant upon a single pair of structure-dependent correction terms, $\delta_{\rm NS}$ and δ_C, without there being any independent verification of those terms' validity; so, in practice, as many transitions as possible are measured and their resultant $\mathcal{F}t$ values compared. If they satisfy CVC by being statistically consistent with each another, then one is justified in taking an average value of $\mathcal{F}t$, from which G_V and $V_{\rm ud}$ can then be derived.

If they are not consistent with each other, then one can proceed no further since inconsistency must signal a failure either of the calculated structure-dependent corrections or else of the CVC hypothesis itself. In either case, an average value of $\mathcal{F}t$ has no defined significance and certainly cannot be used to obtain a value for $V_{\rm ud}$.

Here we find the basis for a test of the calculated structure-dependent correction terms: How well do they do in producing a consistent set of $\mathcal{F}t$ values from the experimental ft values? The latter show very pronounced differences from one transition to another, and the extent to which those differences are successfully removed by a given set of calculated correction terms would be a sensitive measure of the efficacy of the calculations involved. Naturally, such a test is only as good as the CVC hypothesis. However, we believe that most would agree that a persistent scatter in the derived $\mathcal{F}t$ values is more likely to be due to a deficiency in the calculated corrections rather than to a failure of CVC.

III. THE TEST

Our test is based upon the premise that CVC is valid at least to $\pm 0.03\%$, which is the level of precision currently attained by the best ft-value measurements. Under that condition, a valid set of structure-dependent correction terms should produce a statistically consistent set of $\mathcal{F}t$ values, the average of which we can write as $\overline{\mathcal{F}t}$. It then follows from Eq. (3) that, for each individual transition in the set, we can write

$$\delta_C - \delta_{\rm NS} = 1 - \frac{\overline{\mathcal{F}t}}{ft(1 + \delta'_R)}. \quad (4)$$

For any set of corrections to be acceptable, the calculated value of $\delta_C - \delta_{\rm NS}$ for each superallowed transition must satisfy this equation, where ft is the measured result for that transition and $\overline{\mathcal{F}t}$ has the same value for all of them. Thus, to test a set of correction terms for n superallowed transitions, one can treat $\overline{\mathcal{F}t}$ as a single adjustable parameter and use it to bring the n results from the right side of Eq. (4), which are based predominantly on experiment, into the best possible agreement with the corresponding n calculated values for $\delta_C - \delta_{\rm NS}$. The normalized χ^2, minimized by this process, then provides a figure of merit for that set of calculations.

As it happens, there is only one set of calculations available for $\delta_{\rm NS}$ [3,14] but many for the isospin-symmetry-breaking term δ_C. It therefore becomes more useful to rearrange Eq. (4) to read

$$\delta_C = 1 + \delta_{\rm NS} - \frac{\overline{\mathcal{F}t}}{ft(1 + \delta'_R)}. \quad (5)$$

The same least-squares minimization process can, of course, be used in the application of this equation.

IV. AVAILABLE CALCULATIONS FOR δ_C

There have been a number of methods used over the years to calculate the isospin-symmetry-breaking correction to superallowed β decay. We describe some of them here, in chronological order.

A. Damgaard model

The first model was proposed in 1969 by Damgaard [4] and was improved eight years later by Towner *et al.* [15]. The idea is that the proton involved in beta decay has a different radial wave function than the neutron into which it transforms because it is influenced by the Coulomb interaction with all

the other protons in the nucleus. If the other protons present a uniform charge distribution of radius R, then the Coulomb interaction for a proton at $r < R$ is

$$V_c(r) = -\frac{Ze^2}{R^3}\sum_{i=1}^{A}\left(\frac{1}{2}r_i^2 - \frac{3}{2}R^2\right)\left[\frac{1}{2} - t_z(i)\right]\delta(r - r_i), \tag{6}$$

where $t_z(i) = -\frac{1}{2}$ if nucleon i is a proton and $+\frac{1}{2}$ if it is a neutron.

Using an oscillator model as a basis, Damgaard expanded the proton radial function in terms of a complete set of neutron oscillator functions. The set comprised states of the same orbital angular momentum ℓ but differing numbers of radial nodes n. Most of the mixing turned out to be with the state with one more radial node, so

$$u_\ell^{\text{proton}}(r) \approx (1-\alpha^2)^{1/2}u_{n,\ell}(r) + \alpha u_{n+1,\ell}(r.) \tag{7}$$

The mixing amplitude comes from first-order perturbation theory,

$$\begin{aligned}\alpha &= \langle u_{n+1,\ell}|V_c|u_{n,\ell}\rangle(\Delta E)^{-1},\\ \Delta E &= E_{n+1,\ell} - E_{n,\ell} = 2\hbar\omega,\end{aligned} \tag{8}$$

and the Fermi matrix element between $T = 1$ states is given by

$$|M_F|^2 = 2(1-\alpha^2). \tag{9}$$

Upon evaluating the Coulomb matrix element using oscillator functions with V_c taken from Eq. (6), Damgaard obtained

$$\delta_C = \alpha^2 = \frac{Z^2}{(\hbar\omega)^4R^6}\frac{e^4\hbar^4}{16m^2}(n+1)\left(n+\ell+\tfrac{3}{2}\right). \tag{10}$$

If we adopt the relationships $\hbar\omega = 41A^{-1/3}$ MeV and $R = 1.2A^{1/3}$ fm, this expression becomes

$$\delta_C = 0.2645Z^2A^{-2/3}(n+1)\left(n+\ell+\tfrac{3}{2}\right), \tag{11}$$

which, for the light nuclei we are interested in, exhibits the general behavior $\delta_C \propto A^{4/3}$ with some shell structure superimposed through the choice of oscillator quantum numbers n and ℓ. In particular, a proton radial function with one radial node gets a factor of 2 enhancement in its δ_C value over one that has no radial nodes simply from the factor $(n+1)$ in Eq. (11).

We have used Eq. (11) to derive δ_C values for the 13 best known superallowed transitions. These transitions are listed by parent nucleus in the first column of Table I, and the δ_C results for this model appear in the fifth column. We will use these results in our comparative tests of all models.

B. Shell model with Saxon-Woods radial wave functions (SM-SW)

The shell model with Saxon-Woods radial wave functions (SM-SW) approach was introduced by Towner *et al.* in 1977 [15] and improved upon several times since then [3,16]. In their approach, the Fermi matrix element is defined by

$$M_F = \sum_\alpha \langle f|a_\alpha^\dagger b_\alpha|i\rangle = \sum_{\alpha,\pi}\langle f|a_\alpha^\dagger|\pi\rangle\langle\pi|b_\alpha|i\rangle, \tag{12}$$

where $a_\alpha^\dagger$ creates a neutron and b_α annihilates a proton in state α. Here $|i\rangle$ and $|f\rangle$ are the *exact* A-body state vectors for the full Hamiltonian, and $|\pi\rangle$ represents a complete set of $(A-1)$-body parent states. If this Hamiltonian commutes with the isospin operators, then $|i\rangle$ and $|f\rangle$ are exact isospin analogues of each other, and the symmetry-limit matrix element is

$$M_0 = \sum_{\alpha,\pi}|\langle f|a_\alpha^\dagger|\pi\rangle|^2, \tag{13}$$

which for $T = 1$ states corresponds to $M_0 = \sqrt{2}$. However, with isospin not being an exact symmetry, $|i\rangle$ and $|f\rangle$ are not exact isospin analogues; nevertheless, the resulting matrix element M_F is not very different from M_0, the relationship between them being given by Eq. (2): viz., $M_F^2 = M_0^2(1-\delta_C)$, where δ_C is small.

Ideally, to obtain δ_C one would compute Eq. (12) using the shell model, and introduce Coulomb and other charge-dependent terms into the shell-model Hamiltonian. However, the shell-model space would have to be huge to include all the potential states with which the Coulomb interaction might potentially connect. Since this is not a practical proposition, a model approach was developed in which δ_C is divided into two parts:

$$\delta_C = \delta_{C1} + \delta_{C2}. \tag{14}$$

For δ_{C1}, one computes

$$\sum_{\alpha,\pi}\langle\overline{f}|a_\alpha^\dagger|\pi\rangle\langle\pi|b_\alpha|\overline{i}\rangle = M_0(1-\delta_{C1})^{1/2}, \tag{15}$$

where $|\overline{i}\rangle$ and $|\overline{f}\rangle$ are not the exact eigenstates that appear in Eq. (12) but are the shell-model eigenstates of an effective Hamiltonian (including charge-dependent terms) evaluated in a tractable shell-model space. However, this space is not large enough to allow for mixing with functions having a different number of radial nodes, so the term δ_{C2} is introduced to compensate for that limitation. This second term is derived from

$$\sum_{\alpha,\pi}|\langle\overline{f}|a_\alpha^\dagger|\pi\rangle|^2 r_\alpha^\pi = M_0(1-\delta_{C2})^{1/2}, \tag{16}$$

where r_α^π is a radial overlap integral of proton and neutron radial functions. If the proton and neutron radial functions were identical, then it would follow that $r_\alpha^\pi = 1$ and $\delta_{C2} = 0$. But since they are not identical, a finite correction δ_{C2} is obtained. The idea is that nodal mixing mainly impacts on the radial functions, as demonstrated by the Damgaard model, and so its impact is best modeled by Eq. (16).

The wave function for the decaying A-body state $|\overline{i}\rangle$ is expanded in a set of parent states of $(A-1)$ nucleons $|\pi\rangle$ plus a proton, while that of the daughter A-body state $|\overline{f}\rangle$ is expanded in terms of the same set of parent states plus a neutron. The expansion coefficients are obtained from a shell-model calculation. Isospin-symmetry breaking is introduced by allowing the radial function for the proton

TABLE I. Input data for the tests of the isospin-symmetry-breaking corrections δ_C obtained from the various models described in Sec. IV. The experimental *ft* values come from Table IX in the most recent survey of world data [1]; however, in order to ensure undiluted normal statistics, we have set all the "scale factors" used in that reference equal to 1, with the consequence that the uncertainties quoted for most cases are smaller than those listed in Ref. [1]. The calculated values of δ'_R and δ_{NS} come from Table VII of Ref. [3]. The δ_C values tabulated in the last six columns were obtained as follows: The Damgaard values were derived from our Eq. (11); those labeled SM-SW came from Table VII of Ref. [3]; the SM-HF values were obtained by adding the "adopted" δ_{C1} numbers from Table III of Ref. [3] and the "HF" δ_{C2} numbers from Table XI of Ref. [1]; the RHF-RPA values were taken for the PKO1 effective interaction given in Table I of Ref. [8]; the RH-RPA numbers, which correspond to the density-dependent DD-ME2 effective interaction, were taken from the same table and reference; the IVMR values were calculated from our Eq. (22), which is the same as Eq. (32) in Ref. [9].

Parent nucleus	Experimental			δ_C (%)					
	ft value (s)	δ'_R (%)	δ_{NS} (%)	Damgaard	SM-SW	SM-HF	RHF-RPA	RH-RPA	IVMR
$T_z = -1$									
^{10}C	3041.7(43)	1.679(4)	−0.345(35)	0.046	0.175	0.225	0.082	0.150	0.008
^{14}O	3042.3(11)	1.543(8)	−0.245(50)	0.111	0.330	0.310	0.114	0.197	0.015
^{22}Mg	3052.0(70)	1.466(17)	−0.225(20)	0.153	0.380	0.260			0.031
^{34}Ar	3052.7(82)	1.412(35)	−0.180(15)	0.285	0.665	0.540	0.268	0.376	0.064
$T_z = 0$									
^{26}Al	3036.9(9)	1.478(20)	0.005(20)	0.182	0.310	0.440	0.139	0.198	0.041
^{34}Cl	3049.4(11)	1.443(32)	−0.085(15)	0.326	0.650	0.695	0.234	0.307	0.064
^{38}K	3051.9(5)	1.440(39)	−0.100(15)	0.370	0.655	0.745	0.278	0.371	0.077
^{42}Sc	3047.6(12)	1.453(47)	0.035(20)	0.414	0.665	0.640	0.333	0.448	0.091
^{46}V	3049.5(8)	1.445(54)	−0.035(10)	0.524	0.620	0.600			0.106
^{50}Mn	3048.4(7)	1.444(62)	−0.040(10)	0.550	0.655	0.620			0.122
^{54}Co	3050.8(10)	1.443(71)	−0.035(10)	0.613	0.770	0.685	0.319	0.393	0.139
^{62}Ga	3074.1(11)	1.459(87)	−0.045(20)	1.339	1.48	1.21			0.175
^{74}Rb	3084.9(77)	1.50(12)	−0.075(30)	1.422	1.63	1.42	1.088	1.258	0.235
χ^2/n_d (statistical experimental uncertainties only)				8.3	1.2	8.3	7.2	6.0	48
Confidence level (%)				0	26	0	0	0	0
χ^2/n_d (uncertainties on experiment, δ'_R and δ_{NS})				1.7	0.4	2.2	2.7	2.1	11
χ^2/n_d (uncertainties on experiment, δ'_R, δ_{NS}, and δ_C)				0.9	0.3	1.1	1.6	1.3	4.5

in these expansions to differ from that of the neutron. In this model, these radial functions are taken to be eigenfunctions of a Saxon-Woods potential. The well depths of the proton and neutron potentials are adjusted so that the asymptotic forms of the radial function go as $e^{-\alpha r}$, where $\alpha^2 = 2mS/\hbar^2$, with m being the nucleon mass and S being the experimental separation energy for the proton (or neutron) in the A-body state. Further details can be found in [3].

It is important to realize that this model is really semi-phenomenological in its application. In addition to the match with experimental separation energies in the calculation of δ_{C2}, the radius of the Saxon-Woods potential in each case was set to the value determined experimentally for the charge radius by electron scattering [16], and the shell-model parentage was linked to measured single-nucleon transfer reactions [3]. The value of δ_{C1} was also constrained by comparison with experiment. First, for each superallowed transition the single-particle energies of the proton orbits were shifted relative to the neutrons, the exact amount being determined from the spectrum of single-particle states in the closed-shell-plus-proton nucleus versus the closed-shell-plus-neutron nucleus. Second, the two-body Coulomb interaction among the valence protons was adjusted in strength for each decay so that the measured b coefficient in the isobaric multiplet mass equation (IMME) was exactly reproduced for the multiplet involved in that decay. Third, the charge-dependent nuclear interaction, which had been incorporated by a ∼2% increase in all the $T = 1$ proton-neutron matrix elements relative to the neutron-neutron ones, was tuned to give agreement with the measured c coefficient of the IMME.

The current best values for δ_C as calculated with this model are listed in the fifth column of Table VI in Ref. [3] and are reproduced here in the sixth column of Table I.

C. Shell model with Hartree-Fock radial wave functions (SM-HF)

Beginning in 1985, Ormand and Brown [5,6] adopted the same general procedure as the one just described, splitting δ_C into two components, the first of which, δ_{C1}, incorporated configuration mixing within a restricted shell-model space, and the second, δ_{C2}, accounted for mixing with all other states by evaluating the mismatch in the parent and daughter radial wave functions. The shell-model aspects of their model were the same as the shell model with Saxon-Woods radial wave functions, but their radial functions were taken to be eigenfunctions of a mean-field Hartree-Fock potential rather than of a Saxon-Woods potential (SM-HF). As in the SM-SW model, the strength of this mean field was readjusted so that the

asymptotic forms of the proton and neutron radial functions were matched to their respective separation energies.

Ormand and Brown's [5,6] protocol was to perform two Hartree-Fock calculations with a Skyrme interaction: one for the decaying A-body state, whose mean field provided the proton function, and the other for the daughter A-body state, whose mean field provided the neutron function. However, it was noted more recently by Hardy and Towner [1] that there is a problem with this protocol: the Coulomb part of the proton mean field has asymptotically the wrong form, falling off as $(Z+1)e^2/r$ rather than Ze^2/r. They therefore modified the protocol to just a single Skyrme-Hartree-Fock calculation performed in the $(A-1)$-body state, whose mean field provided for both the proton and neutron radial functions. In this procedure, the Coulomb interaction automatically has the right asymptotic form. Further details can be found in [1].

We have obtained the δ_C values for this model by adding the "adopted" δ_{C1} numbers from Table III of Ref. [3] (the same as we used for the SM-SW model) and the "HF" δ_{C2} numbers from Table XI of Ref. [1]. The results appear in column seven of our Table I.

D. Hartree-Fock with random phase approximation (RHF-RPA and RH-RPA)

In 1996, Sagawa *et al.* [7] introduced a new model, in which a Skyrme-Hartree-Fock calculation was performed for each even-even A-body system: the parent for the cases in which the superallowed decay proceeds from a $T_z=-1$ parent nucleus and the daughter for cases of decay from a $T_z=0$ parent nucleus. The odd-odd nucleus was then treated as a particle-hole excitation built on the even-even Hartree-Fock state. The particle-hole calculation was carried out in the charge-exchange random-phase approximation (RPA) in a model space extending up to $10\hbar\omega$ excitation, with radial functions up to five nodes. The lowest state in the RPA spectrum was identified as the isobaric analog state, the state actually involved in the superallowed Fermi beta decay. Unlike the previous two methods, there was no adjustment to reproduce exactly the energy of the analog state, but the authors did check that their results were typically within 500 keV of the experimental value. Isospin-symmetry breaking was introduced by the presence of a Coulomb interaction, augmented by explicit charge-symmetry-breaking and charge-independence-breaking interactions included in the two-body force used in the Hartree-Fock calculation.

Since first results from this model appeared [7], significant progress has been made in self-consistent RPA calculations in charge-exchange channels. Skyrme zero-range interactions have been replaced by finite-range meson-exchange potentials involving σ, ω, ρ, and π mesons, and a relativistic rather than a nonrelativistic treatment can be used. In 2009, Liang *et al.* [8] published improved results from relativistic Hartree-Fock with random phase approximation (RHF-RPA) calculations with three different effective interactions as well as from relativistic Hartree (only) (RH-RPA) calculations with density-dependent meson-nucleon couplings and nonlocal interactions. The results were not particularly sensitive to the interaction used, so in performing our tests, we just use one interaction for each type of calculation: PKO1 for RHF-RPA and DD-ME2 for RH-RPA [8]. The corresponding values for δ_C were taken from Table I in Ref. [8] and are reproduced here in columns eight and nine of Table I. Note that the authors of Ref. [8] only calculated δ_C values for 8 of the 13 well-known superallowed transitions.

E. Isovector monopole resonance (IVMR)

In 2009, Auerbach [9] introduced a model in which he assumed that isospin-symmetry breaking in superallowed β decay is due entirely to mixing with the giant monopole state.

The isovector part of the Coulomb interaction, which appears in Eq. (6), is defined to be the isovector monopole operator $M_0^{(1)}$; thus, we have

$$M_0^{(1)} = \sum_i r_i^2 t_z(i), \tag{17}$$

where $M_0^{(1)}$ is a spherical tensor in isospin space of rank 1, with its z component equal to 0. We write $|M\rangle$ to be the giant monopole state, which is created by the application of operator $M_0^{(1)}$ to the ground state. If the ground state has $N=Z$ with isospin quantum numbers $T=0$ and $T_z=0$, then the giant monopole state is a unique state with quantum numbers $T=1$ and $T_z=0$. But if the ground state has a neutron excess, with $T=T_z=\frac{1}{2}(N-Z)$, then the monopole state is split into two components, one with isospin T and the other with isospin $T+1$. In this case the ground-state wave function is designated by $|T,T\rangle$, and the two components of the monopole state are designated by $|M_{T,T}\rangle$ and $|M_{T+1,T}\rangle$. Furthermore, the isobaric analog of this ground state, $|T,T-1\rangle$, has its giant monopole state split into three isospin components, $|M_{T-1,T-1}\rangle$, $|M_{T,T-1}\rangle$, and $|M_{T+1,T-1}\rangle$.

By assuming that the giant monopole state is the sole source of isospin-symmetry breaking in superallowed decays, Auerbach [9] could write the wave functions for the two states involved in the β decay as

$$\begin{aligned}|\Psi_1\rangle &= (|T,T\rangle + \epsilon_0|M_{T,T}\rangle + \epsilon_1|M_{T+1,T}\rangle)N_1^{-1},\\ |\Psi_2\rangle &= (|T,T-1\rangle + \eta_{-1}|M_{T-1,T-1}\rangle\\ &\quad + \eta_0|M_{T,T-1}\rangle + \eta_1|M_{T+1,T-1}\rangle)N_2^{-1},\end{aligned} \tag{18}$$

where

$$\begin{aligned}N_1 &= \left(1+\epsilon_0^2+\epsilon_1^2\right)^{1/2},\\ N_2 &= \left(1+\eta_{-1}^2+\eta_0^2+\eta_1^2\right)^{1/2},\end{aligned} \tag{19}$$

and amplitudes ϵ_i and η_i can be expressed via perturbation theory in terms of Coulomb matrix elements between the ground state and the respective components of the isovector monopole state. Based on this result, he then derived the corresponding isospin-symmetry-breaking correction to superallowed β decay, which he wrote to order $O(\epsilon^2,\eta^2)$ as

$$\delta_C = \eta_{-1}^2 + (\epsilon_0-\eta_0)^2 + \epsilon_1^2 + \eta_1^2 - 2\epsilon_1\eta_1\left(\frac{2T+1}{T}\right)^{1/2}. \tag{20}$$

Auerbach next argued that the Coulomb matrix elements of differing isospins are all related to each other via isospin

Clebsch-Gordan coefficients. He thus found that the coefficients ϵ_0, η_{-1}, η_0, and η_1 could all be expressed in terms of the one isospin-mixing amplitude ϵ_1. In this way, the expression for δ_C, Eq. (20), reduces to

$$\delta_C = 8(T+1)\frac{V_1}{\xi\hbar\omega A}\epsilon_1^2, \tag{21}$$

where ξ is related to the particle-hole interaction energy required to place the centroid of the giant monopole resonance at the appropriate energy and V_1 is related to the strength of the symmetry potential that sets the energy splitting between the components of the monopole state. Auerbach chose $V_1 = 100$ MeV, $\xi = 3$, and $\hbar\omega = 41A^{-1/3}$ MeV and estimated ϵ_1^2 by appealing to a number of "gross" models discussed in Ref. [17]: a hydrodynamical model, models based on non-energy-weighted and energy-weighted sum rules, and a microscopic model. Each enabled him to obtain a simple expression for δ_C as a function of the mass number A. As an example, his expression in the microscopic model was

$$\delta_C = 18.0 \times 10^{-7} A^{5/3}. \tag{22}$$

We calculated the values of δ_C for the isovector monopole resonance model (IVMR) from this equation for the 13 best known superallowed transitions. The results, which are listed in column ten of Table I, were used as part of our comparative tests of all models.

V. TEST RESULTS

We have now set the stage for applying the test described in Sec. III. As explained there, our procedure for each of the six models is to compare that model's set of calculated δ_C values (listed in Table I) with the set of values obtained from Eq. (5) and, using the method of least squares with $\overline{\mathcal{F}t}$ as the adjustable parameter, to optimize the agreement between them. In effect, the δ_C values from Eq. (5) can be thought of as the "experimental" values: they incorporate the experimental ft values from Ref. [1] as well as the small calculated correction terms, δ'_R and $\delta_{\rm NS}$, from Ref. [3] (also listed in Table I). The parameter $\overline{\mathcal{F}t}$ is a normalizer that allows each model to be tested for its success in obtaining a constant $\mathcal{F}t$ value (i.e., in agreement with CVC), without regard for whether or not that $\mathcal{F}t$ value ultimately satisfies CKM unitarity. The normalized χ^2 for each least-squares fit, expressed as χ^2/n_d, where n_d is the number of degrees of freedom, thus yields a figure of merit for the model used, with smaller χ^2/n_d values indicating better agreement.

Although we take the measured ft values from Ref. [1], strictly speaking the ft-value uncertainties quoted in that reference do not correspond to normal distributions. Each ft value has three experimental inputs—energy, half-life, and branching ratio—and each of these inputs typically includes a number of measurements of that quantity. The survey authors adopted the procedures used by the Particle Data Group [13] and, for any cases in which the measurements when averaged yielded a normalized χ^2 greater than 1, they increased the uncertainty on the average by a scale factor equal to the square root of the normalized χ^2. This conservative approach leads to uncertainties on the ft and $\mathcal{F}t$ values that are larger than would be the case for purely statistical results. For our present purposes we have set all the scale factors in Ref. [1] equal to 1 and obtained new uncertainties on the ft values, which are normally distributed (at least to the extent that the uncertainties assigned by the authors of the original measurements were predominantly statistical). It is these redetermined uncertainties that appear in the second column of Table I.

Obviously, the uncertainties assigned to the theoretical radiative corrections δ'_R, $\delta_{\rm NS}$, and δ_C (if any) are not normally distributed statistical quantities. Therefore, in our first least-squares test, we used only the redetermined uncertainties for the ft values and no uncertainties at all for any of the theoretical terms. The results for χ^2/n_d appear in the first row below the main body of Table I, labeled "statistical experimental uncertainties only." Since this analysis uses only normally distributed uncertainties, we can proceed to evaluate a confidence level for each model.

We follow the Particle Data Group [13] in defining the confidence level (or p value) as being

$$p = \int_{\chi_0^2}^{\infty} P_{n_d}(\chi^2)\, d\chi^2, \tag{23}$$

where $P_{n_d}(\chi^2)$ is the χ^2 probability distribution function for n_d degrees of freedom and χ_0^2 is the value of χ^2 obtained for a particular hypothesis: in our case, for a particular isospin-symmetry-breaking model. With this definition, the confidence level represents the probability that χ^2 for a valid hypothesis could exceed the value χ_0^2 actually obtained for the specific hypothesis being tested. More loosely, in our application the confidence level quoted for a particular model can be interpreted as the probability of that model being a valid one, i.e., of it being consistent with CVC. We express each confidence level as a percent in Table I.

We then present the results of a second least-squares analysis, in which we included uncertainties on the theoretical radiative corrections, δ'_R and $\delta_{\rm NS}$; of course, we retained the redetermined uncertainties already incorporated for the ft values. The resulting χ^2/n_d values appear in the next-to-last row in Table I. These results are also illustrated in Fig. 1.

Finally, we list the results from a third least-squares analysis. Although two of the models, SM-SW and SM-HF, include theoretical uncertainties on δ_C values in their original publications, the other four do not, so to test them all on an equal footing, we have not used uncertainties on any of the model calculations in our first two analyses. We consider this to be the fairest approach. However, we have also examined what happens to our intercomparison if all the calculated δ_C values are assigned the same uncertainties as those originally quoted for the SM-SW calculations [3]. The values for χ^2/n_d resulting from this third analysis appear in the last row of Table I.

The most obvious outcome of these analyses is that only one model, SM-SW, produces satisfactory agreement with CVC, having $\chi^2/n_d = 1.2$ and a confidence level of 26% in the properly statistical analysis. All of the other five models have

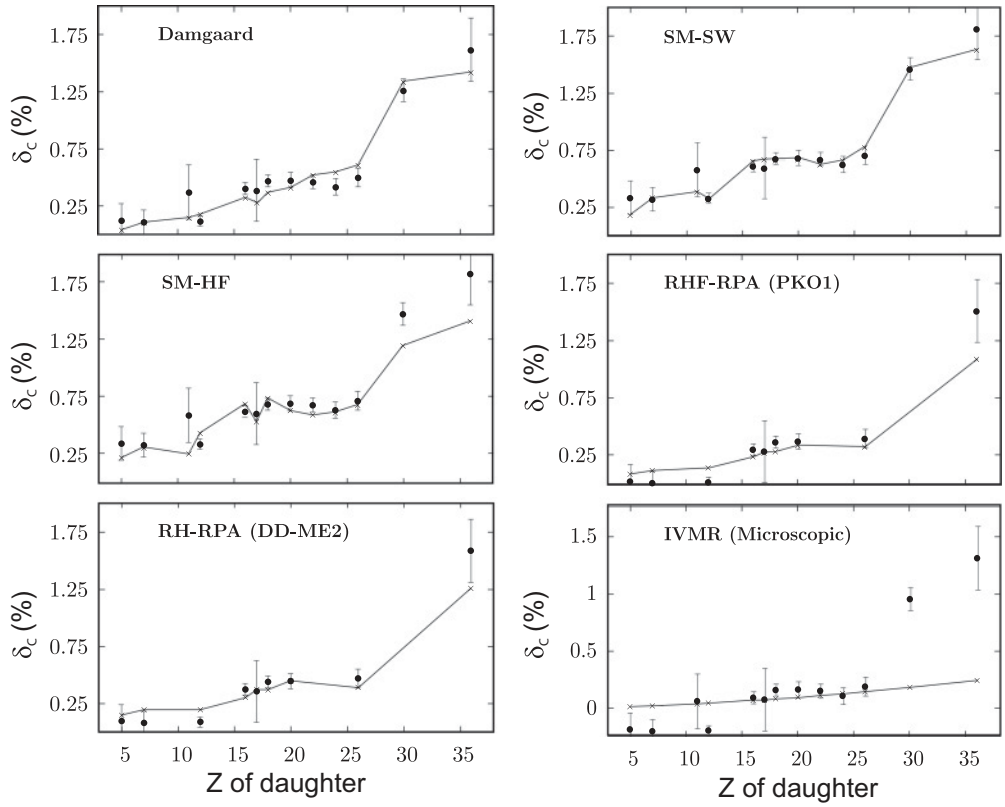

FIG. 1. Isospin-symmetry-breaking correction δ_C, in percent, plotted as a function of atomic number Z of the daughter nucleus. The solid circular points with error bars are the values of δ_C obtained from Eq. (5), with the experimental ft values and the values of δ'_R and δ_{NS} (and their uncertainties) all taken from Table I. In effect, we treat these as the "experimental" δ_C values. The X's joined by lines represent the δ_C values calculated by the various models described in the text and identified in the top left of each graph. The value of $\overline{\mathcal{F}t}$ in Eq. (5) has been adjusted in each case by least-squares fitting to optimize the agreement between the experimental δ_C values and the calculated ones. The corresponding values of χ^2/n_d are listed in the next-to-last row of Table I.

confidence levels well below 0.5%. Because the two other analyses included nonstatistical uncertainties on the theoretical correction terms in addition to the statistical experimental ones, their values of χ^2/n_d are substantially lower, but the relative ranking of the six models is approximately preserved: in all cases the SM-SW model is by far the best. It is remarkable that the model which becomes second best when the theoretical uncertainties are included is the earliest and arguably the most primitive one. Its success evidently stems from its treatment of the radial mismatch between the parent and daughter states, which accounts rather well for the sharp increase in δ_C between $Z = 12$ and $Z = 16$ and between $Z = 26$ and $Z = 30$. It is perhaps equally striking that the most recent IVMR model fails to reproduce the trend of the data or any of its characteristic features.

VI. CONCLUSIONS

Evidently, the shell model with Saxon-Woods radial wave functions, SM-SW, is the only model tested that yields isospin-symmetry-breaking corrections which, when combined with the experimental ft values, produce $\mathcal{F}t$ values that agree with the CVC hypothesis over the full range of Z values. This, of course, does not prove that the SM-SW model is correct in every way; however, it does demonstrate that the other models in their present form cannot be used to extract a number for V_{ud} and to test CKM unitarity. As we note in Sec. II, if the $\mathcal{F}t$ values are not consistent with one another, then their average has no defined significance since either the symmetry-breaking model is wrong or CVC itself has failed.

There is a second model, SM-HF, which has many promising features. As can be appreciated from an examination of Fig. 1, its relatively large χ^2 is due to its failure to match the experimental δ_C values for the cases with $Z \geqslant 30$. If we were to restrict ourselves only to the lighter cases, then the model would agree well with CVC. This difference at the highest Z values between the SM-SW and SM-HF model calculations has been known for 15 years, having first been pointed out by Ormand and Brown [6] even before the decays of the highest-Z emitters, ^{62}Ga and ^{74}Rb, had yet been precisely measured. Prompted by the results reported here, we are currently examining whether this feature of the SM-HF model (as described in Sec. IV C) is sensitive to the particular Skyrme interaction used [18]. We have, by now, sampled 12 different interactions and have also added a pairing term to the interaction, turning the calculation into a Hartree-Fock-Bogolyubov one. However, under no circumstances have we

been able to produce agreement with experiment over the full range of Z values. It is important to realize that both the SM-SW and SM-HF models use identical spectroscopic input, so it would appear that the high-Z discrepancy is inherent to the SM-HF model itself. However, it must be admitted that too little spectroscopic information is known in this region to fully characterize the required model space. Calculations with larger model spaces and improved Hamiltonians are certainly to be encouraged.

Fortunately, it is the successful SM-SW model that has principally been employed to calculate the δ_C values used in the most recent data survey [1]. As was argued in that survey, the consistency of the $\mathcal{F}t$ values was a powerful validation of those calculated correction terms and justified the subsequent derivation of $V_{\rm ud}$. However, in actually deriving $V_{\rm ud}$ and its uncertainty, we incorporated the SM-HF calculations as well, even though we knew that model had a much poorer χ^2. Our rationale was one of conservatism. We enlarged the uncertainty assigned to the average $\overline{\mathcal{F}t}$ value to cover both sets of δ_C calculations in order to be safe by including some provision for systematic theoretical uncertainties. Whether we continue this practice in future is not yet decided.

For now, though, we know that there are, as yet, no comparably successful competitive models. More important, we also have a protocol for testing future models, which is evidently very sensitive to the validity of the model. Furthermore, even though it is only the *relative* Z-dependent variations in δ_C that are being tested, it would surely require a pathological fault indeed in the theory to allow the observed nucleus-to-nucleus variations in δ_C to be reproduced in such detail while failing to obtain the *absolute* values to comparable precision.

With this perspective, it is now informative to consider the points raised recently by Miller and Schwenk [10], who claim that the SM-SW model is based on a formally incorrect interpretation of the isospin ladder operator. They claim that this "incorrect" usage must have led to incomplete results for δ_C, but they do not produce any "exact" calculations with which to compare. Instead, they identify a term involving radial excitations, which they consider to be missing from the SM-SW model, and proceed to evaluate this term under simplifying assumptions. They assume that the radial excitations are dominated by mixing with states having one more radial node (cf. the Damgaard model) and, further, that the relevant excitations are dominated by the isovector monopole resonance (cf. the IVMR model). Under these conditions they find that this "missing" term almost completely cancels the SM-SW-model result, and although they produce no numbers, they state that this would result in δ_C values comparable in magnitude to the IVMR-model results or even smaller. Clearly, such a result would disagree at least as strongly with CVC as does the IVMR model. Therefore, if any term is really missing from the SM-SW-model calculations, the test results presented here show that it must either be independent of Z or else very small; otherwise, the data would become inconsistent with the CVC hypothesis. Considering that the Coulomb force is the principal source of isospin symmetry breaking, it is highly unlikely that any large component of δ_C could be Z independent.

From an experimental point of view, the results in Fig. 1 clearly demonstrate the importance of precisely measured *ft* values. For example, the very precise values for $^{26}\mathrm{Al}^m$ (plotted at $Z = 12$, the atomic number of its daughter) and $^{34}\mathrm{Cl}$ (see $Z = 16$) contribute very significantly to the overall χ^2 for each model fit. Equally important, though, are the *ft* values for transitions that exhibit large values for δ_C. The most obvious examples are the decays of $^{62}\mathrm{Ga}$ (see $Z = 30$) and $^{74}\mathrm{Zr}$ (see $Z = 36$): Their δ_C values differ enormously from those for the transitions with $Z \leqslant 54$, and this difference plays an important role in differentiating one symmetry-breaking model from another. More measurements of both types would be much welcomed in this context.

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