Examining the equivalence of Bakamjian-Thomas mass operators in different forms of dynamics

W. N. Polyzou
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA
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We discus the proof of the equivalence of relativistic quantum mechanical models based on the generalized Bakamjian-Thomas construction in all of Dirac's forms of dynamics. Explicit representations of the equivalent mass operators are given in all three of Dirac's forms of dynamics.

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I. INTRODUCTION

Relativistic invariance in a quantum theory means that experiments cannot make an absolute determination of inertial reference frames. In a quantum theory experiments measure probabilities, expectation values, and ensemble averages. In a relativistic theory inertial reference frames are related by Lorentz transformations and space-time translations. The group generated by these transformations is the Poincaré group. Wigner [1] proved that a necessary and sufficient condition for relativistic invariance in a quantum theory is that the correspondence between equivalent states in different inertial reference frames be given by a unitary ray representation of the Poincaré group. This ensures that probabilities, expectation values, and ensemble averages for equivalent experiments in different inertial frames are identical.

Since Lorentz transformations mix position and time coordinates and translations change position coordinates, time evolution can be expressed directly using only Lorentz transformations and translations. This leads to a problem in how to add interactions to obtain an internally consistent initial value problem. The nature of the problem can be appreciated by considering the commutation relations:

$$[P^i, K^j] = i\delta_{ij} H, \tag{1.1}$$

where H is the generator of time translations (the Hamiltonian), $\mathbf{P}$ are the generators of spatial translations, and $\mathbf{K}$ are the generators of rotationless Lorentz transformations. These commutation relations are satisfied for systems of free particles. If an interaction is added to H on the right side of these equations, then consistency requires that interactions also be added to some operators on the left. The full Poincaré commutation relations imply additional nonlinear constraints on these interactions.

In quantum field theory the resolution of this problem follows as a consequence of the local commutation relations of the fields. Dirac [2] studied the problem of satisfying the Poincaré commutation for interacting systems from an algebraic point of view (using Poisson brackets in classical mechanics). His analysis showed that there are maximal subgroups of the Poincaré group, called kinematic subgroups, that can be made free of interaction. Dirac identified the three largest kinematic subgroups: (1) the subgroup generated by translations and rotations (the three-dimensional Euclidean group), (2) the Lorentz group, and (3) the subgroup of the Poincaré group that leaves a hyperplane tangent to the light-cone invariant. Dirac called the dynamical models with these different kinematic symmetries instant-form dynamics, point-form dynamics, and light-front dynamics, respectively. While each of these choices reduces the complexity of the final solution, none of them provides an explicit realization of a dynamical representation of the Poincaré group.

Bakamjian and Thomas [3] gave the first explicit representation of the Poincaré Lie algebra with interactions for a two-particle system in Dirac's instant form of the dynamics. Their construction was generalized to the three-body system by Coester [4]. The Bakamjian-Thomas construction can also be applied to systems of four or more particles, but the resulting unitary representation of the Poincaré group violates spacelike cluster properties. This deficiency was resolved by Sokolov [5], who with Shatnyi also established the equivalence [6] of the models in each of Dirac's forms of dynamics. While Sokolov provided a systematic construction for restoring cluster properties, the construction is sufficiently complicated that it has not been implemented in any application.

For systems of two or three particles Sokolov's systematic construction is not necessary. Many applications where a relativistic treatment is needed utilize the Bakamjian-Thomas construction. Applications based on each of Dirac's forms of dynamics have been applied to model realistic systems. Calculations based on models using different forms of dynamics are sometimes compared, and the advantages of one form over another are sometimes inferred. In light of Sokolov and Shatnyi's proof of the equivalence of different kinematic subgroups, it is apparent that models with different kinematic subgroups that give different results are due to the inequivalence of the models. Nevertheless, differences seem to arise naturally because certain model assumptions may appear more natural in the theory with one kinematic subgroup over another kinematic subgroup.

In addition, the unitary transformations that relate models based on different forms of dynamics must be dynamical because they necessarily generate interactions in some kinematic generators. For example, any unitary transformation that maps an instant-form representation of the Poincaré group to a point-from representation transforms linear momentum operators with no interaction to momentum operators with interactions. In three- or more-body models one would expect that if one of these interacting unitary transformations is applied to a Hamiltonian with only two-body interactions, then it will generate a transformed Hamiltonian that has three-body interactions. This would suggest some forms of dynamics may be preferable because some may not need strong many-body interactions. If one considers the mass Casimir operator rather than the Hamiltonian, it is possible to find classes of unitary transformations that relate the different forms of dynamics where this does not happen. Specifically, with these transformations the dynamical equations in any of Dirac's from of dynamics are essentially identical for systems with any number of particles, including models that do not conserve particle number.

The important consequence of this result is that there are no real advantages to any one form of the dynamics; in fact, the different forms of dynamics are simply different representations of a more-universal Poincaré invariant quantum mechanics. The second consequence of this result is that it provides a framework where models based on different kinematic subgroups can be compared.

In Sec. II we discuss the Bakamjian-Thomas construction of dynamical unitary representations of the Poincaré group. In Sec. III we introduce Ekstein's [7] notion of scattering equivalence and establish the scattering equivalence of specific Bakamjian-Thomas constructions in each of Dirac's forms of dynamics. Since the equivalence is best understood using an abstract treatment, in Sec. IV we discuss an application to the three-body system in more detail. The results are summarized in Sec. V.

II. THE GENERALIZED BAKAMJIAN-THOMAS CONSTRUCTION

One of the most straightforward constructions of exactly Poincaré invariant quantum mechanical models of systems of a finite number of degrees of freedom is based on a method introduced by Bakamjian and Thomas [3]. The construction can be summarized as follows. The Hilbert space for a particle of mass m and spin j is the mass m, spin j irreducible representation space of the Poincaré group. The model Hilbert space for a system is determined by the particle content of the system. It is the direct sum of tensor products of irreducible representation spaces for the Poincaré group. The kinematic (noninteracting) unitary representation of the Poincaré group $U_0(\Lambda, a)$ on this space is the direct sum of tensor products of single-particle unitary irreducible representations of the Poincaré group.

The kinematic representation of the Poincaré group is decomposed into a direct integral of irreducible representations of the Poincaré group using Poincaré group Clebsch-Gordan coefficients [4,8,9]. Wave functions in this direct integral representation are square integrable functions of the eigenvalues of (1) the Casimir operators (m, j) of the Poincaré group (2) commuting observables $\mathbf{v}$ that label different vectors in an irreducible subspace (3) invariant degeneracy operators $\mathbf{d}$ that distinguish multiple copies of the same irreducible representation. Wave functions in this representation are square integrable functions $\psi(m, j, \mathbf{v}, \mathbf{d}) = \langle (m, j), \mathbf{v}, \mathbf{d}|\psi\rangle$ of the eigenvalues of these operators.

Typical choices for $\mathbf{v}$ for a two-body system are the three components of the linear momentum (continuous variables) and one component of the canonical spin (discrete variable), the four velocity components (continuous variables) and one component of the canonical spin (discrete variable), or the generators of space-time translations on a light front (continuous spectrum) and one component of the light-front spin (discrete variable). A common choice of the degeneracy parameters $\mathbf{d}$ for a two-particle system is l^2 and s^2, which are kinematically invariant angular momenta.

The Poincaré transformation properties of wave functions in one of these bases is given by

$$\langle (m, j)\mathbf{v}, \mathbf{d}|U_0(\Lambda, a)|\psi\rangle = \sum\!\!\!\!\!\!\int' d\mathbf{v}' \mathcal{D}^{m,j}_{\mathbf{v};\mathbf{v}'}[\Lambda, a]\psi(m, j, \mathbf{v}', \mathbf{d}), \tag{2.1}$$

where

$$\mathcal{D}^{m,j}_{\mathbf{v};\mathbf{v}'}[\Lambda, a] = \langle (m, j)\mathbf{v}|U(\Lambda, a)|(m, j)\mathbf{v}'\rangle \tag{2.2}$$

is the known mass m, spin j irreducible representation of the Poincaré group in the basis $\mathbf{v}$. Note that the Poincaré group Wigner function $\mathcal{D}^{m,j}_{\mathbf{v};\mathbf{v}'}[\Lambda, a]$ does not depend on the degeneracy quantum numbers $\mathbf{d}$ and is the same for either a system or a free particle with the same m and j as the system.

The goal of the Bakamjian-Thomas construction is to add interactions to the Poincaré generators in a manner that preserves the Poincaré Lie algebra. This is nontrivial because the commutation relation (1.1) cannot be satisfied for an interacting H unless some combination of $\mathbf{P}$ and $\mathbf{K}$ also include interactions.

Bakamjian and Thomas solve this problem by adding interactions to the mass Casimir operator m of the noninteracting system. The allowed interactions in the Bakamjian-Thomas construction are represented by kernels that have the form

$$\begin{aligned}\langle (m, j), \mathbf{v}, \mathbf{d}|V|(m', j'), \mathbf{v}', \mathbf{d}'\rangle \\ = \delta(\mathbf{v} : \mathbf{v}')\delta_{jj'}\langle m, \mathbf{d}\|V^j\|m', \mathbf{d}'\rangle\end{aligned} \tag{2.3}$$

in the kinematic irreducible representation, where $\delta(\mathbf{v} : \mathbf{v}')$ denotes a product of Dirac delta functions in the continuous variables and Kronecker delta functions in the discrete variables. For these interactions, $\{m, j, \mathbf{v}, \Delta\mathbf{v}\}$ and $\{m_d = m + V, j, \mathbf{v}, \Delta\mathbf{v}\}$, where $\Delta\mathbf{v}$ are four operators conjugate to $\mathbf{v}$, have the same commutation relations. Since all 10 Poincaré generators can be expressed as functions of these operators, then if $m_d = m_d^\dagger := m + V > 0$, m_d becomes the mass Casimir operator for a dynamical representation of the Poincaré group. The structure of the interaction and the requirement $m_d > 0$ imply that simultaneous eigenstates of m_d, j^2, and $\mathbf{v}$, denoted by $|(\lambda, j), \mathbf{v}\rangle$, are complete and transform irreducibly with respect to a dynamical representation of the Poincaré group.

Simultaneous eigenfunctions of $\{m_d, j, \mathbf{v}\}$ in the kinematic irreducible basis have the form

$$\langle (m, j), \mathbf{v}, \mathbf{d}|(\lambda', j'), \mathbf{v}'\rangle = \delta(\mathbf{v} : \mathbf{v}')\delta_{jj'}\psi_{\lambda',j'}(m, \mathbf{d}), \tag{2.4}$$

where the internal wave function $\psi_{\lambda,j}(m, \mathbf{d})$ is the solution of the mass eigenvalue equation:

$$\begin{aligned}(\lambda - m)\psi_{\lambda,j}(m, \mathbf{d}) \\ = \sum\!\!\!\!\!\!\int \langle m, \mathbf{d}\|V^j\|m', \mathbf{d}'\rangle dm' d\mathbf{d}' \psi_{\lambda,j}(m', \mathbf{d}').\end{aligned} \tag{2.5}$$

Note that the variables $\mathbf{v}$, which define the choice of basis on each irreducible subspace, *do not* appear in Eq. (2.5) for the internal wave function $\psi_{\lambda,j}(m,\mathbf{d})$. In addition, the variables $\mathbf{v}$ play no role in formulating the asymptotic conditions for scattering solutions of Eq. (2.5). On the other hand, the operator V in Eq. (2.3) is different for each choice of $\mathbf{v}$ because Eq. (2.3) implies that the operators $\mathbf{v}$ commute with the interaction V.

The structure of the interaction means that the internal wave function $\psi_{\lambda,j}(m,\mathbf{d})$ is independent of the choice of basis for the kinematic irreducible representation. The dynamical unitary representation of the Poincaré group in this complete set of eigenstates is

$$\begin{aligned}&\langle(m,j),\mathbf{v},\mathbf{d}|U(\Lambda,a)|(\lambda,j),\mathbf{v}'\rangle\\&\quad=\sum\!\!\!\!\!\!\int{}'' \langle(m,j),\mathbf{v},\mathbf{d}|(\lambda,j),\mathbf{v}''\rangle d\mathbf{v}''\mathcal{D}^{\lambda,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a],\end{aligned}\tag{2.6}$$

where

$$\mathcal{D}^{\lambda,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a]:=\langle(\lambda,j),\mathbf{v}''|U(\Lambda,a)|(\lambda,j),\mathbf{v}'\rangle\tag{2.7}$$

is identical to Eq. (2.2) with m replaced by the mass eigenvalue λ of the dynamical mass operator m_d. This representation is dynamical because the Wigner function depends on the mass eigenvalue λ, which requires solving Eq. (2.5).

This is a short summary of the Bakamjian-Thomas construction. This construction gives an explicit representation, Eq. (2.6), of finite Poincaré transformations. Dynamical generators can be constructed by differentiating with respect to the group parameters. Bakamjian and Thomas actually construct the generators, but they are difficult to exponentiate, while the finite transformations discussed here can be used directly in applications.

The Bakamjian-Thomas construction summarized here is not limited to systems with two particles or fixed numbers of particles. In more complex systems the interaction is a sum of interactions that may be more naturally expressed in bases with the same $\mathbf{v}$ but different degeneracy parameters. For example, in the three-body problem it is natural to construct three-body kinematic irreducible representations using successive pairwise coupling. Different orders of pairwise coupling lead to irreducible representations with the same overall $\mathbf{v}$ but different choices of degeneracy parameters $\mathbf{d}$. For example, interactions involving the i-j pair of particles are most naturally described in a representation where the i-j pair is coupled first.

Because the degeneracy parameters are kinematically invariant, the coefficients of the transformation that relates bases with degeneracy parameters $\mathbf{d}_b$ to bases with degeneracy parameters $\mathbf{d}_a$ necessarily have the form

$$\begin{aligned}&\langle(m,j),\mathbf{v},\mathbf{d}_a|(m',j'),\mathbf{v}',\mathbf{d}'_b\rangle\\&\quad=\delta(\mathbf{v}:\mathbf{v}')\delta_{jj'}\delta(m-m')R^{jm}(\mathbf{d}_a,\mathbf{d}'_b).\end{aligned}\tag{2.8}$$

The coefficients $R^{jm}(\mathbf{d}_a,\mathbf{d}'_b)$ of the unitary operator that transforms invariant degeneracy parameters are Racah coefficients for the Poincaré group. The important observation is that these coefficients commute with and are independent of the variables $\mathbf{v}$ that label different vectors in an irreducible subspace.

In the general case the interaction kernel, Eq. (2.1), has the form

$$\begin{aligned}&\langle(m,j),\mathbf{v},\mathbf{d}|V|(m',j'),\mathbf{v}',\mathbf{d}'\rangle\\&\quad=\delta(\mathbf{v}:\mathbf{v}')\delta_{jj'}\sum\!\!\!\!\!\!\int{}' R^{jm}(\mathbf{d},\mathbf{d}_b)d\mathbf{d}_b\\&\qquad\times\langle m,\mathbf{d}_b\|V^j_b\|m',\mathbf{d}'_b\rangle d\mathbf{d}'_bR^{jm'}(\mathbf{d}'_b,\mathbf{d}').\end{aligned}\tag{2.9}$$

The relevant observation is that, in general, the interaction still has the form (2.1) with

$$\begin{aligned}&\langle m,\mathbf{d}\|V^j\|m',\mathbf{d}'\rangle\\&\quad:=\sum\!\!\!\!\!\!\int R^{jm}(\mathbf{d},\mathbf{d}_b)d\mathbf{d}_b\langle m,\mathbf{d}_b\|V^j_b\|m',\mathbf{d}'_b\rangle d\mathbf{d}'_bR^{jm'}(\mathbf{d}'_b,\mathbf{d}').\end{aligned}\tag{2.10}$$

To make the connection with Dirac's forms of dynamics, note that for some choice of bases $|(m,j)\mathbf{v},\mathbf{d}\rangle$ the Poincaré group Wigner function $\mathcal{D}^{\lambda,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a]$ is independent of the mass λ when (Λ,a) is restricted to a subgroup of the Poincaré group. The subgroup only depends on the choice of basis $\mathbf{v}$. This is because the Poincaré group Wigner function does not depend on the degeneracy parameters $\mathbf{d}$. This subgroup is called the kinematic subgroup associated with the basis $\mathbf{v}$. Dirac identified the three largest kinematic subgroups, which are the three-dimensional Euclidean group (instant-form dynamics), the Lorentz group (point-form dynamics), and the subgroup that leaves a plane tangent to the light-cone invariant (front-form dynamics). In our presentation, each kinematic subgroup is uniquely associated with a preferred basis for irreducible subspaces. This characterization exists even in the absence of interactions. The kinematic subgroup becomes a kinematic subgroup of the dynamical model when the interaction (2.9) also commutes with this subgroup.

The natural bases for the irreducible subspaces associated with Dirac's [2] forms of dynamics are simultaneous eigenstates given in Table I. Where $\mathbf{p}$ are momentum operators, p^0 is the Hamiltonian, and $\mathbf{j}_c$ and $\mathbf{j}_f$ are the canonical and light-front spin operators, which are related by a momentum-dependent rotation [10].

The connection with Dirac's notion of forms of dynamics is that when (Λ,a) is an element of the kinematic subgroup, then $U(\Lambda,a)$ can either act to the right, on the parameters of the state vector, or to the left, on the arguments of the wave function:

$$\begin{aligned}&\langle(m,j)\mathbf{v},\mathbf{d}|U[\Lambda,a]|(\lambda,j)\mathbf{v}'\rangle\\&\quad=\sum\!\!\!\!\!\!\int{}''\mathcal{D}^{m,j}_{\mathbf{v},\mathbf{v}''}[\Lambda,a]d\mathbf{v}''\langle(m,j)\mathbf{v}'',\mathbf{d}|(\lambda,j)\mathbf{v}'\rangle\\&\quad=\sum\!\!\!\!\!\!\int{}''\langle(m,j)\mathbf{v},\mathbf{d}|(\lambda,j)\mathbf{v}''\rangle d\mathbf{v}''\mathcal{D}^{\lambda,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a].\end{aligned}\tag{2.11}$$

TABLE I. Simultaneous eigenstates.

Form	Vector variables
Instant	$(\mathbf{v}\to\mathbf{p},\mathbf{j}_c\cdot\hat{\mathbf{z}})$
Point	$\mathbf{v}\to(\mathbf{u}:=\mathbf{p}/m,\mathbf{j}_c\cdot\hat{\mathbf{z}})$
Front	$\mathbf{v}\to(p^+:=p^0+p^3,p^1,p^2,\mathbf{j}_f\cdot\hat{\mathbf{z}})$

In this case $\mathcal{D}^{\lambda,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a]=\mathcal{D}^{m,j}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a]$ because the Wigner functions are independent of m and λ for kinematic (Λ,a).

Thus, while the computation of a general Poincaré transformation,

$$\begin{aligned}\langle(m,j)\mathbf{v},\mathbf{d}|U[\Lambda,a]|\Psi\rangle &= \sum\!\!\!\!\!\!\int \psi_{\lambda',j'}(m,\mathbf{d})\mathcal{D}^{\lambda',j'}_{\mathbf{v},\mathbf{v}'}[\Lambda,a]\psi^{*}_{\lambda',j'}(m',\mathbf{d}')\\ &\quad\times d\lambda' dm' d\mathbf{v}' d\mathbf{d}'\langle(m',j)\mathbf{v}',\mathbf{d}'|\Psi\rangle,\end{aligned} \tag{2.12}$$

requires solutions of the eigenvalue problem (2.5), for (Λ,a) in the kinematic subgroup we get an equivalent but simpler result that does not require the solution of Eq. (2.5):

$$\langle(m,j)\mathbf{v},\mathbf{d}|U[\Lambda,a]|\Psi\rangle=\sum\!\!\!\!\!\!\int d\mathbf{v}'\mathcal{D}^{m,j}_{\mathbf{v},\mathbf{v}'}[\Lambda,a]\langle(m,j)\mathbf{v}',\mathbf{d}'|\Psi\rangle. \tag{2.13}$$

III. THE EQUIVALENCE

We call two Poincaré invariant theories scattering equivalent if (1) the unitary representations of the Poincaré group are related by a unitary transformation and (2) both theories have the same S-matrix elements. The nontrivial property of a scattering equivalence is that the unitary transformation is *not* used to transform the free-particle states that are used to formulate the scattering asymptotic conditions.

Note the that unitarity alone does not imply the equivalence of the scattering matrices. Two-body models with different repulsive potentials are unitarily equivalent, but they do not necessarily have identical phase shifts. Ekstein [7,11] showed that a necessary and sufficient condition for a unitary transformation A to be a scattering equivalence is

$$\lim_{t\to\pm\infty}\|(A-I)U_0(t)|\psi\rangle\|=0 \tag{3.1}$$

for *both* time limits.

Different bases for the irreducible representation spaces $|(m,j)\mathbf{v}_a\rangle$ and $|(m,j)\mathbf{v}_b\rangle$ are related by a matrix of the form

$$\langle(m,j)\mathbf{v}_a|(m',j')\mathbf{v}'_b\rangle=\delta(m:m')\delta_{jj'}C^{mj}(\mathbf{v}_a;\mathbf{v}'_b). \tag{3.2}$$

These transformations can be used to construct Poincaré group Clebsch-Gordan coefficients in different bases with the same sets of degeneracy quantum numbers.

Tensor products of irreducible representations of the Poincaré group can be decomposed into a direct integral of irreducible representations using the Poincaré group Clebsch-Gordan coefficients. We use the following notation for the Clebsch-Gordan coefficients in the a basis:

$$\langle(m_1,j_1),\mathbf{v}_{a1},(m_2,j_2)\mathbf{v}_{a2}|(m,j),\mathbf{v}_a,\mathbf{d}\rangle. \tag{3.3}$$

The Clebsch-Gordan coefficients have the intertwining property

$$\begin{aligned}&\sum\!\!\!\!\!\!\int{}'' \langle(m_1,j_1),\mathbf{v}_{a1},(m_2,j_2)\mathbf{v}_{a2}|(m',j'),\mathbf{v}''_a,\mathbf{d}''\rangle d\mathbf{v}''\mathcal{D}^{m',j'}_{\mathbf{v}'',\mathbf{v}'}[\Lambda,a]\\ &\quad=\sum\!\!\!\!\!\!\int{}''\prod\mathcal{D}^{m_1,j_1}_{\mathbf{v}_{a1},\mathbf{v}''_{a1}}[\Lambda,a]\mathcal{D}^{m_2,j_2}_{\mathbf{v}_{a2},\mathbf{v}''_{a2}}[\Lambda,a]d\mathbf{v}''_{a1}d\mathbf{v}''_{a2}\\ &\quad\quad\times\langle(m_1,j_1),\mathbf{v}''_{a1},(m_2,j_1)\mathbf{v}''_{a2}|(m,j),\mathbf{v}',\mathbf{d}\rangle.\end{aligned} \tag{3.4}$$

Since the Poincaré group Wigner d functions have the same variables $(\mathbf{v}_a)$ on both sides of Eq. (3.4), the mass-independent subgroups of the Poincaré group are the same in both the tensor product and the irreducible representation. This means that the kinematic subgroups associated with vector variables $\mathbf{v}$ are mass independent on both the tensor product and irreducible representations. It turns out that the irreducible representations are more useful for comparing Bakamjian-Thomas constructions with different kinematic subgroups. The mass-independent subgroups associated with the basis choices are given in Table I, and when the interactions commute with these subgroups, they become the kinematic subgroups in Dirac's forms of dynamics.

Depending on details of the construction of the Clebsch-Gordan coefficients, there are different possible choices of degeneracy quantum numbers $\mathbf{d}$. For the purpose of comparing two dynamical models it is useful to have a common choice of degeneracy parameters.

For any fixed choice of $\mathbf{d}$ in the a basis we can construct a Clebsch-Gordan coefficient in the b basis that has the same degeneracy quantum numbers $\mathbf{d}$ using the transformation (3.2) in the Clebsch-Gordan coefficient (3.3):

$$\begin{aligned}&\langle(m_1,j_1),\mathbf{v}_{b1},(m_2,j_2)\mathbf{v}_{b2}|(m,j),\mathbf{v}_b,\mathbf{d}\rangle\\ &\quad=\sum\!\!\!\!\!\!\int d\mathbf{v}'_{a1}d\mathbf{v}'_{a2}d\mathbf{v}'_a C^{m_1j_1}(\mathbf{v}_{b1};\mathbf{v}'_{a1})C^{m_2j_2}(\mathbf{v}_{b2};\mathbf{v}'_{a2})\\ &\quad\quad\times\langle(m_1,j_1),\mathbf{v}'_{a1},(m_2,j_2)\mathbf{v}'_{a2}|(m,j),\mathbf{v}'_a,\mathbf{d}\rangle C^{mj}(\mathbf{v}'_a;\mathbf{v}_b).\end{aligned} \tag{3.5}$$

This follows from the identity

$$\begin{aligned}&\sum\!\!\!\!\!\!\int{}' d\mathbf{v}'_a\mathcal{D}^{m,j}_{\mathbf{v}_a,\mathbf{v}'_a}[\Lambda,a]C^{mj}(\mathbf{v}'_a;\mathbf{v}_b)\\ &\quad=\sum\!\!\!\!\!\!\int{}' C^{mj}(\mathbf{v}_a;\mathbf{v}'_b)\mathcal{D}^{m,j}_{\mathbf{v}'_b,\mathbf{v}_b}[\Lambda,a].\end{aligned} \tag{3.6}$$

Using either coefficient (3.3) or (3.5), we can successively use pairwise coupling to construct irreducible bases for the Hilbert space in the a or b basis with *identical degeneracy parameters* $\mathbf{d}$. We write these bases as

$$|(m,j)\mathbf{v}_a,\mathbf{d}\rangle,\quad |(m,j)\mathbf{v}_b,\mathbf{d}\rangle. \tag{3.7}$$

What Sokolov and Shatnyi established [6] was that the Bakamjian-Thomas constructions using the interactions

$$\begin{aligned}&\langle(m,j),\mathbf{v}_a,\mathbf{d}|V_a|(m',j'),\mathbf{v}'_b,\mathbf{d}'\rangle\\ &\quad=\delta(\mathbf{v}_a:\mathbf{v}'_a)\delta_{jj'}\langle m,\mathbf{d}\|V^j\|m',\mathbf{d}'\rangle\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\langle(m,j),\mathbf{v}_b,\mathbf{d}|V_b|(m',j'),\mathbf{v}'_b,\mathbf{d}'\rangle\\ &\quad=\delta(\mathbf{v}_b:\mathbf{v}'_b)\delta_{jj'}\langle m,\mathbf{d}\|V^j\|m',\mathbf{d}'\rangle\end{aligned} \tag{3.9}$$

in the mass operator are scattering equivalent. At first glance, it looks like V_a and V_b are related by a simple variable change; however, this is not the case because V_a and V_b commute with different kinematic subgroups. This property cannot be changed by a change of variables. V_a and V_b are

TABLE II. Functions for the three cases of interest.

Form	Function
Instant	$H = \sqrt{m^2 + \mathbf{p}^2}$
Point	$H = \sqrt{1 + \mathbf{v}^2} m$
Front	$H = \frac{1}{2}(p^+ + \frac{\mathbf{p}_\perp^2 + m^2}{p^+})$

different operators that have a common kernel in *different* representations.

It follows from Eqs. (3.8) and (3.9) that both representations have identical internal wave functions. They are related by the unitary transformation:

$$A := \sum\!\!\!\!\!\!\!\int_j d\lambda d\mathbf{v}_a d\mathbf{v}_b |(\lambda, j)\mathbf{v}_a\rangle C^{\lambda j}(\mathbf{v}_a; \mathbf{v}_b)\langle(\lambda, j)\mathbf{v}_b|, \quad (3.10)$$

which involves a sum over the eigenvalues λ of the internal mass operator (2.5). This is a dynamical operator because the transformation depends on the eigenvalues λ of Eq. (2.5). In principle, A might depend on the scattering asymptotic conditions used to define the complete set of eigenstates in the expansions in Eq. (3.9); however, Ekstein's theorem implies that this cannot happen if both representations give the same S matrix.

That both models give identical S matrices can be established by showing that both Hamiltonians can be replaced by the corresponding mass operators when computing the S matrix. This is because in the Bakamjian-Thomas construction interactions are added to the mass operator or some function of the mass operator. This identification is plausible because the mass is the relativistic generalization of the center of momentum Hamiltonian, which can be used to construct the S matrix in nonrelativistic quantum mechanics. To see how this arises in the relativistic case, observe that in any of Dirac's forms of dynamics the Bakamjian-Thomas Hamiltonian is a function of the mass operator and kinematic variables, j^2, $\mathbf{v}$, and Δv. For the three cases of interest these functions are given in Table II.

In each case H has the form $H = f(m, \mathbf{v})$, where $\mathbf{v}$ are kinematic operators that commute with m. The scattering wave operators have the form

$$\Omega_{a\pm} = s - \lim_{t\to\pm\infty} e^{iH_a t}\Phi_a e^{-iH_f t}$$
$$= s - \lim_{t\to\pm\infty} e^{if(m_a,\mathbf{v}_a)t}\Phi_a e^{-if(m_f,\mathbf{v}_a)t}, \quad (3.11)$$

where

$$S_a = \Omega_{+a}^\dagger \Omega_{-a} \quad (3.12)$$

and

$$\Phi_a = \delta(\mathbf{v}_a - \mathbf{v}_a')\hat{\Phi}, \quad (3.13)$$

where $\hat{\Phi}$ is the projection that provides the scattering asymptotic conditions for the scattering solutions of Eq. (2.5). Equation (2.5) and hence $\hat{\Phi}$ are independent of the choice of basis $\mathbf{v}_a$. m_f is the invariant mass of the system of free asymptotic particles.

The Kato-Birman invariance principle [12–14] allows the replacement of $f(m)$ by m in Eq. (3.11) provided f is in a class of admissible functions [12–14], which holds for f in Table II.

The invariance principle implies

$$\Omega_{a\pm} = s - \lim_{t\to\pm\infty} e^{iH_a t}\Phi_a e^{-iH_f t}$$
$$= s - \lim_{\tau\to\pm\infty} e^{im_a\tau}\Phi_a e^{-im_f\tau}. \quad (3.14)$$

Using the interactions in Eq. (2.3) gives S matrices of the forms

$$\langle(m, j), \mathbf{v}_a, \mathbf{d}|S_a|(m', j'), \mathbf{v}_a', \mathbf{d}'\rangle$$
$$= \delta(\mathbf{v}_a, \mathbf{v}_a')\delta_{jj'}\delta(m - m')\langle\mathbf{d}\|S^{mj}\|\mathbf{d}'\rangle \quad (3.15)$$

and

$$\langle(m, j), \mathbf{v}_b, \mathbf{d}|S_b|(m', j'), \mathbf{v}_b', \mathbf{d}'\rangle$$
$$= \delta(\mathbf{v}_b, \mathbf{v}_b')\delta_{jj'}\delta(m - m')\langle\mathbf{d}\|S^{mj}\|\mathbf{d}'\rangle, \quad (3.16)$$

where the reduced S matrices $\langle\mathbf{d}\|S^{mj}\|\mathbf{d}'\rangle$ are identical.

If we change irreducible basis variables in Eq. (3.15), we get

$$\langle(m, j), \mathbf{v}_b, \mathbf{d}|S_a|(m', j'), \mathbf{v}_b', \mathbf{d}'\rangle$$
$$= \sum\!\!\!\!\!\!\!\int C^{mj}(\mathbf{v}_b; \mathbf{v}_a)d\mathbf{v}_a\langle(m, j), \mathbf{v}_a, \mathbf{d}|S_a|$$
$$\times (m', j'), \mathbf{v}_a', \mathbf{d}'\rangle d\mathbf{v}_a' C^{m'j'}(\mathbf{v}_a'; \mathbf{v}_b')$$
$$= \sum\!\!\!\!\!\!\!\int C^{mj}(\mathbf{v}_b; \mathbf{v}_a)d\mathbf{v}_a\delta(\mathbf{v}_a, \mathbf{v}_a')\delta_{jj'}\delta(m - m')$$
$$\times d\mathbf{v}_a' C^{m'j'}(\mathbf{v}_a'; \mathbf{v}_b')\langle\mathbf{d}\|S^{mj}\|\mathbf{d}'\rangle$$
$$= \delta(\mathbf{v}_b; \mathbf{v}_b')\delta_{jj'}\delta(m - m')\langle\mathbf{d}\|S^{mj}\|\mathbf{d}'\rangle$$
$$= \langle(m, j), \mathbf{v}_b, \mathbf{d}|S_b|(m', j'), \mathbf{v}_a', \mathbf{d}'\rangle, \quad (3.17)$$

which proves the equivalence of the full S matrices. $\delta(m - m')$ is needed for the cancelation of $C^{mj}(\mathbf{v}_y; \mathbf{v}_x)$.

It then follows from Ekstein's theorem that H_a and H_b are related by a unitary scattering equivalence A. The operator A necessarily has the form

$$A = \Omega_{a\pm}\Omega_{b\pm}^\dagger, \quad (3.18)$$

where these wave operators include the one-body channels (bound states) so they are unitary. The intertwining properties of the wave operators imply

$$AM_b = \Omega_{a\pm}\Omega_{b\pm}^\dagger M_b \Omega_{a\pm} M_f \Omega_{b\pm}^\dagger M_a \Omega_{a\pm}\Omega_{b\pm}^\dagger = M_a A. \quad (3.19)$$

If these relations are combined with the kinematic symmetries, it follows that

$$A\{H_b, \mathbf{P}_b, \mathbf{J}_b, \mathbf{K}_b\} = \{H_a, \mathbf{P}_a, \mathbf{J}_a, \mathbf{K}_a\}A \quad (3.20)$$

or, equivalently,

$$AU_b(\Lambda, a) = U_a(\Lambda, a)A, \quad (3.21)$$

as desired.

We are now in a position to summarize the main result. Consider a Bakamjian-Thomas model dynamics with a specific choice of kinematic subgroup. The mass operator has a kernel in the free-particle irreducible basis $|(m, j)\mathbf{v}_a, \mathbf{d}\rangle$ of the form

$$\langle (m', j')\mathbf{v}'_a, \mathbf{d}'|M_a|(m, j)\mathbf{v}_a, \mathbf{d}\rangle = \delta(\mathbf{v}'_a : \mathbf{v}_a)\delta_{j'j}\langle m', \mathbf{d}'\|M^j\|m, \mathbf{d}\rangle. \quad (3.22)$$

This is a Bakamjian-Thomas mass operator with the kinematic subgroup associated with the irreducible vector variables $\mathbf{v}_a$. A new Bakamjian-Thomas mass operator in the free-particle irreducible basis $|(m, j)\mathbf{v}_b, \mathbf{d}\rangle$, related to $|(m, j)\mathbf{v}_b, \mathbf{d}\rangle$ by Eq. (3.2), is defined by the kernel:

$$\langle (m', j')\mathbf{v}'_b, \mathbf{d}'|M_b|(m, j)\mathbf{v}_b, \mathbf{d}\rangle = \delta(\mathbf{v}'_b : \mathbf{v}_b)\delta_{j'j}\langle m', \mathbf{d}'\|M^j\|m, \mathbf{d}\rangle. \quad (3.23)$$

This operator commutes with $\mathbf{v}_b$, $\Delta\mathbf{v}_b$ and is the mass operator for a scattering-equivalent Bakamjian-Thomas dynamics with the kinematic subgroup associated with the choice of irreducible variables $\mathbf{v}_b$.

Each of the two scattering-equivalent representations has a different kinematic subgroup. Both models can be expressed in terms of tensor products of single-particle irreducible representations where the equivalence is not as obvious. While the operator A is a complicated dynamical operator, the internal mass eigenvalue problems in both representations are identical and given by Eq. (2.5).

IV. EXAMPLE

In this section I illustrate the equivalence with a three-body example. Specifically, I give explicit formulas for the abstract operators needed to relate an instant-form Bakamjian-Thomas three-body model to an equivalent front-form Bakamjian-Thomas model.

The natural variables $\mathbf{v}_i$ for an instant-form dynamics are $\mathbf{p}$ and $\mathbf{j}_c \cdot \hat{\mathbf{z}}$, where $\mathbf{j}_c$ is the canonical spin.

The single-particle basis (2.1) and Poincaré group Wigner d function in this basis (2.2) are

$$|(m, j)\mathbf{p}, \mu\rangle \quad (4.1)$$

and

$$\mathcal{D}^{j,m}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda, a] = \delta(\mathbf{p}' - \Lambda p)\sqrt{\frac{\omega_m(\mathbf{p}')}{\omega_m(\mathbf{p}')}}e^{-i\omega_m(\mathbf{p}')a^0 + i\mathbf{p}\cdot\mathbf{a}}D^j_{\mu'\mu} \times \left[B_c^{-1}(\mathbf{p}'/m)\Lambda B_c(\mathbf{p}/m)\right], \quad (4.2)$$

where $\omega_m(\mathbf{p})$ is the energy and $B_c(\mathbf{p}/m)$ is the rotationless Lorentz transformation that transforms $(m, \mathbf{0})$ to $(\omega_m(\mathbf{p}), \mathbf{p})$. This Wigner function is independent of m for (Λ, a) corresponding to a spatial translation or rotation. Rotations are kinematic in this basis because the rotationless boosts satisfy

$$R = B_c^{-1}(R\mathbf{p}/m)RB_c(\mathbf{p}/m) \quad (4.3)$$

independent of m.

The tensor product bases for two- and three-particle systems in this basis are

$$|(m_1, j_1)\mathbf{p}_1, \mu_1(m_2, j_2)\mathbf{p}_2, \mu_2\rangle \quad (4.4)$$

and

$$|(m_1, j_1)\mathbf{p}_1, \mu_1(m_2, j_2)\mathbf{p}_2, \mu_2(m_3, j_3)\mathbf{p}_3, \mu_3\rangle. \quad (4.5)$$

The noninteracting irreducible bases for the two- and three-body systems are related to the single-particle bases by the Poincaré group Clebsch-Gordan coefficients in this basis. Because the two-body invariant mass has a continuous spectrum, it is convenient to replace m by the equivalent operator k, which is defined implicitly by

$$m = m(k) = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2}. \quad (4.6)$$

With this modification the Poincaré group Clebsch-Gordan coefficients in this basis are

$$\begin{aligned}
&{}_c\langle (m_1, j_1)\mathbf{p}_1, \mu_1|(m_2, j_2)\mathbf{p}_2, \mu_2|[k(m), j]\mathbf{p}, \mu, l, s, j_1, j_2\rangle_c \\
&= \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p})\frac{\delta[m(\mathbf{p}_1, \mathbf{p}_2) - m]}{k^2} \\
&\times \sqrt{\frac{\omega_{m_1}(k^2)}{\omega_{m_1}(\mathbf{p}_1^2)}\frac{\omega_{m_2}(k^2)}{\omega_{m_2}(\mathbf{p}_2^2)}\frac{\omega_m(\mathbf{p}^2)}{m}} \\
&\times \sum D^{j_1}_{\mu_1\mu_1}[B_c^{-1}(\mathbf{p}_1/m_1)B_c(\mathbf{p}/m)B_c \\
&\times(\mathbf{k}_1/m_1)]D^{j_2}_{\mu_2\mu_2}\left[B_c^{-1}(\mathbf{p}_2/m_2)B_c(\mathbf{p}/m)B_c(\mathbf{k}_2/m_2)\right] \\
&\times Y^l_{m_l}[\hat{\mathbf{k}}(\mathbf{p}_1, \mathbf{p}_2)]\langle j_1, \mu_1, j_2, \mu_2|s, \mu_s\rangle\langle l, m_l, s, \mu_s|j, \mu\rangle,
\end{aligned} \quad (4.7)$$

where

$$(\omega_m(\mathbf{k}_i), \mathbf{k}_i) = B_c^{-1}(\mathbf{p}_i/m_i)\begin{pmatrix}\omega_m(\mathbf{p}_i)\\ \mathbf{p}_i\end{pmatrix}, \quad (4.8)$$

$Y^l_{m_l}(\hat{\mathbf{k}})$ is a spherical harmonic, and $\mathbf{k}_i^2 = k^2$. The quantities l and s in this expression are the degeneracy parameters $\mathbf{d}$:

$$|(m, j)\mathbf{p}, \mu, \mathbf{d}\rangle := |[k(m), j]\mathbf{p}, \mu, l, s, j_1, j_2\rangle_c. \quad (4.9)$$

For the three-body system irreducible basis vectors are constructed using sequential pairwise coupling. The degeneracy parameters depend on the order of coupling:

$$\begin{aligned}
&|(q, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\rangle \\
&= \sum\int |(k_{12}, j_{12})\mathbf{p}_{12}, \mu_{12}; (m_3, j_3)\mathbf{p}_3, \mu_3\rangle d\mathbf{p}_{12}d\mathbf{p}_3 \\
&\times \langle (k_{12}, j_{12})\mathbf{p}_{12}, \mu_{12}; (m_3, j_3)\mathbf{p}_3, \mu_3| \\
&\times (m, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}\rangle.
\end{aligned} \quad (4.10)$$

The degeneracy parameters

$$\{\mathbf{d}\} = \{L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\} \quad (4.11)$$

are associated with the coupling order $[(1 + 2) + (3)]$.

Racah coefficients that change the order of coupling are constructed by taking the overlap of states of the form (4.10) corresponding to different orders of coupling. Explicit expressions for the Racah coefficients involve four Clebsch-Gordan coefficients. The explicit expressions are not very illuminating, but symmetry considerations imply that the Racah coefficients in this basis must have the structure

$$\begin{aligned}&\langle (q_{12;3}, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}, m_{12}, j_{12}, l_{12}, s_{12}|(q_{23,1}, j')\\ &\quad\times \mathbf{p}', \mu'; L_{23,1}, S_{23,1}, m_{23}, j_{23}, l_{23}, s_{23}\rangle\\ &\quad= \delta[m(q_{12;3}, k_{12}) - m(q_{23;1}, k_{23})]\delta_{jj'}\delta(\mathbf{p}-\mathbf{p}')\delta_{\mu\mu'}\\ &\quad\times R^{jm}(q_{12;3}, L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}; q_{23;1},\\ &\quad\times L_{23,1}, S_{23,1}, k_{23}, j_{23}, l_{23}, s_{23}). \qquad (4.12)\end{aligned}$$

A two-body Bakamjian-Thomas interaction between particles 1 and 2 in the irreducible three-body basis has the structure

$$\begin{aligned}&\langle (q_{12;3}, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}|V_{12}|(q'_{12;3}, j')\\ &\quad\times \mathbf{p}', \mu'; L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle\\ &= \delta(\mathbf{p}-\mathbf{p}')\delta_{jj'}\delta_{\mu\mu'}\frac{\delta(q_{12;3}-q'_{12;3})}{q^2_{12;3}}\delta_{L_{12,3}L'_{12,3}}\delta_{S_{12,3}S'_{12,3}}\delta_{j_{12}j'_{12}}\\ &\quad\times \langle k_{12}, l_{12}, s_{12}\| v^{j_{12}}_{12}\| k'_{12}, l'_{12}, s'_{12}\rangle. \qquad (4.13)\end{aligned}$$

Since this interaction appears in the two-body mass operator, when it is used in the three-body mass operator, it is added as

$$V_{12,3} = \sqrt{q^2_{12,3} + (m_{12}+V_{12})^2} - \sqrt{q^2_{12,3}+m^2_{12}}. \qquad (4.14)$$

The full three-body Bakamjian-Thomas mass operator is

$$M = M_0 + V_{12,3} + V_{23,1} + V_{31,2}. \qquad (4.15)$$

Each of the operators $V_{ij,k}$ is naturally computed using a different order of coupling. By using the Racah coefficients (4.12) we can write the full Bakamjian-Thomas mass operator as a kernel in an irreducible basis associated with a fixed order of coupling. For example, the kernel of the mass operator in the basis

$$|(q_{12;3}, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\rangle \qquad (4.16)$$

has the form

$$\begin{aligned}&\langle (q_{12;3}, j)\mathbf{p}, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}|M|\\ &(q'_{12;3}, j')\mathbf{p}', \mu'; L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle\\ &\quad= \delta(\mathbf{p}-\mathbf{p}')\delta_{jj'}\delta_{\mu\mu'}\langle q_{12;3}, L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\\ &\|\hat{M}^j\| q'_{12;3}, L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle. \qquad (4.17)\end{aligned}$$

This is the kernel of the operator that appears in mass eigenvalue equation (2.5). Solving for the eigenstates of M in this basis gives simultaneous eigenstates of M,j^2,$\mathbf{p}$, $\mathbf{j}_z$,

$$|(\lambda, j)\mathbf{p}, \mu\rangle \qquad (4.18)$$

that are complete and transform like Eq. (2.6) with $\mathcal{D}^{j,\lambda}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda, a]$ given by Eq. (4.2).

This defines the dynamical model with an instant-form kinematic symmetry because the Poincaré group Wigner d functions $\mathcal{D}^{j,\lambda}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda, a]$ are independent of the mass eigenvalue λ for $[\Lambda, a]$ corresponding to a pure rotation or translation.

For the equivalent front-form model the irreducible basis variables $\mathbf{v}_a := \{\mathbf{p}, \mathbf{j}_{cz}\}$ are replaced by $\{\mathbf{v}_b := p^+, \mathbf{p}_\perp, \mathbf{j}_{fz}\}$.

The light-front momenta are related to the instant-form momenta by

$$p^+ = \sqrt{m^2+\mathbf{p}^2} + p_3, \quad p_1 = p_1, \quad p_2 = p_2, \qquad (4.19)$$

while the light-front spins are related to the canonical (instant form) spins by

$$\mathbf{j}_c = B^{-1}_c(p/m)B_f(p/m)\mathbf{j}_f, \qquad (4.20)$$

where $B_f(p/m)$ is a light-front-preserving boost that transforms a particle of mass m at rest to momentum $\mathbf{p}$. In this expression the boosts are interpreted as 4×4 matrices of operators. The combination $B^{-1}_c(p/m)B_f(p/m)$, which leaves $(m,0,0,0)$ unchanged, is a momentum-dependent three-rotation (called a Melosh [10] rotation). Since $\mathbf{j}^2$ is rotationally invariant, it follows that $j^2_f = j^2_c$.

In the notation of Sec. III the kernel of the operator that transforms the instant-form irreducible basis to the light-front irreducible basis is

$$\begin{aligned}C^{mj}_{p^+,\mathbf{p}_\perp,\mu;\mathbf{p}',\mu'} &= \delta[p^+ - p'_3 - \omega_m(\mathbf{p}')]\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\\ &\quad\times\sqrt{\frac{p^+}{\omega_m(\mathbf{p})}}D^j_{\mu\mu'}[B^{-1}_f(p/m)B_c(p'/m)]. \qquad (4.21)\end{aligned}$$

The Wigner function in the basis

$$|(m, j)p^+, \mathbf{p}_\perp, \mu\rangle \qquad (4.22)$$

with $p' := \Lambda p$ is

$$\begin{aligned}&\mathcal{D}^{j,m}_{p'^+,\mathbf{p}'_\perp,\mu';p^+,\mathbf{p}_\perp,\mu}[\Lambda, a]\\ &\quad= \delta(p'^+ - \Lambda^+ p)\delta(\mathbf{p}'_\perp - \Lambda_\perp p)\\ &\quad\times\sqrt{\frac{p^{+\prime}}{p^+}}e^{-i\frac{1}{2}(p'^+a^- + p'^-a^+) + i\mathbf{p}'_\perp\cdot\mathbf{a}_\perp}D^j_{\mu'\mu}\\ &\quad\times\left[B^{-1}_f(\mathbf{p}'/m)\Lambda B_f(\mathbf{p}/m)\right], \qquad (4.23)\end{aligned}$$

where $p^- = \frac{\mathbf{p}^2_\perp + m^2}{p^+}$. This Wigner function has the property that if (Λ, a) leaves the light-front invariant, then it does not depend on m.

Clebsch-Gordan coefficients in the light-front basis (4.23) with the same degeneracy parameters as Eq. (4.16) are obtained by applying $C^{mj}_{p^+,\mathbf{p}_\perp,\mu;\mathbf{p}',\mu'}$ to both sizes of the instant-form Poincaré group Clebsch-Gordan coefficient:

$$\begin{aligned}&\sum\!\!\!\!\!\!\int C^{m_1j_1}_{p^+_1,\mathbf{p}_{1\perp},\mu_1;\mathbf{p}'_1,\mu'_1}C^{m_2j_2}_{p^+_2,\mathbf{p}_{2\perp},\mu_2;\mathbf{p}'_2,\mu'_2}{}_c\langle (m_1, j_1)\mathbf{p}'_1, \mu'_1|(m_2, j_2)\mathbf{p}'_2,\\ &\quad\times\mu'_2|[k(m), j]\mathbf{p}_{12}, \mu, l_{12}, s_{12}, j_1, j_2\rangle_c C^{mj*}_{\mu;\mathbf{p}',\mu';p^+,\mathbf{p}_\perp}. \qquad (4.24)\end{aligned}$$

This gives the Clebsch-Gordan coefficient in the light-front basis:

$$\begin{aligned}&{}_f\langle (m_1, j_1)p^+_1, \mathbf{p}_{1\perp}, \mu_1|(m_2, j_2)p^+_2,\\ &\quad\times\mathbf{p}_{2\perp}, \mu_2|(k, j)p^+, \mathbf{p}_\perp, \mu, l, s, j_1, j_2\rangle_f\\ &\quad= \delta(\mathbf{p}_{1\perp}+\mathbf{p}_{2\perp}-\mathbf{p}_\perp)\delta(p^+_1+p^+_2-p^+)\frac{\delta[k(p_1,p_2)-k]}{k^2}\\ &\quad\times\left|\sqrt{\frac{\omega_{m_1}(k^2)\omega_{m_2}(k^2)P^+}{p^+_1p^+_2m}}\right|^{1/2}\\ &\quad\times\sum D^{j_1}_{\mu_1\mu_1}[B^{-1}_f(k_1/m_1)B_c(k_1/m_1)]D^{j_2}_{\mu_2\mu_2}\\ &\quad\times\left[B^{-1}_f(k_2/m_2)B_c(k_2/m_2)\right]Y^l_{m_l}[\hat{\mathbf{k}}(p_1,p_2)]\\ &\quad\times\langle j_1, \mu_1, j_2, \mu_2, |s, \mu_s\rangle\langle l, m_l, s, \mu_s|j, \mu\rangle. \qquad (4.25)\end{aligned}$$

In arriving at expression (4.25) we have used the property that light-front-preserving boosts form a group, so there are no Wigner rotations associated with the light-front-preserving boosts; however, the Wigner rotations in Eq. (4.7) get replaced by Melosh rotations in Eq. (4.25). The irreducible kinematic basis that replaces Eq. (4.10) is

$$\begin{aligned}&|(q_{12,3}, j)p^+, \mathbf{p}_\perp, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\rangle\\&\quad= \sum\int |(k_{12}, j_{12})p^+_{12}, \mathbf{p}_{12\perp}, \mu_{12}, l_{12}, s_{12};\\&\quad\times (m_3, j_3)p^+_3, \mathbf{p}_{3\perp}, \mu_3\rangle dp^+_{12} d\mathbf{p}_{12\perp} dp^+_3 d\mathbf{p}_{3\perp}\\&\quad\times \langle (k_{12}, j_{12})p^+_{12}, \mathbf{p}_{12\perp}, \mu_{12}; (m_3, j_3)p^+_3,\\&\quad\times \mathbf{p}_{3\perp}, \mu_3|(m, j)p^+, \mathbf{p}_\perp, \mu; L_{12,3}, S_{12,3}\rangle \end{aligned} \tag{4.26}$$

with

$$\begin{aligned}&|(k_{12}, j_{12})p^+_{12}, \mathbf{p}_{12\perp}, \mu_{12}, l_{12}, s_{12}\rangle\\&\quad= \sum\!\!\!\!\!\!\int (m_1, j_1)p^+_1, \mathbf{p}_{1\perp}, \mu'_1; (m_2, j_2)p^+_2, \mathbf{p}_{2\perp}, \mu'_2\rangle\\&\quad\times \langle dp^+_2 d\mathbf{p}_{2\perp} dp^+_1 d\mathbf{p}_{1\perp}(m_1, j_1)p^+_1, \mathbf{p}_{1\perp}, \mu'_1; (m_2, j_2)p^+_2,\\&\quad\times \mathbf{p}_{2\perp}, \mu'_2|(k_{12}, j_{12})p^+_{12}, \mathbf{p}_{12\perp}, \mu_{12}, l_{12}, s_{12}\rangle. \end{aligned} \tag{4.27}$$

Note that the degeneracy parameters

$$\{\mathbf{d}\} = \{L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\} \tag{4.28}$$

in Eq. (4.26) are identical to the degeneracy parameters in Eq. (4.10). The interaction that replaces Eq. (4.13) in the scattering-equivalent light-front dynamics model is

$$\begin{aligned}&\langle (q, j)p^+, \mathbf{p}_\perp, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}|V_{12}|\\&\quad\times (q', j')p'^+, \mathbf{p}'_{1\perp}\mu'; L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle\\&\quad= \delta(p^+ - p'^+)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\delta_{jj'}\delta_{\mu\mu'}\frac{\delta(q - q')}{q^2}\\&\quad\times \delta_{L_{12,3}L'_{12,3}}\delta_{S_{12,3}S'_{12,3}}\delta_{j_{12}j'_{12}}\langle k_{12}, l_{12}, s_{12}\|v^{j_{12}}_{12}\|k'_{12}, l'_{12}, s'_{12}\rangle. \end{aligned} \tag{4.29}$$

The Bakamjian-Thomas mass operator that replaces Eq. (4.17) in the scattering-equivalent light-front dynamics model is

$$\begin{aligned}&\langle (q_{12;3}, j)p^+, \mathbf{p}_\perp, \mu; L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}|M|\\&\quad\times (q'_{12;3}, j')p'^+, \mathbf{p}'_\perp, \mu'; L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle\\&\quad= \delta(p^+ - p'^+)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\delta_{jj'}\delta_{\mu\mu'}\langle q_{12;3}, L_{12,3}, S_{12,3}, k_{12},\\&\quad\times j_{12}, l_{12}, s_{12}\|\hat{M}^j\|q'_{12;3}, L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle. \end{aligned} \tag{4.30}$$

Note that $\hat{M}$ in Eq. (4.30) is identical to $\hat{M}$ in Eq. (4.17) provided both operators have the same internal interactions $\langle k_{12}, l_{12}, s_{12}\|v^{j_{12}}_{12}\|k'_{12}, l'_{12}, s'_{12}\rangle$.

Equation (4.30) defines the mass operator for the scattering-equivalent model with the light-front kinematic symmetry. Any dynamical three-body calculation will give complete sets of bound-state and scattering eigenstates of Eq. (2.5) with the internal kernel

$$\begin{aligned}&\langle q_{12;3}, L_{12,3}, S_{12,3}, k_{12}, j_{12}, l_{12}, s_{12}\|\hat{M}^j\|\\&\quad\times q'_{12;3}, L'_{12,3}, S'_{12,3}, k'_{12}, j'_{12}, l'_{12}, s'_{12}\rangle, \end{aligned} \tag{4.31}$$

which appears in both Eqs. (4.30) and (4.17). Either solution can be expressed in the kinematic tensor product representation using the appropriate Poincaré group Clebsch-Gordan coefficients. Since the Clebsch-Gordan coefficients in a given basis preserve the kinematic subgroup for that basis, the tensor product representations also have the same kinematic symmetries as the corresponding mass operators in either Eq. (4.30) or Eq. (4.31). Both mass operators give the same scattering matrix, even though the dynamical models have different kinematic symmetries.

V. SUMMARY

To summarize, Bakamjian-Thomas constructions of dynamical representations of the Poincaré group have the general form

$$U_b(\Lambda, a)|(\lambda, j), \mathbf{v}_b\rangle = \sum\!\!\!\!\!\!\int d\mathbf{v}'|(\lambda, j), \mathbf{v}'_b\rangle \mathcal{D}^{\lambda', j}_{\mathbf{v}'_b, \mathbf{v}_b}[\Lambda, a], \tag{5.1}$$

where, in a kinematically irreducible basis $|(m, j), \mathbf{v}_b, \mathbf{d}\rangle$,

$$\langle (m, j), \mathbf{v}_b|(\lambda, j'), \mathbf{v}'_b\rangle = \delta(\mathbf{v}_b, \mathbf{v}'_b)\psi_{\lambda, j}(m, \mathbf{d}) \tag{5.2}$$

and $\psi_{\lambda, j}(m, \mathbf{d})$ is the solution of the mass eigenvalue equation:

$$\begin{aligned}(\lambda - m)\psi_{\lambda', j'}(m, \mathbf{d}) &= \sum\!\!\!\!\!\!\int \langle m, \mathbf{d}\|V^j\|m', \mathbf{d}'\rangle\\&\quad\times dm' d\mathbf{d}'\psi_{\lambda', j'}(m', \mathbf{d}'), \end{aligned} \tag{5.3}$$

which is identical in all forms of dynamics. Equivalent models with different kinematic symmetries differ only in the choice of the variables $\mathbf{v}_b$ in Eqs. (5.1) and (5.2). While different choices of $\mathbf{v}_b$ lead to different interactions (2.1) with different kinematic symmetries, the resulting dynamical models are all scattering equivalent.

Irreducible vectors in the different forms of dynamics are related by

$$|(\lambda, j), \mathbf{v}_b\rangle = \sum\!\!\!\!\!\!\int' |(\lambda, j), \mathbf{v}'_c\rangle d\mathbf{v}'_c C^{\lambda j}(\mathbf{v}'_c; \mathbf{v}'_b), \tag{5.4}$$

and the Wigner functions in different representations are related by

$$\begin{aligned}\mathcal{D}^{\lambda, j}_{\mathbf{v}'_c, \mathbf{v}_c}[\Lambda, a] &= \sum\!\!\!\!\!\!\int d\mathbf{v}_b d\mathbf{v}'_b C^{\lambda j}\\&\quad\times (\mathbf{v}'_c; \mathbf{v}'_b)\mathcal{D}^{\lambda, j}_{\mathbf{v}'_b, \mathbf{v}_b}[\Lambda, a]C^{\lambda j}(\mathbf{v}_{b'}; \mathbf{v}_c). \end{aligned} \tag{5.5}$$

The transformation relating the different kinematic subgroups is dynamical because the mass eigenvalues λ that appear in both $C^{\lambda j}(\mathbf{v}_b; \mathbf{v}_c)$ and $\mathcal{D}^{\lambda, j}_{\mathbf{v}'_b, \mathbf{v}_b}[\Lambda, a]$ are determined by solving the dynamical equation. The important observation is that the physical observables (binding energies, S-matrix elements) are obtained by solving Eq. (5.3), which is independent of the choice of kinematic subgroup.

The conclusion of this work is that Poincaré invariant quantum models should be considered as being defined without reference to any specific kinematic subgroup and any Poincaré invariant model can be transformed to a representation that has the same S matrix and exhibits any chosen kinematic symmetry. This conclusion is not limited to two-body models or models that conserve particle number, nor is it limited to the maximal kinematic subgroups discussed by Dirac. The nontrivial dynamical equation that must be solved is the internal mass eigenvalue problem (5.3), which is the same in all cases. The different choices of representation have no effect on bound-state or scattering observables.

The one class of applications where using different forms of dynamics has dynamical consequences is when they are used to calculate electromagnetic observables in the one-photon-exchange approximation. This is because the initial and final hadronic states are in different frames and have different invariant masses. The S-matrix equivalence proof requires $m \neq m'$. This in not true when the photon transfers energy and/or momentum to a dynamical system. While the equivalence can be recovered by transforming the impulse current in one representation to another representation, the equivalent transformed current will generally have many-body contributions.

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