The $n + n + \alpha$ system in a continuum Faddeev formulation

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(Received 30 June 2010; revised manuscript received 16 September 2010; published 10 November 2010)

The continuum Faddeev equations for the neutron-neutron-alpha ($n + n + \alpha$) system are formulated for a general interaction as well as for finite rank forces. In addition, the capture process $n + n + \alpha \rightarrow {}^6\text{He} + \gamma$ is derived.

DOI: 10.1103/PhysRevC.82.054002 PACS number(s): 21.45.−v, 25.10.+s, 25.40.Lw

I. INTRODUCTION

In recent years the study of quantum halo systems experienced increased interest in the nuclear as well as the atomic few-body community. For a recent review see Ref. [1]. The nucleus ^{6}He is of particular interest since it constitutes the lightest two-neutron halo nucleus with a ^{4}He core. Being an effective three-body system, the properties of the ground state have been explored using either Faddeev [2–4] or hyperspherical harmonics (HH) [5–8]. More recently the ground state has also been calculated with multicluster methods [9–17] as well as with Green's function Monte Carlo (GFMC) [18]. Those multicluster methods include various techniques such as the microscopic dynamical multiconfiguration three-cluster model [9], the stochastic variational method [10], the multicluster dynamic model (MDMP and AMDMP), the hybrid-TV model, a combination of the cluster orbital shell model (COSM) [15] and the extended cluster model (ECM) [16], the refined resonating group method [11,14], and the coupled-rearrangement-channel variational method with Gaussian basis functions [17]. In addition, the beta decay to the $\alpha + d$ continuum was studied [19,20]. Of interest is also the two-neutron capture process $^4\text{He}(2n,\gamma)^6\text{He}$ as a possible route bridging the instability gap at nuclear mass $A = 5$ [21].

The situation is quite different in the continuum of two neutrons and an α particle. There is a well established 2^+ resonance [22], but further resonant structures are still under debate [7,23–30]. Up to now more indirect approaches in understanding the resonance structure have been carried out (e.g., the four-body distorted wave approach) leading to three-body continuum excitations of two-neutron Borromean halo nuclei [23,28]. Furthermore, complex scaling in Coulomb breakup reactions was employed [31]. In addition, an extension of the HH method on a Lagrange mesh [30] was used to study three-body continuum states. This is at least a four-body problem with great uncertainties about the reaction mechanisms and the interactions entering these much more complicated systems.

Thus the currently predominant approach to continuum calculations for the pure $n + n + \alpha$ system is the HH method [7,8,26–28,32]. A Faddeev approach is, to the best of our knowledge, still missing. Only for the ^{6}Li nucleus, a Faddeev treatment of the deuteron-alpha (d-α) system has been employed [33], which however, did not have to face the challenge of three-to-three scattering. This also refers to the pioneering work by Koike [34,35] on d-α scattering.

The aim of this investigation is to fill that gap. For the $n + n + \alpha$ system one faces the situation of three free particles being in the initial channel and leading again to three free particles in the exit channel. In other words, one has to deal with three-to-three scattering. Scattering of three free incoming particles to three free outgoing ones in a Faddeev approach was initiated in Refs. [36,37] in the context of the three-body photodisintegration of ^{3}He. This path was also followed in the same context and in a Faddeev approach by Meijgaard and Tjon [38]. Into the matrix element for the photodisintegration enters the three-nucleon to three-nucleon scattering wave function, which was evaluated in Ref. [38] and then inserted into the photodisintegration matrix element. However, evaluating the wave function is a completely unnecessary complication since this process is initiated by the three-nucleon bound state. One can directly derive a Faddeev equation for the three-body breakup amplitude, in which the driving term contains the action of the current operator on the ^{3}He ground state. Then the complete final-state interaction is generated by a Faddeev integral kernel for the amplitude. This considerably simplifies the technical part of a calculation since no disconnected processes occur. This very procedure was pioneered in Refs. [36,37] and is being applied in state-of-the-art calculations (see, e.g., Refs. [39,40]).

The same procedure can trivially be adapted to the capture process $n + n + \alpha \rightarrow {}^6\text{He} + \gamma$, as will be displayed in the present investigation. This capture process is relevant for the production rate of ^{6}He in astrophysical environments [41] characterized by high neutron and alpha densities (e.g., those related to supernova shock fronts). In Ref. [21] this three-body process is approximated by sequential two-body processes, whereas, in principle, a genuine three-body reaction needs to be calculated. Furthermore, the $nn\alpha \rightarrow nn\alpha$ amplitude is relevant for determining the next order coefficient [42] in the virial equation of state in low-density matter [43].

From a technical point of view, the Faddeev approach to the $n + n + \alpha$ continuum is strongly needed since the currently predominant approach, namely the HH approach, still faces open challenges. It is already known that in the breakup process $n + d \rightarrow n + n + p$ a strong final-state interactions (FSI) peak appears for the $n - n$ subsystem. In the Faddeev approach using Jacobi momenta this can be mapped out correctly, whereas when changing to the hyperspherical angle, the convergence is quite poor for this particular configuration. In the HH method, the control of the coupling potentials can be a painful exercise, whereas in the Faddeev approach using Jacobi variables the dynamics are perfectly well under control in all details. This same situation must be expected in $n+n+\alpha$ scattering, where the three-body S matrix is characterized by continuous quantum numbers describing how the energy is distributed over the relative motion. There are strong initial-state and final-state interaction peaks, which in a discrete representation through hyperspherical K quantum numbers are difficult to map out correctly. As stated earlier, only a technically reliable approach such as the Faddeev one will guarantee the validity of the results when searching for ^{6}He resonances.

The article is organized as follows. In Sec. II we derive the coupled Faddeev equations for the three-to-three scattering amplitude, followed by a partial wave decomposition in Sec. III. The Faddeev equations will be solved by iteration yielding a multiple scattering series. This will be outlined in Sec. IV. Since most of the Faddeev-type investigations of the $n+n+\alpha$ system are based on finite rank forces, we also present in Sec. V a continuum formulation based on separable forces. Furthermore, we discuss the unitarity relation for the three-to-three amplitude in Sec. VI. Finally, the capture process $n+n+\alpha \rightarrow {}^6\text{He}+\gamma$ will be discussed for the Faddeev scheme in Sec. VII. Then we summarize in Sec. VII. Technical details about the partial wave decomposition and an efficient way of treating the three-body singularities are given in the Appendices.

II. FADDEEV EQUATIONS FOR THE $nn\alpha$ SYSTEM

In developing the formal expression for the transition amplitude between three free particles interacting with short-range, strong interactions, we start from the triad of Lippmann-Schwinger (LS) equations [44,45] acting on a three-particle initial state given by

$$\Phi_\alpha^{(+)} = |\mathbf{p}_\alpha\rangle^{(+)}|\mathbf{q}_\alpha\rangle, \tag{2.1}$$

where $|\mathbf{p}_\alpha\rangle^{(+)}$ is a two-body scattering state and the index $\alpha = 1, 2, 3$ indicates the three choices of pairs characterized by the third particle, the spectator. Furthermore, $V^\alpha = \sum_{\beta\neq\alpha} V_\beta$, where V_β ($\beta = 1, 2, 3$) are the pair forces. Three-body forces can, in principle, be incorporated in a straightforward fashion. However, we will only concentrate on two-body forces here. The triad of LS equations,

$$\Psi_0^{(+)} = \Phi_\alpha^{(+)} + G_\alpha V^\alpha \Psi_0^{(+)}, \tag{2.2}$$

define the scattering wave uniquely. The channel Green's function is given by $G_\alpha^{-1} = (E + i\varepsilon - H_0 - V_\alpha)^{-1}$. We use standard Jacobi momenta $\mathbf{p}_\alpha$ and $\mathbf{q}_\alpha$ and their quantum numbers as basis states.

By suitable multiplication of the three equations in the triad from the left by V_β one obtains the transition operators $U_{\alpha 0} \equiv (V_\beta + V_\gamma)\Psi_0^{(+)}$, with $\beta \neq \alpha, \gamma \neq \alpha$, which fulfill the set of equations

$$U_{\alpha 0} = \sum_{\beta\neq\alpha} t_\beta \Phi_0 + \sum_{\beta\neq\alpha} t_\beta G_0 U_{\beta 0}, \tag{2.3}$$

where $\Phi_0 = |\mathbf{p}\rangle|\mathbf{q}\rangle$ is the free three-particle state.

The three-body breakup operator is given by

$$U_{00} \equiv \sum_\gamma V_\gamma \Psi_0^{(+)}. \tag{2.4}$$

Again, from the triad follows

$$U_{00} = \sum_\gamma t_\gamma \Phi_0 + \sum_\gamma t_\gamma G_0 U_{\gamma 0}. \tag{2.5}$$

Iterating Eq. (2.3) one obtains the multiple scattering series

$$U_{00} = \sum_\gamma t_\gamma \Phi_0 + \sum_\gamma t_\gamma G_0 \sum_{\beta\neq\gamma} t_\beta \Phi_0 + \sum_\gamma t_\gamma G_0 \sum_{\beta\neq\gamma} t_\beta G_0 \sum_{\alpha\neq\beta} t_\alpha \Phi_0 + \cdots. \tag{2.6}$$

Instead of working with the coupled set of Eq. (2.3) and the relation of Eq. (2.5) for the three-body breakup operator, one can generate the multiple scattering series directly by decomposing U_{00} as

$$U_{00} \equiv \sum_\gamma U_\gamma, \tag{2.7}$$

and choosing U_γ to obey the coupled set of Faddeev equations

$$U_\gamma = t_\gamma + t_\gamma G_0 \sum_{\alpha\neq\gamma} U_\alpha. \tag{2.8}$$

Indeed, iterating Eq. (2.8) and inserting the result into Eq. (2.7) leads exactly to the multiple scattering series from above. Explicitly, we have a set of three coupled equations

$$\begin{aligned} U_1 &= t_1 + t_1 G_0 (U_2 + U_3), \\ U_2 &= t_2 + t_2 G_0 (U_3 + U_1), \\ U_3 &= t_3 + t_3 G_0 (U_1 + U_2). \end{aligned} \tag{2.9}$$

We also observe that comparing Eqs. (2.7) and (2.4) leads to

$$U_\gamma \equiv V_\gamma \Psi_0^{(+)}. \tag{2.10}$$

When considering the $n+n+\alpha$ system, we need to incorporate the identity of the two neutrons. Fixing arbitrarily the α particle as spectator and labelling it as "1" and the two neutrons as particles "2" and "3", the scattering wave function $\Psi_0^{(+)}$ must be antisymmetric under the exchange of particles "2" and "3". Thus, defining the transposition operator P_{23}, the scattering wave function must fulfill $P_{23}\Psi_0^{(+)} = -\Psi_0^{(+)}$. Using this in Eq. (2.10) leads to

$$U_3 = -P_{23} U_2. \tag{2.11}$$

Thus, for the $n+n+\alpha$ system we only have two coupled equations

$$\begin{aligned} U_1 &= t_1 + t_1 G_0 \,(1 - P_{23})\, U_2, \\ U_2 &= t_2 + t_2 G_0 \,(-P_{23} U_2 + U_1). \end{aligned} \tag{2.12}$$

More precisely, one has to apply the driving terms to the free state $\Phi_{0,a}$, which is antisymmetric under the exchange of the two neutrons

$$\Phi_{0,a} \equiv (1-P_{23})|\mathbf{p}_1\mathbf{q}_1\rangle|0m_2m_3\rangle\big|0\tfrac{1}{2}\tfrac{1}{2}\big\rangle, \tag{2.13}$$

leading to

$$\begin{aligned} U_1\Phi_{0,a} &= t_1\Phi_{0,a} + t_1G_0(1-P_{23})U_2\Phi_{0,a},\\ U_2\Phi_{0,a} &= t_2\Phi_{0,a} + t_2G_0(-P_{23}U_2\Phi_{0,a} + U_1\Phi_{0,a}). \end{aligned} \tag{2.14}$$

The full breakup operator is given by

$$U_{00}\Phi_{0,a} = U_1\Phi_{0,a} + (1-P_{23})U_2\Phi_{0,a}. \tag{2.15}$$

For the on-shell breakup amplitude one has to evaluate the matrix element $\langle\Phi'_{0,a}|U_{00,a}|\Phi_{0,a}\rangle$, where in the final-state momenta as well as spin magnetic quantum numbers are changed

$$\Phi'_{0,a} = (1-P_{23})|\mathbf{p}'_1\mathbf{q}'_1\rangle|0m'_2m'_3\rangle\big|0\tfrac{1}{2}\tfrac{1}{2}\big\rangle. \tag{2.16}$$

III. PARTIAL WAVE DECOMPOSITION

To solve the coupled equations, Eq. (2.14), two sets of partial wave basis states are needed

$$\begin{aligned} |p_1q_1\alpha_1\rangle &\equiv \sum_{\mu_1} C(j_1\lambda_1J,\mu_1M_1-\mu_1)|p_1(l_1s_1)j_1\mu_1\rangle\\ &\quad\times|q_1\lambda_1M_1-\mu_1\rangle\left|\left(\frac{1}{2}\frac{1}{2}\right)1\right\rangle,\\ |p_2q_2\alpha_2\rangle &\equiv \sum_{\mu_2} C(j_2I_2J,\mu_2M_2-\mu_2)|p_2(l_2s_2)j_2\mu_2\rangle\\ &\quad\times\left|q_2\left(\lambda_2\frac{1}{2}\right)I_2M_2-\mu_2\right\rangle\left|\left(\frac{1}{2}\frac{1}{2}\right)1\right\rangle. \end{aligned} \tag{3.1}$$

The details of a partial wave decomposition of Eq. (2.14) is well known (see, e.g., Ref. [39]), and we refer to Ref. [46] for details. Employing the states of Eq. (3.1), the coupled equations, Eq. (2.14) read

$$\begin{aligned} \langle p'_1q'_1\alpha'_1|U_{1,a} &= \frac{\delta(q'_1-q_1)}{q_1^2}\,t_{\alpha'_1}(p'_1p_1,E_{q_1})\,C_{\alpha'_1}^{m_2+m_3}(\theta_1)\\ &\quad+\big(1+(-1)^{l'_1+s'_1}\big)\int dx\int dq'_2q'^2_2\\ &\quad\times t_{\alpha'_1}(p'_1\pi_1(q'_1q'_2x),E_{q'_1})\,G_0(\pi_1(q'_1q'_2x),q'_1)\\ &\quad\times\sum_{\alpha'_2}G_{\alpha'_1\alpha'_2}(q'_1q'_2x)\langle\pi_2(q'_1q'_2x)q'_2\alpha'_2|U_{2,a} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \langle p'_2q'_2\alpha'_2|U_{2,a} &= \frac{\delta(q'_2-q_2)}{q_2^2}\,t_{\alpha'_2}(p'_2p_2,E_{q_2})\,D_{\alpha'_2}^{m_2,m_3}(\theta_1)\\ &\quad-\frac{\delta(q'_2-\tilde{q}_2)}{\tilde{q}_2^2}\,t_{\alpha'_2}(p'_2\tilde{p}_2,E_{\tilde{q}_2})\,\tilde{D}_{\alpha'_2}^{m_2,m_3}(\theta_1)\\ &\quad+\int dx\int dq'_1q'^2_1\,t_{\alpha'_2}(p'_2\pi_3(q'_2q'_1x),E_{q'_2})\\ &\quad\times G_0(\pi_3(q'_2q'_1x),q'_2)\\ &\quad\times\sum_{\alpha'_1}H_{\alpha'_2\alpha'_1}(q'_2q'_1x)\langle\pi_4(q'_2q'_1x)q'_1\alpha'_1|U_{1,a}\\ &\quad-\int dx\int dq'''_2q'''^2_2\,t_{\alpha'_2}(p'_2\pi_5(q'_2q'''_2x),E_{q'_2})\\ &\quad\times G_0(\pi_5(q'_2q'''_2x),q'_2)\\ &\quad\times\sum_{\alpha'''_2}I_{\alpha'_2\alpha'''_2}(q'_2q'''_2x)\langle\pi_6(q'_2q'''_2x)q'''_2\alpha'''_2|U_{2,a}, \end{aligned}$$

where

$$\begin{aligned} C_{\alpha_1}^{m_2+m_3}(\theta_1) &= (1+(-)^{l_1+s_1})\left(\frac{1}{2}\frac{1}{2}s_1,m_2m_3\right)\sum_{m_{l_1}}(l_1s_1j_1,m_{l_1},m_2+m_3)\\ &\quad\times(j_1\lambda_1J,m_{l_1}+m_2+m_3,0,M)Y_{l_1m_{l_1}}(\theta_1,0)\sqrt{\frac{\hat{\lambda}_1}{4\pi}}, \end{aligned} \tag{3.3}$$

with $\hat{\lambda}_1 = 2\lambda_1+1$.

$$\begin{aligned} D_{\alpha'_2}^{m_2,m_3}(\theta_1) &\equiv D_{\alpha'_2}^{m_2,m_3}(\theta_{p_2}\theta_{q_2})\\ &= \delta_{s'_2\frac{1}{2}}\sum_{\mu}(j'_2I'_2J',\mu,M'-\mu)\\ &\quad\times\left(l'_2\frac{1}{2}j'_2,\mu-m_3,m_3\right)\,Y^*_{l'_2\mu-m_3}(\hat{p}_2)\\ &\quad\times\left(\lambda'_2\frac{1}{2}I'_2,M'-\mu-m_2,m_2\right)Y^*_{\lambda'_2M'-\mu-m_2}(\hat{q}_2), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \tilde{D}_{\alpha'_2}^{m_2,m_3}(\theta_1) &\equiv \tilde{D}_{\alpha'_2}^{m_2,m_3}(\theta_{p_2}\theta_{q_2})\\ &= \delta_{s'_2\frac{1}{2}}\sum_{\mu}(j'_2I'_2J',\mu M'-\mu)\\ &\quad\times\left(l'_2\frac{1}{2}j'_2,\mu-m_2,m_2\right)\,Y^*_{l'_2\mu-m_2}(\hat{\tilde{p}}_2)\\ &\quad\times\left(\lambda'_2\frac{1}{2}I'_2,M'-\mu-m_3,m_3\right)\,Y^*_{\lambda'_2M'-\mu-m_3}(\hat{\tilde{q}}_2). \end{aligned} \tag{3.5}$$

The "shifted" momenta π_i are given as

$$\begin{aligned} \pi_1 &= \sqrt{\alpha^2q'^2_1+q'^2_2+2\alpha q'_1q'_2x},\\ \pi_2 &= \sqrt{q'^2_1+\beta^2q'^2_2+2\beta q'_1q'_2x},\\ \pi_3 &= \sqrt{q'^2_2+\beta^2q'^2_1+2\beta q'_2q'_1x},\\ \pi_4 &= \sqrt{\alpha^2q'^2_2+q'^2_1+2\alpha q'_2q'_1x},\\ \pi_5 &= \sqrt{\overline{\beta}^2q'^2_2+q'''^2_2+2\overline{\beta}q'_2q'''_2x},\\ \pi_6 &= \sqrt{q'^2_2+\overline{\beta}^2q'''^2_2+2\overline{\beta}q'_2q_2'''x}, \end{aligned} \tag{3.6}$$

where

$$\alpha = \frac{1}{2}, \quad \beta = \frac{m_\alpha}{m + m_\alpha}, \qquad \gamma = \frac{2m + m_\alpha}{2(m + m_\alpha)}, \quad \bar{\beta} = \frac{m}{m + m_\alpha}. \tag{3.7}$$

Here m is the neutron mass and m_α the mass of the ^{4}He nucleus.

We refer to Appendix A for some details of the derivation and the expressions of the purely geometric quantities $G_{\alpha_1'\alpha_2'}(q_1'q_2'x)$, $H_{\alpha_2'\alpha_1'}(q_2'q_1'x)$, and $I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)$. Furthermore, $t_{\alpha_1'}(p_1'p_1, E_{q_1})$ is the two-neutron t matrix and $t_{\alpha_2'}(p_2'p_2, E_{q_2})$ the one for the neutron-α pair.

Due to the free Green's functions G_0 and the x integration over it, one encounters the well-known logarithmic singularities of any three-body problem. These singularities can be reliably treated [39,47]. However, the method suggested in Refs. [48,49] appears to be beneficial here since not only kernels contain logarithmic singularities, but also the driving terms. We illustrate this new method with an example in Appendix B.

IV. MULTIPLE SCATTERING SERIES

A well-established way to solve a coupled set of Faddeev equations is to generate the multiple scattering series. For the 3N system it is laid out in Ref. [39]. Schematically Eq. (3.2) has the form

$$U = U^{(0)} + KU, \tag{4.1}$$

which, when iterated, yield

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots, \tag{4.2}$$

with

$$U^{(n)} = KU^{(n-1)}, \quad n = 1, 2, \ldots. \tag{4.3}$$

The first few terms of this series are depicted in Fig. 1. The driving terms of Eq. (3.2), sketched in the upper row of Fig. 1 are necessarily disconnected since a two-body t matrix cannot act on three particles.

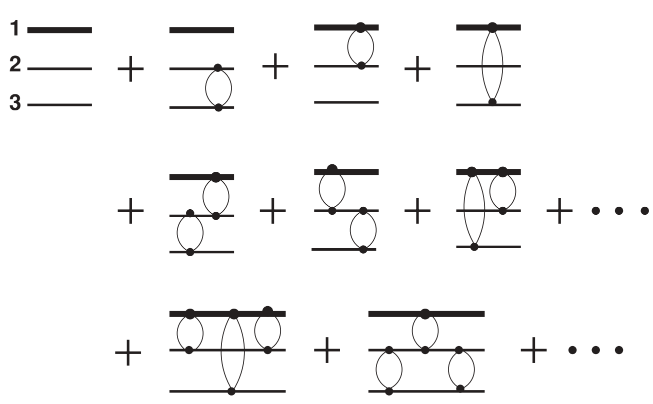

FIG. 1. Diagrammatic representation of the first few terms of the multiple scattering series for the neutron-neutron-α system. Here the alpha particle (1) is indicated by the thicker line.

Let us consider the terms of second order in the two-body t matrix (indicated in the second row of Fig. 1)

$$\begin{aligned}
\langle p_1'q_1'\alpha_1'|U_{1,a}^{(1)} &\equiv (1+(-1)^{l_1'+s_1'}) \int dx \int dq_2' q_2'^2\, t_{\alpha_1'}(p_1'\pi_1(q_1'q_2'x), E_{q_1'}) \\
&\quad \times G_0(\pi_1(q_1'q_2'x), q_1') \sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x) \\
&\quad \times \left[\frac{\delta(q_2'-q_2)}{q_2^2}\, t_{\alpha_2'}(\pi_2(q_1'q_2'x)p_2, E_{q_2})\, D_{\alpha_2'}^{m_2,m_3}(\theta_1) \right. \\
&\quad \left. - \frac{\delta(q_2'-\tilde{q}_2)}{\tilde{q}_2^2}\, t_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2x)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}_{\alpha_2'}^{m_2,m_3}(\theta_1) \right] \\
&= (1+(-1)^{l_1'+s_1'}) \int dx \sum_{\alpha_2'} [t_{\alpha_1'}(p_1'\pi_1(q_1'q_2x), E_{q_1'}) \\
&\quad \times G_0(\pi_1(q_1'q_2x), q_1') G_{\alpha_1'\alpha_2'}(q_1'q_2x)\, t_{\alpha_2'}(\pi_2(q_1'q_2x)p_2, E_{q_2}) \\
&\quad \times D_{\alpha_2'}^{m_2,m_3}(\theta_1) - t_{\alpha_1'}(p_1'\pi_1(q_1'\tilde{q}_2x), E_{q_1'})\, G_0(\pi_1(q_1'\tilde{q}_2x), q_1') \\
&\quad \times G_{\alpha_1'\alpha_2'}(q_1'\tilde{q}_2x) t_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2x)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}_{\alpha_2'}^{m_2,m_3}(\theta_1)].
\end{aligned} \tag{4.4}$$

The only singular function under the x integral is the free Green's function, which leads in the $q_1' - q_2$ and $q_1' - \tilde{q}_2$ planes of external momenta to the well-known logarithmic singularities. The same is true for

$$\begin{aligned}
\langle p_2'q_2'\alpha_2'|U_{2,a}^{(1)} &\equiv \int dx \int dq_1' q_1'^2 t_{\alpha_2'}(p_2'\pi_3(q_2'q_1'x), E_{q_2'})\, G_0(\pi_3(q_2'q_1'x), q_2') \\
&\quad \times \sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\, \frac{\delta(q_1'-q_1)}{q_1^2}\, t_{\alpha_1'}(\pi_4(q_2'q_1'x)p_1, E_{q_1}) \\
&\quad \times C_{\alpha_1'}^{m_2+m_3}(\theta_1) - \int dx \int dq_2''' q_2'''^2\, t_{\alpha_2'}(p_2'\pi_5(q_2'q_2'''x), E_{q_2'}) \\
&\quad \times G_0(\pi_5(q_2'q_2'''x), q_2') \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x) \\
&\quad \times \left[\frac{\delta(q_2'''-q_2)}{q_2^2}\, t_{\alpha_2'''}(\pi_6(q_2'q_2'''x)p_2, E_{q_2})\, D_{\alpha_2'''}^{m_2,m_3}(\theta_1) \right. \\
&\quad \left. - \frac{\delta(q_2'''-\tilde{q}_2)}{\tilde{q}_2^2}\, t_{\alpha_2'''}(\pi_6(q_2'q_2'''x)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}_{\alpha_2'''}^{m_2,m_3}(\theta_1) \right] \\
&= \int dx\, t_{\alpha_2'}(p_2'\pi_3(q_2'q_1x), E_{q_2'})\, G_0(\pi_3(q_2'q_1x), q_2') \\
&\quad \times \sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1x)\, t_{\alpha_1'}(\pi_4(q_2'q_1x)p_1, E_{q_1})\, C_{\alpha_1'}^{m_2+m_3}(\theta_1) \\
&\quad - \int dx \left[t_{\alpha_2'}(p_2'\pi_5(q_2'q_2x), E_{q_2'})\, G_0(\pi_5(q_2'q_2x), q_2') \right. \\
&\quad \times \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2x)\, t_{\alpha_2'''}(\pi_6(q_2'q_2x)p_2, E_{q_2})\, D_{\alpha_2'''}^{m_2,m_3}(\theta_1)
\end{aligned}$$

$$- t_{\alpha_2'}(p_2'\pi_5(q_2'\tilde{q}_2x), E_{q_2'})\, G_0(\pi_5(q_2'\tilde{q}_2x), q_2') \times \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'\tilde{q}_2x)\, t_{\alpha_2'''}(\pi_6(q_2'\tilde{q}_2x)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}^{m_2,m_3}_{\alpha_2'''}(\theta_1)\Bigg]. \quad (4.5)$$

Indeed, the free Green's functions G_0 lead in the only remaining x integral to logarithmic singularities.

To safely apply the kernel to the previous amplitude that amplitude has to be a smooth function. This is only the case for the next higher order, being of third order in t, sketched in the third row of Fig. 1. The third-order term reads

$$\begin{aligned}
\langle p_1'q_1'\alpha_1'|U^{(2)}_{1,a} \equiv{}& (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2\, t_{\alpha_1'}(p_1'\pi_1(q_1'q_2'x), E_{q_1'})G_0(\pi_1(q_1'q_2'x), q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\\
&\times\Bigg[\int dy\, t_{\alpha_2'}(\pi_2(q_1'q_2'x)\pi_3(q_2'q_1y), E_{q_2'})G_0(\pi_3(q_2'q_1y), q_2')\sum_{\alpha_1''} H_{\alpha_2'\alpha_1''}(q_2'q_1y)t_{\alpha_1''}(\pi_4(q_2'q_1y)p_1, E_{q_1})\, C^{m_2+m_3}_{\alpha_1''}(\theta_1)\\
&-\int dy\,\Bigg[t_{\alpha_2'}(\pi_2(q_1'q_2'x)\pi_5(q_2'q_2y), E_{q_2'})\, G_0(\pi_5(q_2'q_2y), q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2y)\, t_{\alpha_2'''}(\pi_6(q_2'q_2y)p_2, E_{q_2})\, D^{m_2,m_3}_{\alpha_2'''}(\theta_1)\\
&- t_{\alpha_2'}(\pi_2(q_1'q_2'x)\pi_5(q_2'\tilde{q}_2y), E_{q_2'})\, G_0(\pi_5(q_2'\tilde{q}_2y), q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'\tilde{q}_2y)\, t_{\alpha_2'''}(\pi_6(q_2'\tilde{q}_2y)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}^{m_2,m_3}_{\alpha_2'''}(\theta_1)\Bigg]\Bigg].
\end{aligned} \quad (4.6)$$

Correspondingly one obtains

$$\begin{aligned}
\langle p_2'q_2'\alpha_2'|U^{(2)}_{2,a} \equiv{}& \sum_{\alpha_1'}\int dx \int dq_1' q_1'^2\, t_{\alpha_2'}(p_2'\pi_3(q_2'q_1'x), E_{q_2'})G_0(\pi_3(q_2'q_1'x), q_2')H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\,(1+(-1)^{l_1'+s_1'})\\
&\times\int dy\,\Bigg[t_{\alpha_1'}(\pi_4(q_2'q_1'x)\pi_1(q_1'q_2y), E_{q_1'})G_0(\pi_1(q_1'q_2y), q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2y)\, t_{\alpha_2'}(\pi_2(q_1'q_2y)p_2, E_{q_2})\, D^{m_2,m_3}_{\alpha_2'}(\theta_1)\\
&- t_{\alpha_1'}(\pi_4(q_2'q_1'x)\pi_1(q_1'\tilde{q}_2y), E_{q_1'})\, G_0(\pi_1(q_1'\tilde{q}_2y), q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'\tilde{q}_2y)\, t_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2y)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_1)\Bigg]\\
&-\int dx\int dq_2'''(q_2''')^2\, t_{\alpha_2'}(p_2'\pi_5(q_2'q_2'''x), E_{q_2'})\, G_0(\pi_5(q_2'q_2'''x), q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\\
&\times\Bigg[\int dy\, t_{\alpha_2'''}(\pi_6(q_2'q_2'''x)\pi_3(q_2'''q_1y), E_{q_2'})\, G_0(\pi_3(q_2'''q_1y), q_2''')\sum_{\alpha_1'} H_{\alpha_2'''\alpha_1'}(q_2'''q_1y)\, t_{\alpha_1'}(\pi_4(q_2'''q_1y)p_1, E_{q_1})\\
&\times C^{m_2+m_3}_{\alpha_1'}(\theta_1) - \int dy\,\Bigg[t_{\alpha_2'''}(\pi_6(q_2'q_2'''x)\pi_5(q_2'''q_2y), E_{q_2'''})\, G_0(\pi_5(q_2'''q_2y), q_2''')\sum_{\alpha_2''''} I_{\alpha_2'''\alpha_2''''}(q_2'''q_2y)\\
&\times t_{\alpha_2''''}(\pi_6(q_2'''q_2y)p_2, E_{q_2})\, D^{m_2,m_3}_{\alpha_2''''}(\theta_1) - t_{\alpha_2'''}(\pi_6(q_2'q_2'''x)\pi_5(q_2'''\tilde{q}_2y), E_{q_2'''})\, G_0(\pi_5(q_2'''\tilde{q}_2y), q_2''')\\
&\times\sum_{\alpha_2''''} I_{\alpha_2'''\alpha_2''''}(q_2'''\tilde{q}_2y)\, t_{\alpha_2''''}(\pi_6(q_2'''\tilde{q}_2y)\tilde{p}_2, E_{\tilde{q}_2})\, \tilde{D}^{m_2,m_3}_{\alpha_2''''}(\theta_1)\Bigg]\Bigg].
\end{aligned} \quad (4.7)$$

All three-fold integrals in Eqs. (4.6) and (4.7) are of the same type: two angular integrations, where each one leads to logarithmic singularities in the corresponding momenta, one of which is external and the other the intermediate integration variable. It is not difficult to see that the intermediate momentum integration over products of logarithms leads to smooth functions in the external momenta. Therefore the third-order amplitudes in t can serve as driving terms for the application of the kernels, and thus leading to all higher-order amplitudes

$$\langle p_1'q_1'\alpha_1'|U_{1,a}^{(n)} = (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \times t_{\alpha_1}(p_1'\pi_1(q_1'q_2'x), E_{q_1'})G_0(\pi_1(q_1'q_2'x), q_1') \times \sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\ \langle \pi_2(q_1'q_2'x)q_2'\alpha_2'|U_{2,a}^{(n-1)}$$

$$\langle p_2'q_2'\alpha_2'|U_{2,a}^{(n)} = \int dx \int dq_1' q_1'^2\ t_{\alpha_2'}(p_2'\pi_3(q_2'q_1'x), E_{q_2'}) \times G_0(\pi_3(q_2'q_1'x), q_2') \sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x) \times \langle \pi_4(q_2'q_1'x)q_1'\alpha_1'|U_{1,a}^{(n-1)} - \int dx \int dq_2''' q_2'''^2 \times t_{\alpha_2'}(p_2'\pi_5(q_2'q_2'''x), E_{q_2'})\ G_0(\pi_5(q_2'q_2'''x), q_2') \times \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\ \langle \pi_6(q_2'q_2'''x)q_2'''\alpha_2'''|U_{2,a}^{(n-1)}, \tag{4.8}$$

with $n = 3, 4, \ldots$.

The resulting series $\sum_{n=3}^{\infty}\langle p_1'q_1'\alpha_1'|U_{1,a}^{(n)}$ and $\sum_{n=3}^{\infty}\langle p_2'q_2'\alpha_2'|U_{2,a}^{(n)}$ can safely be summed via Padé summation. For the corresponding three-nucleon amplitudes the previous considerations were made in Ref. [38].

V. FINITE RANK FORCES

So far, Faddeev-type studies of light nuclei treating the discrete structures were based on finite rank forces [33]. Therefore, it appears useful to also formulate the $nn\alpha$ system in the continuum in this fashion. For the sake of a simple notation we choose a rank-1 separable t matrix,

$$t_\alpha(pp', E_q) = h_\alpha(p)\tau_\alpha(q)h_\alpha(p'). \tag{5.1}$$

Then Eq. (3.2) takes the form

$$\langle p_1'q_1'\alpha_1'|U_{1,a} = \frac{\delta(q_1'-q_1)}{q_1^2}\ h_{\alpha_1'}(p_1')\tau_{\alpha_1'}(E_{q_1})h_{\alpha_1'}(p_1)\ C_{\alpha_1'}^{m_2+m_3}(\theta_1) + (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} h_{\alpha_1'}(p_1') \times \tau_{\alpha_1'}(E_{q_1'})h_{\alpha_1'}(\pi_1(q_1'q_2'x))G_0(\pi_1(q_1'q_2'x), q_1') \times G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\ \langle \pi_2(q_1'q_2'x)q_2'\alpha_2'|U_{2,a} \equiv h_{\alpha_1'}(p_1')Z_{\alpha_1'}(q_1'), \tag{5.2}$$

where the new unknown single variable amplitude is

$$Z_{\alpha_1'}(q_1') = \frac{\delta(q_1'-q_1)}{q_1^2}\ \tau_{\alpha_1'}(E_{q_1})h_{\alpha_1'}(p_1)\ C_{\alpha_1'}^{m_2+m_3}(\theta_1) + (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} \tau_{\alpha_1'}(E_{q_1'}) \times h_{\alpha_1'}(\pi_1(q_1'q_2'x))G_0(\pi_1(q_1'q_2'x), q_1') \times G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\ \langle \pi_2(q_1'q_2'x)q_2'\alpha_2'|U_{2,a}. \tag{5.3}$$

Similarly, the second equation, Eq. (3.2), becomes

$$\langle p_2'q_2'\alpha_2'|U_{2,a} = \frac{\delta(q_2'-q_2)}{q_2^2}\ h_{\alpha_2'}(p_2')\tau_{\alpha_2'}(E_{q_2})h_{\alpha_2'}(p_2)\ D_{\alpha_2'}^{m_2,m_3}(\theta_1) + \sum_{\alpha_1'}\int dx \int dq_1' q_1'^2\ h_{\alpha_2'}(p_2')\tau_{\alpha_2'}(E_{q_2'})h_{\alpha_2'}(\pi_3(q_2'q_1'x)) \times G_0(\pi_3(q_2'q_1'x), q_2')\ H_{\alpha_2'\alpha_1'}(q_2'q_1'x) \times \langle \pi_4(q_2'q_1'x)q_1'\alpha_1'|U_{1,a} - \sum_{\alpha_2'''}\int dx \int dq_2''' q_2'''^2 \times h_{\alpha_2'}(p_2')\tau_{\alpha_2'}(E_{q_2'})h_{\alpha_2'}(\pi_5(q_2'q_2'''x))G_0(\pi_5(q_2'q_2'''x), q_2') \times I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\ \langle \pi_6(q_2'q_2'''x)\, q_2'''\alpha_2'''|U_{2,a} \equiv h_{\alpha_2'}(p_2')V_{\alpha_2'}(q_2'), \tag{5.4}$$

with

$$V_{\alpha_2'}(q_2') = \frac{\delta(q_2'-q_2)}{q_2^2}\tau_{\alpha_2'}(E_{q_2})h_{\alpha_2'}(p_2)\ D_{\alpha_2'}^{m_2,m_3}(\theta_1) + \sum_{\alpha_1'}\int dx \int dq_1' q_1'^2\ \tau_{\alpha_2'}(E_{q_2'})h_{\alpha_2'}(\pi_3(q_2'q_1'x)) \times G_0(\pi_3(q_2'q_1'x), q_2')\ H_{\alpha_2'\alpha_1'}(q_2'q_1'x) \times \langle \pi_4(q_2'q_1'x)q_1'\alpha_1'|U_{1,a} - \sum_{\alpha_2'''}\int dx \int dq_2''' q_2'''^2 \times \tau_{\alpha_2'}(E_{q_2'})h_{\alpha_2'}(\pi_5(q_2'q_2'''x))G_0(\pi_5(q_2'q_2'''x), q_2') \times I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\ \langle \pi_6(q_2'q_2'''x)q_2'''\alpha_2'''|U_{2,a}. \tag{5.5}$$

Then we insert the functions Z and V under the integrals

$$Z_{\alpha_1'}(q_1') = \frac{\delta(q_1'-q_1)}{q_1^2}\tau_{\alpha_1'}(q_1)h_{\alpha_1'}(p_1)C_{\alpha_1'}^{m_2+m_3}(\theta_1) + (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \times \sum_{\alpha_2'} \tau_{\alpha_1'}(E_{q_1'})h_{\alpha_1'}(\pi_1(q_1'q_2'x))G_0(\pi_1(q_1'q_2'x), q_1') \times G_{\alpha_1'\alpha_2'}(q_1'q_2'x)h_{\alpha_2'}(\pi_2(q_1'q_2'x))V_{\alpha_2'}(q_2'), \tag{5.6}$$

and

$$V_{\alpha_2}(q_2') = \frac{\delta(q_2'-q_2)}{q_2^2}\tau_{\alpha_2'}(q_2)h_{\alpha_2'}(p_2)\ D_{\alpha_2'}^{m_2,m_3}(\theta_1) + \sum_{\alpha_1'}\int dx \int dq_1' q_1'^2\ \tau_{\alpha_2'}(E_{q_2'})h_{\alpha_2'}(\pi_3(q_2'q_1'x)) \times G_0(\pi_3(q_2'q_1'x), q_2')\ H_{\alpha_2'\alpha_1'}(q_2'q_1'x)h_{\alpha_1'}(\pi_4(q_2'q_1'x)) \times Z_{\alpha_1'}(q_1') - \sum_{\alpha_2'''}\int dx \int dq_2''' q_2'''^2\ \tau_{\alpha_2'}(E_{q_2'}) \times h_{\alpha_2'}(\pi_5(q_2'q_2'''x))G_0(\pi_5(q_2'q_2'''x), q_2')\ I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x) \times h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))V_{\alpha_2'''}(q_2'''). \tag{5.7}$$

These two equations, Eqs. (5.6) and (5.7), form a set of coupled one-dimensional integral equations. The low-order iterations exhibit the same features as discussed in detail in the previous section. Thus, we write

$$Z_{\alpha_1'}(q_1') = Z^{(0)}_{\alpha_1'}(q_1') + Z^{(1)}_{\alpha_1'}(q_1') + Z^{(2)}_{\alpha_1'}(q_1') + Z^{(3)}_{\alpha_1'}(q_1') + \cdots, \tag{5.8}$$

and

$$V_{\alpha_2'}(q_2') = V^{(0)}_{\alpha_2'}(q_2') + V^{(1)}_{\alpha_2'}(q_2') + V^{(2)}_{\alpha_2'}(q_2') + V^{(3)}_{\alpha_2'}(q_2') + \cdots. \tag{5.9}$$

From Eqs. (5.6) and (5.7) we can read off the different orders. For the lowest order we obtain

$$Z^{(0)}_{\alpha_1'}(q_1') = \frac{\delta(q_1'-q_1)}{q_1^2}\,\tau_{\alpha_1'}(q_1)\,h_{\alpha_1'}(p_1)\,C^{m_2+m_3}_{\alpha_1'}(\theta_1), \tag{5.10}$$

and

$$V^{(0)}_{\alpha_2'}(q_2') = \frac{\delta(q_2'-q_2)}{q_2^2}\,\tau_{\alpha_2'}(q_2)\,h_{\alpha_2'}(p_2)\,D^{m_2,m_3}_{\alpha_2'}(\theta_1) - \frac{\delta(q_2'-\tilde{q}_2)}{\tilde{q}_2^2}\,\tau_{\alpha_2'}(\tilde{q}_2)\,h_{\alpha_2'}(\tilde{p}_2\,)\tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_1). \tag{5.11}$$

The second order is given by

$$\begin{aligned}
Z^{(1)}_{\alpha_1'}(q_1') &= (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} \tau_{\alpha_1'}(E_{q_1'})\,h_{\alpha_1'}(\pi_1(q_1'q_2'x))\,G_0(\pi_1(q_1'q_2'x),q_1')G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\,h_{\alpha_2'}(\pi_2(q_1'q_2'x))\,V^{(0)}_{\alpha_2'}(q_2')\\
&= (1+(-1)^{l_1'+s_1'})\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} \tau_{\alpha_1'}(E_{q_1'})\,h_{\alpha_1'}(\pi_1(q_1'q_2'x))\,G_0(\pi_1(q_1'q_2'x),q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\,h_{\alpha_2'}(\pi_2(q_1'q_2'x))\\
&\quad\times\left[\frac{\delta(q_2'-q_2)}{q_2^2}\,\tau_{\alpha_2'}(q_2)\,h_{\alpha_2'}(p_2)\,D^{m_2,m_3}_{\alpha_2'}(\theta_1) - \frac{\delta(q_2'-\tilde{q}_2)}{\tilde{q}_2^2}\,\tau_{\alpha_2'}(\tilde{q}_2)\,h_{\alpha_2'}(\tilde{p}_2)\,\tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_1)\right]\\
&= (1+(-1)^{l_1'+s_1'})\,\tau_{\alpha_1'}(q_1')\int dx\,\Bigg[h_{\alpha_1'}(\pi_1(q_1'q_2x))\,G_0(\pi_1(q_1'q_2x),q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2x)\,h_{\alpha_2'}(\pi_2(q_1'q_2x))\,\tau_{\alpha_2'}(q_2)\,h_{\alpha_2'}(p_2)\\
&\quad\times D^{m_2,m_3}_{\alpha_2'}(\theta_1) - h_{\alpha_1'}(\pi_1(q_1'\tilde{q}_2x))\,G_0(\pi_1(q_1'\tilde{q}_2x),q_1')\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'\tilde{q}_2x)\,h_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2x))\,\tau_{\alpha_2'}(\tilde{q}_2)\,h_{\alpha_2'}(\tilde{p}_2)\,\tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_1)\Bigg]\\
&= (1+(-1)^{l_1'+s_1'})\,\tau_{\alpha_1'}(q_1')\sum_{\alpha_2'}\Bigg[\tau_{\alpha_2'}(q_2)\,h_{\alpha_2'}(p_2)\,D^{m_2,m_3}_{\alpha_2'}(\theta_1)\int dx\,h_{\alpha_1'}(\pi_1(q_1'q_2x))\,G_0(\pi_1(q_1'q_2x),q_1')\,G_{\alpha_1'\alpha_2'}(q_1'q_2x)\\
&\quad\times h_{\alpha_2'}(\pi_2(q_1'q_2x)) - \tau_{\alpha_2'}(\tilde{q}_2)\,h_{\alpha_2'}(\tilde{p}_2)\,\tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_1)\int dx\,h_{\alpha_1'}(\pi_1(q_1'\tilde{q}_2x))\,G_0(\pi_1(q_1'\tilde{q}_2x),q_1')\,G_{\alpha_1'\alpha_2'}(q_1'\tilde{q}_2x)\,h_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2x))\Bigg],
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
V^{(1)}_{\alpha_2'}(q_2') &= \int dx \int dq_1' q_1'^2\,\tau_{\alpha_2'}(q_2')\,h_{\alpha_2'}(\pi_3(q_2'q_1'x))\,G_0(\pi_3(q_2'q_1'x),q_2')\sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\,h_{\alpha_1'}(\pi_4(q_2'q_1'x))\,Z^{(0)}_{\alpha_1'}(q_1')\\
&\quad-\int dx\int dq_2''' q_2'''^2\,\tau_{\alpha_2'}(q_2')\,h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\,G_0(\pi_5(q_2'q_2'''x),q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\,h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))\,V^{(0)}_{\alpha_2'''}(q_2''')\\
&= \int dx\int dq_1' q_1'^2\,\tau_{\alpha_2'}(q_2')\,h_{\alpha_2'}(\pi_3(q_2'q_1'x))G_0(\pi_3(q_2'q_1'x),q_2')\sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)h_{\alpha_1'}(\pi_4(q_2'q_1'x))\,\frac{\delta(q_1'-q_1)}{q_1^2}\,\tau_{\alpha_1'}(q_1)h_{\alpha_1'}(p_1)\\
&\quad\times C^{m_2+m_3}_{\alpha_1'}(\theta_1) - \int dx\int dq_2''' q_2'''^2\,\tau_{\alpha_2'}(q_2')\,h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\,G_0(\pi_5(q_2'q_2'''x),q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\,h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))\\
&\quad\times\left[\frac{\delta(q_2'''-q_2)}{q_2^2}\,\tau_{\alpha_2'''}(q_2)\,h_{\alpha_2'''}(p_2)\,D^{m_2,m_3}_{\alpha_2'''}(\theta_1) - \frac{\delta(q_2'''-\tilde{q}_2)}{\tilde{q}_2^2}\,\tau_{\alpha_2'''}(\tilde{q}_2)\,h_{\alpha_2'''}(\tilde{p}_2)\,\tilde{D}^{m_2,m_3}_{\alpha_2'''}(\theta_1)\right]\\
&= \int dx\,\tau_{\alpha_2'}(q_2')\,h_{\alpha_2'}(\pi_3(q_2'q_1x))\,G_0(\pi_3(q_2'q_1x),q_2')\sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1x)\,h_{\alpha_1'}(\pi_4(q_2'q_1x))\,\tau_{\alpha_1'}(q_1)\,h_{\alpha_1'}(p_1)\,C^{m_2+m_3}_{\alpha_1'}(\theta_1)
\end{aligned}$$

$$- \int dx \left[\tau_{\alpha_2'}(q_2')\, h_{\alpha_2'}(\pi_5(q_2'q_2x))\, G_0(\pi_5(q_2'q_2x), q_2') \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2x)\, h_{\alpha_2'''}(\pi_6(q_2'q_2x))\, \tau_{\alpha_2'''}(q_2)\, h_{\alpha_2'''}(p_2)\, D_{\alpha_2'''}^{m_2,m_3}(\theta_1) \right.$$

$$\left. - \tau_{\alpha_2'}(q_2')\, h_{\alpha_2'}(\pi_5(q_2'\tilde{q}_2x))\, G_0(\pi_5(q_2'\tilde{q}_2x), q_2') I_{\alpha_2'\alpha_2'''}(q_2'\tilde{q}_2x)\, h_{\alpha_2'''}(\pi_6(q_2'\tilde{q}_2x))\, \tau_{\alpha_2'''}(\tilde{q}_2)\, h_{\alpha_2'''}(p_2)\, \tilde{D}_{\alpha_2'''}^{m_2,m_3}(\theta_1) \right]$$

$$= \tau_{\alpha_2'}(q_2') \sum_{\alpha_1'} \tau_{\alpha_1'}(q_1)\, h_{\alpha_1'}(p_1)\, C_{\alpha_1'}^{m_2+m_3}(\theta_1) \int dx\, h_{\alpha_2'}(\pi_3(q_2'q_1x))\, G_0(\pi_3(q_2'q_1x), q_2')\, H_{\alpha_2'\alpha_1'}(q_2'q_1x)\, h_{\alpha_1'}(\pi_4(q_2'x))$$

$$- \tau_{\alpha_2'}(q_2') \sum_{\alpha_2'''} \left[\tau_{\alpha_2'''}(q_2)\, h_{\alpha_2'''}(p_2)\, D_{\alpha_2'''}^{m_2,m_3}(\theta_1) \int dx\, h_{\alpha_2'}(\pi_5(q_2'q_2x))\, G_0(\pi_5(q_2'q_2x), q_2')\, I_{\alpha_2'\alpha_2'''}(q_2'q_2x)\, h_{\alpha_2'''}(\pi_6(q_2'q_2x)) \right.$$

$$\left. - \tau_{\alpha_2'''}(\tilde{q}_2)\, h_{\alpha_2'''}(\tilde{p}_2)\, \tilde{D}_{\alpha_2'''}^{m_2,m_3}(\theta_1) \int dx\, h_{\alpha_2'}(\pi_5(q_2'\tilde{q}_2x))\, G_0(\pi_5(q_2'\tilde{q}_2x), q_2')\, I_{\alpha_2'\alpha_2'''}(q_2'\tilde{q}_2x) h_{\alpha_2'''}(\pi_6(q_2'\tilde{q}_2x)) \right]. \tag{5.13}$$

As for general forces, the x integration leads to logarithmic singularities in the external momenta. The next order, however, gives smooth functions

$$Z_{\alpha_1'}^{(2)}(q_1') = (1 + (-1)^{l_1'+s_1'})\, \tau_{\alpha_1'}(q_1') \int dx \int dq_2' q_2'^2\, h_{\alpha_1'}(\pi_1(q_1'q_2'x))\, G_0(\pi_1(q_1'q_2'x), q_1') \sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\, h_{\alpha_2'}(\pi_2(q_1'q_2'x))\, V_{\alpha_2'}^{(1)}(q_2')$$

$$= (1 + (-1)^{l_1'+s_1'})\, \tau_{\alpha_1'}(q_1') \int dq_2' q_2'^2 \int dx\, h_{\alpha_1'}(\pi_1(q_1'q_2'x))\, G_0(\pi_1(q_1'q_2'x), q_1') \sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\, h_{\alpha_2'}(\pi_2(q_1'q_2'x))\, \tau_{\alpha_2'}(q_2')$$

$$\times \left[\sum_{\alpha_1''} \tau_{\alpha_1''}(q_1)\, h_{\alpha_1''}(p_1)\, C_{\alpha_1''}^{m_2+m_3}(\theta_1) \int dy\, h_{\alpha_2'}(\pi_3(q_2'q_1y))\, G_0(\pi_3(q_2'q_1y), q_2')\, H_{\alpha_2'\alpha_1'}(q_2'q_1y)\, h_{\alpha_1'}(\pi_4(q_2'y)) \right.$$

$$- \sum_{\alpha_2'''} \left[\tau_{\alpha_2'''}(q_2)\, h_{\alpha_2'''}(p_2)\, D_{\alpha_2'''}^{m_2,m_3}(\theta_1) \int dy\, h_{\alpha_2'}(\pi_5(q_2'q_2y))\, G_0(\pi_5(q_2'q_2y), q_2')\, I_{\alpha_2'\alpha_2'''}(q_2'q_2y)\, h_{\alpha_2'''}(\pi_6(q_2'q_2y)) \right.$$

$$\left.\left. - \tau_{\alpha_2'''}(\tilde{q}_2)\, h_{\alpha_2'''}(\tilde{p}_2)\, \tilde{D}_{\alpha_2'''}^{m_2,m_3}(\theta_1) \int dy\, h_{\alpha_2'}(\pi_5(q_2'\tilde{q}_2y))\, G_0(\pi_5(q_2'\tilde{q}_2y), q_2')\, I_{\alpha_2'\alpha_2'''}(q_2'\tilde{q}_2y)\, h_{\alpha_2'''}(\pi_6(q_2'\tilde{q}_2y)) \right] \right], \tag{5.14}$$

and

$$V_{\alpha_2'}^{(2)}(q_2') = \int dx \int dq_1' q_1'^2\, \tau_{\alpha_2'}(q_2')\, h_{\alpha_2'}(\pi_3(q_2'q_1'x))\, G_0(\pi_3(q_2'q_1'x), q_2') \sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\, h_{\alpha_1'}(\pi_4(q_2'q_1'x))\, Z_{\alpha_1'}^{(1)}(q_1')$$

$$- \int dx \int dq_2''' q_2'''^2\, \tau_{\alpha_2'}(q_2')\, h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\, G_0(\pi_5(q_2'q_2'''x), q_2') \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\, h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))\, V_{\alpha_2'''}^{(1)}(q_2''')$$

$$= \tau_{\alpha_2'}(q_2') \int dq_1' q_1'^2 \int dx\, h_{\alpha_2'}(\pi_3(q_2'q_1'x))\, G_0(\pi_3(q_2'q_1'x), q_2') \sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\, h_{\alpha_1'}(\pi_4(q_2'q_1'x))(1 + (-1)^{l_1'+s_1'})\, \tau_{\alpha_1'}(q_1')$$

$$\times \sum_{\alpha_2''} \left[\tau_{\alpha_2''}(q_2)\, h_{\alpha_2''}(p_2)\, D_{\alpha_2''}^{m_2,m_3}(\theta_1) \int dy\, h_{\alpha_1'}(\pi_1(q_1'q_2y))\, G_0(\pi_1(q_1'q_2y), q_1')\, G_{\alpha_1'\alpha_2''}(q_1'q_2y)\, h_{\alpha_2''}(\pi_2(q_1'q_2y)) \right.$$

$$\left. - \tau_{\alpha_2''}(\tilde{q}_2)\, h_{\alpha_2''}(\tilde{p}_2)\, \tilde{D}_{\alpha_2''}^{m_2,m_3}(\theta_1) \int dy\, h_{\alpha_1'}(\pi_1(q_1'\tilde{q}_2y))\, G_0(\pi_1(q_1'\tilde{q}_2y), q_1')\, G_{\alpha_1'\alpha_2''}(q_1'\tilde{q}_2y)\, h_{\alpha_2''}(\pi_2(q_1'\tilde{q}_2y)) \right]$$

$$- \tau_{\alpha_2'}(q_2') \int dq_2''' q_2'''^2 \int dx\, h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\, G_0(\pi_5(q_2'q_2'''x), q_2') \sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\, h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))$$

$$\times \left[\tau_{\alpha_2'''}(q_2''') \sum_{\alpha_1'} \tau_{\alpha_1'}(q_1)\, h_{\alpha_1'}(p_1) C_{\alpha_1'}^{m_2+m_3}(\theta_1) \int dy\, h_{\alpha_2'''}(\pi_3(q_2'''q_1y))\, G_0(\pi_3(q_2'''q_1y), q_2''')\, H_{\alpha_2'''\alpha_2'}(q_2'''q_1y)\, h_{\alpha_1'}(\pi_4(q_2'''y)) \right.$$

$$- \tau_{\alpha_2'''}(q_2''') \sum_{\alpha_2''''} \left[\tau_{\alpha_2''''}(q_2)\, h_{\alpha_2''''}(p_2)\, D_{\alpha_2''''}^{m_2,m_3}(\theta_1) \int dy\, h_{\alpha_2'''}(\pi_5(q_2'''q_2y))\, G_0(\pi_5(q_2'''q_2y), q_2''')\, I_{\alpha_2'''\alpha_2''''}(q_2'''q_2y) \right.$$

$$\times\ h_{\alpha_2''''}(\pi_6(q_2'''q_2y)) - \tau_{\alpha_2''''}(\tilde{q}_2)\ h_{\alpha_2''''}(p_2)\ \tilde{D}^{m_2,m_3}_{\alpha_2''''}(\theta_1)\int dy\ h_{\alpha_2'''}(\pi_5(q_2'''\tilde{q}_2y))\ G_0(\pi_5(q_2'''\tilde{q}_2y), q_2''')$$
$$\times\ I_{\alpha_2'''\alpha_2''''}(q_2'''\tilde{q}_2y)h_{\alpha_2''''}(\pi_6(q_2'''\tilde{q}_2y))\Bigg]\Bigg]. \qquad (5.15)$$

As shown in the previous section, there appear three-fold integrals. Two of them are over angles, leading to logarithmic singularities, which are then eliminated by the third intermediate momentum integral.

Thus we end up starting with $n=3$

$$\begin{aligned}
Z^{(n)}_{\alpha_1'}(q_1') &= (1+(-1)^{l_1'+s_1'})\int dx\int dq_2'q_2'^2\ \tau_{\alpha_1'}(q_1')\\
&\quad\times h_{\alpha_1'}(\pi_1(q_1'q_2'x))\ G_0(\pi_1(q_1'q_2'x), q_1')\\
&\quad\times\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\ h_{\alpha_2'}(\pi_2(q_1'q_2'x))\ V^{(n-1)}_{\alpha_2'}(q_2')\\
&= (1+(-1)^{l_1'+s_1'})\ \tau_{\alpha_1'}(q_1')\int dq_2'q_2'^2\\
&\quad\times\int dx\ h_{\alpha_1'}(\pi_1(q_1'q_2'x))\ G_0(\pi_1(q_1'q_2'x), q_1')\\
&\quad\times\sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1'q_2'x)\ h_{\alpha_2'}(\pi_2(q_1'q_2'x))\ V^{(n-1)}_{\alpha_2'}(q_2').
\end{aligned} \qquad (5.16)$$

Correspondingly,

$$\begin{aligned}
V^{(n)}_{\alpha_2'}(q_2') &= \int dx\int dq_1'q_1'^2\ \tau_{\alpha_2'}(q_2')\ h_{\alpha_2'}(\pi_3(q_2'q_1'x))\\
&\quad\times G_0(\pi_3(q_2'q_1'x), q_2')\sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\\
&\quad\times h_{\alpha_1'}(\pi_4(q_2'q_1'x))\ Z^{(n-1)}_{\alpha_1'}(q_1')\\
&\quad-\int dx\int dq_2'''q_2'''^2\ \tau_{\alpha_2'}(q_2')\ h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\\
&\quad\times G_0(\pi_5(q_2'q_2'''x), q_2')\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\\
&\quad\times h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))\ V^{(n-1)}_{\alpha_2'''}(q_2''')\\
&= \tau_{\alpha_2'}(q_2')\int dq_1'q_1'^2\int dx\ h_{\alpha_2'}(\pi_3(q_2'q_1'x))\\
&\quad\times G_0(\pi_3(q_2'q_1'x), q_2')\sum_{\alpha_1'} H_{\alpha_2'\alpha_1'}(q_2'q_1'x)\\
&\quad\times h_{\alpha_1'}(\pi_4(q_2'q_1'x))\ Z^{(n-1)}_{\alpha_1'}(q_1') - \tau_{\alpha_2'}(q_2')\int dq_2'''q_2'''^2\\
&\quad\times\int dx\ h_{\alpha_2'}(\pi_5(q_2'q_2'''x))\ G_0(\pi_5(q_2'q_2'''x), q_2')\\
&\quad\times\sum_{\alpha_2'''} I_{\alpha_2'\alpha_2'''}(q_2'q_2'''x)\ h_{\alpha_2'''}(\pi_6(q_2'q_2'''x))\ V^{(n-1)}_{\alpha_2'''}(q_2''').
\end{aligned} \qquad (5.17)$$

Again, the singular integrals can be rewritten according to the method given in Appendix B.

VI. UNITARITY RELATIONS

The scattering states $\Psi_0^{(+)}$ depend on the initial-state quantum numbers

$$\Psi_0^{(+)} \equiv \Psi^{(0)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}, \qquad (6.1)$$

like the initial state

$$\Phi_{0,a} \equiv \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}. \qquad (6.2)$$

Using the full Green's operator, $G \equiv (E+i\epsilon-H)^{-1}$, to the full Hamiltonian, $\Psi^{(0)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}$ obeys the equation

$$\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} = \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} + GV\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}. \qquad (6.3)$$

A second scattering state is defined by

$$\Psi^{(-)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} = \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} + G^*V\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}. \qquad (6.4)$$

Both are related to each other as

$$\begin{aligned}
\Psi^{(-)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} &= \Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} + (G^*-G)V\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\\
&= \Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} + 2\pi i\delta(E-H)V\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}.
\end{aligned} \qquad (6.5)$$

The S matrix is defined as

$$S^{m_2'm_3',m_2m_2}_{\mathbf{p}_1'\mathbf{q}_1',\mathbf{p}_1\mathbf{q}_1} \equiv \langle\Psi^{(-)}_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle. \qquad (6.6)$$

Inserting Eq. (6.5) leads to

$$\begin{aligned}
S^{m_2'm_3',m_2m_2}_{\mathbf{p}_1'\mathbf{q}_1',\mathbf{p}_1\mathbf{q}_1} &= \langle\Psi^{(+)}_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle - 2\pi i\delta(E'-E)\\
&\quad\times\langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|V|\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1m_2m_3}\rangle.
\end{aligned} \qquad (6.7)$$

Now, we have due to general considerations

$$\langle\Psi^{(+)}_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle = \langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle. \qquad (6.8)$$

Consequently,

$$\begin{aligned}
S^{m_2'm_3',m_2m_2}_{\mathbf{p}_1'\mathbf{q}_1',\mathbf{p}_1\mathbf{q}_1} &= \langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle\\
&\quad-2\pi i\delta(E'-E)\langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|V|\Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle\\
&= \langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle\\
&\quad-2\pi i\delta(E'-E)\langle\Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|U^{00}|\Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle.
\end{aligned} \qquad (6.9)$$

For the last equation we used the definition of the transition operator U^{00}.

Since the scattering states in the definition of S belong to the same Hamiltonian one has to have

$$S^{m_2'm_3',m_2m_2}_{\mathbf{p}_1'\mathbf{q}_1',\mathbf{p}_1\mathbf{q}_1} \equiv \hat{S}^{m_2'm_3',m_2m_2}_{\mathbf{p}_1'\mathbf{q}_1',\mathbf{p}_1\mathbf{q}_1}\ \delta(E'-E), \qquad (6.10)$$

where $E = \frac{p_1^2}{m} + \frac{q_1^2}{2M_1}$ (setting the α particle as spectator), and correspondingly as a similar expression for E'. The unitarity relation simply follows from the completeness relation spanned by the scattering states

$$\langle \Psi^{(+)}_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} | \Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} \rangle = \sum_{m_2''m_3''} \int d^3p_1'' d^3q_1'' \langle \Psi^{(+)}_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} | \Psi^{(-)}_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} \rangle \times \langle \Psi^{(-)}_{\mathbf{p}_1''\mathbf{q}_1''m_2''m_3''} | \Psi^{(+)}_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} \rangle, \quad (6.11)$$

or in terms of the S-matrix elements

$$\langle \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} | \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} \rangle = \sum_{m_2''m_3''} \int d^3p_1'' d^3q_1'' S^{m_2''m_3'',m_2'm_3' *}_{\mathbf{p}_1''\mathbf{q}_1'',\mathbf{p}_1'\mathbf{q}_1'} S^{m_2''m_3'',m_2m_3}_{\mathbf{p}_1''\mathbf{q}_1'',\mathbf{p}_1\mathbf{q}_1}. \quad (6.12)$$

This can be rewritten in terms of the matrix elements of U^{00}. Using the completeness relation

$$\sum_{m_2''m_3''} \int d^3p_1'' d^3q_1'' | \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} \rangle \langle \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} | = 1, \quad (6.13)$$

and

$$\delta(E'' - E') \langle \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} | U^{00} | \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} \rangle = \langle \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} | \delta(H_0 - E') U^{00} | \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} \rangle, \quad (6.14)$$

leads to

$$\delta(E - E') \Big[i \langle \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} | U^{00} | \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} \rangle^* - i \langle \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} | U^{00} | \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} \rangle + 2\pi \sum_{m_2''m_3''} \int d^3p_1'' d^3q_1'' \delta(E'' - E) \times \langle \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} | U^{00} | \Phi^0_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'} \rangle^* \times \langle \Phi^0_{\mathbf{p}_1''\mathbf{q}_1'',m_2''m_3''} | U^{00} | \Phi^0_{\mathbf{p}_1\mathbf{q}_1,m_2m_3} \rangle \Big] = 0. \quad (6.15)$$

More interesting is the partial wave decomposed version for the on-shell matrix element

$$\langle p_1'q_1'\alpha_1' | U^{00} | p_1q_1\alpha_1 \rangle = U^{00,J_1}_{\alpha_1'\alpha_1}(p_1'p_1) \delta_{J_1'J_1} \delta_{M_1'M_1}. \quad (6.16)$$

The dependence on p_1', p_1 is sufficient since on-shell $q_1 = \sqrt{2M_1(E - \frac{p_1^2}{m})}$, $q_1' = \sqrt{2M_1(E' - \frac{p_1'^2}{m})}$.

As shown in Appendix C, the matrix element $U^{00,J_1}_{\alpha_1'\alpha_1}(p_1'p_1)$ obeys the unitarity relation

$$i\, U^{00,J*}_{\alpha_1\alpha_1'}(p_1, p_1') - i\, U^{00,J}_{\alpha_1'\alpha_1}(p_1', p_1) + 2\pi \sum_{\alpha_1''} \int \delta(E'' - E) U^{00,J*}_{\alpha_1''\alpha_1'}(p_1'', p_1') U^{00,J}_{\alpha_1''\alpha_1}(p_1'', p_1) = 0. \quad (6.17)$$

The corresponding relation for the partial-wave projected S-matrix element is

$$\sum_{\alpha_1''} \int \delta(E'' - E)\delta(E'' - E')\, S^{J*}_{\alpha_1'',\alpha_1'}(p_1''p_1')\, S^{J}_{\alpha_1'',\alpha_1}(p_1''p_1) = \delta_{\alpha_1'\alpha_1} \frac{\delta(p_1 - p_1')}{p_1^2} \frac{\delta(q_1 - q_1')}{q_1^2}. \quad (6.18)$$

Note that not only discrete quantum numbers span the columns and rows of the S matrix, but also the continuous quantum numbers $p_1' p_1$, which describe how the energy is continuously distributed among the two relative motions.

VII. CAPTURE PROCESS $n + n + \alpha \to {}^6$He

The matrix element for the capture process is simply related to the time-reversed photodisintegration process of ^{6}He into three free particles. It is well known [40] how to treat photodisintegration of ^{3}He in the Faddeev scheme. In essentially the same manner one can formulate photodisintegration of ^{6}He based on an effective three-particle picture. Let O be the photon absorption operator and $|\Psi_{^6\mathrm{He}}\rangle$ the ^{6}He ground state. The breakup amplitude into $nn\alpha$ can then be written as an infinite series of processes

$$\langle \Phi_{0,a} | U_0 | \Psi_{^6\mathrm{He}} \rangle = \langle \Phi_{0,a} | O | \Psi_{^6\mathrm{He}} \rangle + \sum_i \langle \Phi_{0,a} | V_i G_0 O | \Psi_{^6\mathrm{He}} \rangle + \sum_{ij} \langle \Phi_{0,a} | V_i G_0 V_j G_0 O | \Psi_{^6\mathrm{He}} \rangle + \cdots. \quad (7.1)$$

Here V_i are the pair forces among the nn and $n\alpha$ particles and G_0 is the free propagator. This infinite series in terms of pair forces represents FSI. The first term is the direct breakup process generated by O. Let us define

$$\langle \Phi_{0,a} | U_0 | \Psi_{^6\mathrm{He}} \rangle = \langle \Phi_{0,a} | O | \Psi_{^6\mathrm{He}} \rangle + \sum_i \langle \Phi_{0,a} | U_{0i} | \Psi_{^6\mathrm{He}} \rangle, \quad (7.2)$$

where U_{0i} comprises all terms with V_i to the very left

$$U_{0i} | \Psi_{^6\mathrm{He}} \rangle \equiv V_i G_0 O | \Psi_{^6\mathrm{He}} \rangle + V_i \sum_j G_0 V_j\, G_0\, O | \Psi_{^6\mathrm{He}} \rangle + \cdots. \quad (7.3)$$

Clearly this can be summed up as

$$U_{0i} | \Psi_{^6\mathrm{He}} \rangle = V_i\, G_0\, O | \Psi_{^6\mathrm{He}} \rangle + V_i\, G_0 \sum_j U_{0j} | \Psi_{^6\mathrm{He}} \rangle. \quad (7.4)$$

Separating the terms $U_{0i}|\Psi_{^6\mathrm{He}}\rangle$ to the left and introducing the t matrices t_i leads to three coupled Faddeev equations ($i = 1, 2, 3$)

$$U_{0i} | \Psi_{^6\mathrm{He}} \rangle = t_i G_0 O | \Psi_{^6\mathrm{He}} \rangle + t_i G_0 \sum_{j \neq i} U_{0j} | \Psi_{^6\mathrm{He}} \rangle. \quad (7.5)$$

The photon absorption operator O has to be symmetric under the exchange of the two neutrons, which we number as particles 2 and 3. Thus using the antisymmetry of $|\Psi_{^6\mathrm{He}}\rangle$ with respect to the two neutrons, one finds

$$P_{23}U_{02}|\Psi_{^6\mathrm{He}}\rangle = -U_{03}|\Psi_{^6\mathrm{He}}\rangle. \tag{7.6}$$

This leads to the two coupled equations

$$\begin{aligned} U_{01}|\Psi_{^6\mathrm{He}}\rangle &= t_1G_0O|\Psi_{^6\mathrm{He}}\rangle + t_1G_0(1-P_{23}U_{02})|\Psi_{^6\mathrm{He}}\rangle,\\ U_{02}|\Psi_{^6\mathrm{He}}\rangle &= t_2G_0O|\Psi_{^6\mathrm{He}}\rangle + t_2G_0(U_{01}-P_{23}U_{02})|\Psi_{^6\mathrm{He}}\rangle, \end{aligned} \tag{7.7}$$

corresponding to Eq. (2.12) from Sec. II.

The complete breakup amplitude is given by

$$\begin{aligned} \langle\Phi_{0,a}|U_0|\Psi_{^6\mathrm{He}}\rangle &= \langle\Phi_{0,a}|O|\Psi_{^6\mathrm{He}}\rangle + \langle\Phi_{0,a}|U_{01}|\Psi_{^6\mathrm{He}}\rangle\\ &\quad + \langle\Phi_{0,a}|(1-P_{23})U_{02}|\Psi_{^6\mathrm{He}}\rangle. \end{aligned} \tag{7.8}$$

Using adequate pair forces and photon absorption operators (single-particle currents, two-body currents, and possibly beyond) these coupled equations can be solved by standard techniques [40].

VIII. SUMMARY

The structure inherent in the continuum states of the $n+n+\alpha$ system has so far only been explored in the framework of the HH approach [7,8,26–28,32]. There are strong initial-state and final-state interaction peaks, not only in the nn subsystem but also in the $n-\alpha$ subsystem. This poses a still unsolved challenge for the expansion into the discrete set of K harmonics as already known for the $n+d\rightarrow n+n+p$ system. This is pointed out by the authors of Ref. [7], who note that even for a maximum $K_{\max}=20$ in their calculation, the result is not completely converged.

A corresponding investigation in the Faddeev approach is still missing. The aim of this article is to lay the formal ground to do so.

In the Faddeev approach all the structures in the relative motions of the three particles are mapped out correctly, thus leading to a reliable path to the three-to-three scattering S matrix, which contains the information of the resonance structure of the ^{6}He system.

We derived two coupled Faddeev equations for the three-to-three scattering amplitudes. In a partial wave decomposed representation they form a system of two-dimensional coupled equations for each fixed total angular momentum. The multiple scattering series being arranged in powers in the two-body t matrices is the natural starting point for the solution of this coupled system of integral equations. The term linear in the t matrices is disconnected. The next term, second order in t, has well established logarithmic singularities in the external momenta. Only the term of third order in t is a smooth function of the external momenta and thus can serve as a driving term for the consecutive application of the Faddeev kernels. This provides all higher-order terms that can then be summed up by Padé.

Since up to now nearly all Faddeev-based investigations of the discrete structure of the $n+n+\alpha$ system are based on finite rank forces, we also derived the continuum equations using these types of forces. The unitarity relations are especially interesting since the rows and columns of the S matrix are not only numbered by discrete quantum numbers, but also by continuously varying on-shell momenta.

Finally, we provided Faddeev equations for the $n+n+\alpha$ capture process to the ^{6}He ground state. We pointed out that it is not necessary here to first evaluate the three-to-three wave function as was done in Ref. [38], and that one directly can use a Faddeev form for the entire breakup amplitude with no disconnected terms as is done in modern calculations [39,40], and as it was pioneered in Refs. [36,37].

This capture process is relevant for the production rate of ^{6}He in astrophysical environments [41] characterized by high neutron and alpha densities (e.g., those related to supernova shock fronts). In Ref. [21] this three-body process is approximated by sequential two-body processes, whereas, in principle, a genuine three-body reaction needs to be calculated. Very recently it was pointed out [41] that currently employed two-step mechanisms over intermediate resonances in the three-to-three scattering of the $n+n+\alpha$ system are most likely insufficient since the time delays for those intermediate steps are comparable to the duration of the entire process. This strongly supports the need for the approach we present in this article.

ACKNOWLEDGMENTS

This work was performed, in part, under the auspices of the US Department of Energy under Contract No. DE-FG02-93ER40756 with Ohio University and Grant No. 163.PNE.A912371 from the Ministère de L'enseignement Superieur et de la Recherche Scientifique, Algeria (K.K). We (K.K and Ch.E) thank the Institute for Nuclear Theory at the University of Washington for its hospitality and the US Department of Energy for partial support during the initial phase of this work.

APPENDIX A: PARTIAL WAVE DECOMPOSITION IN THE *nnα* SYSTEM

Partial wave decomposition of three-body wave functions have often been documented, see, for instance, Ref. [39], and focus on the $nn\alpha$ system [46]. Thus we are here relatively brief.

The projection of the free state, Eq. (2.13), onto the partial wave basis states of Eq. (3.1) is given by

$$\begin{aligned} &\langle p_1'q_1'\alpha_1'|\Phi_{\mathbf{p}_1\mathbf{q}_1m_2m_3}\rangle\\ &\quad = (1+(-)^{l_1'+s_1'})\sum_\mu (j_1'\lambda_1'J',\mu,M'-\mu)\\ &\qquad \times (l_1's_1'j_1',\mu-m_2-m_3,m_2+m_3)\left(\frac{1}{2}\frac{1}{2}s_1',m_2m_3\right)\\ &\qquad \times \frac{\delta(p_1'-p_1)}{p_1^2}\frac{\delta(q_1'-q_1)}{q_1^2}\,Y^*_{l_1'\mu-m_2-m_3}(\hat{p}_1)\,Y^*_{\lambda_1'M'-\mu}(\hat{q}_1). \end{aligned} \tag{A1}$$

This leads to

$$\begin{aligned} &\langle p_1''q_1''\alpha_1''|t_1|\Phi_{\mathbf{p}_1\mathbf{q}_1m_2m_3}\rangle\\ &\quad = \int dp_1'\,p_1'^2 t_{\alpha_1''}(p_1''p_1'q_1'')\,\Big[(1+(-)^{l_1''+s_1''}) \end{aligned}$$

$$\times \sum_{\mu} (j_1'' \lambda_1'' J', \mu, M'' - \mu)$$
$$\times (l_1'' s_1'' j_1'', \mu - m_2 - m_3, m_2 + m_3) \left(\frac{1}{2}\frac{1}{2} s_1'', m_2 m_3 \right)$$
$$\times \left. \frac{\delta(p_1' - p_1)}{p_1^2} \frac{\delta(q_1'' - q_1)}{q_1^2} Y^*_{l_1''\mu - m_2 - m_3}(\hat{p}_1) Y^*_{\lambda_1'' M'' - \mu}(\hat{q}_1) \right]$$
$$\equiv \frac{\delta(q_1'' - q_1)}{q_1^2} \, t_{\alpha_1''}(p_1'' p_1, E_{q_1}) \, C^{m_2+m_3}_{\alpha_1''}(\hat{p}_1 \hat{q}_1), \quad \text{(A2)}$$

and gives the driving term of Eq. (3.2) together with the amplitude C given in Eq. (3.3).

In case of the second driving term in Eq. (2.14), it is adequate to rewrite the free state $|\Phi_{\mathbf{p}_1\mathbf{q}_1 m_2 m_3}\rangle$ in terms of the Jacobi momenta of the type 2 (where the neutron is the spectator)

$$\mathbf{p}_2 = -\beta \mathbf{p}_1 - \gamma \mathbf{q}_1, \quad \mathbf{q}_2 = \mathbf{p}_1 - \alpha \mathbf{q}_1 , \quad \text{(A3)}$$

with

$$\alpha = \frac{1}{2}, \quad \beta = \frac{m_\alpha}{m + m_\alpha}, \qquad \gamma = \frac{2m + m_\alpha}{2(m + m_\alpha)}, \quad \text{(A4)}$$

and $\alpha\beta + \gamma = 1$.

Then

$$|\Phi_{\mathbf{p}_1\mathbf{q}_1 m_2 m_3}\rangle = |\mathbf{p}_2 \mathbf{q}_2\rangle |0\rangle_1 |m_2\rangle_2 |m_3\rangle_3 \left| 0\left(\tfrac{1}{2}\tfrac{1}{2}\right)1 \right\rangle - |\tilde{\mathbf{p}}_2 \tilde{\mathbf{q}}_2\rangle |0\rangle_1 |m_2\rangle_3 |m_3\rangle_2 \left| 0\left(\tfrac{1}{2}\tfrac{1}{2}\right)1 \right\rangle, \quad \text{(A5)}$$

where

$$\tilde{\mathbf{p}}_2 = \beta \mathbf{p}_1 - \gamma \mathbf{q}_1, \quad \tilde{\mathbf{q}}_2 = -\mathbf{p}_1 - \alpha \mathbf{q}_1. \quad \text{(A6)}$$

The partial wave projected state in system "2" is given by

$$\langle p_2' q_2' \alpha_2' | \Phi_{\mathbf{p}_1\mathbf{q}_1 m_2 m_3}\rangle$$
$$= \frac{\delta(p_2' - p_2)}{p_2^2} \frac{\delta(q_2' - q_2)}{q_2^2} \, \delta_{s_2' \frac{1}{2}} \sum_{\mu} (j_2' I_2' J', \mu M' - \mu)$$
$$\times \left(l_2' \frac{1}{2} j_2', \mu - m_3, m_3 \right) Y^*_{l_2' \mu - m_3}(\hat{p}_2)$$
$$\times \left(\lambda_2' \frac{1}{2} I_2', M' - \mu - m_2, m_2 \right) Y^*_{\lambda_2' M' - \mu - m_2}(\hat{q}_2)$$
$$- \frac{\delta(p_2' - \tilde{p}_2)}{\tilde{p}_2^2} \frac{\delta(q_2' - \tilde{q}_2)}{\tilde{q}_2^2} \, \delta_{s_2' \frac{1}{2}} \sum_{\mu} (j_2' I_2' J', \mu M' - \mu)$$
$$\times \left(l_2' \frac{1}{2} j_2', \mu - m_2, m_2 \right) Y^*_{l_2' \mu - m_2}(\hat{\tilde{p}}_2)$$
$$\times \left(\lambda_2' \frac{1}{2} I_2', M' - \mu - m_3, m_3 \right) Y^*_{\lambda_2' M' - \mu - m_3}(\hat{\tilde{q}}_2), \quad \text{(A7)}$$

and the second driving term becomes

$$\langle p_2' q_2' \alpha_2' | t_2 | \Phi_{\mathbf{p}_1\mathbf{q}_1 m_2 m_3}\rangle$$
$$\equiv \frac{\delta(q_2' - q_2)}{q_2^2} \, t_{\alpha_2'}(p_2' p_2, E_{q_2'}) \, D^{m_2,m_3}_{\alpha_2'}(\theta_{p_2} \theta_{q_2})$$
$$- \frac{\delta(q_2' - \tilde{q}_2)}{\tilde{q}_2^2} \, t_{\alpha_2'}(p_2' \tilde{p}_2, E_{\tilde{q}_2}) \, \tilde{D}^{m_2,m_3}_{\alpha_2'}(\theta_{\tilde{p}_2} \theta_{\tilde{q}_2}), \quad \text{(A8)}$$

with D and $\tilde{D}$ given in Eqs. (3.4) and (3.5). For the kernel pieces we refer to Ref. [46].

APPENDIX B: AVOIDING LOGARITHMIC SINGULARITIES IN THE INTEGRALS

We illustrate the new manner to rewrite the Faddeev kernel such that only a single pole singularity appears in an example (for more details see Refs. [48,49]) Consider the first kernel in Eq. (3.2), of which the first piece can be rewritten as

$$\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} \int dp_1'' p_1''^2 \, \frac{\delta(p_1'' - \pi_1(q_1' q_2' x))}{p_1''^2}$$
$$\times t_{\alpha_1'}(p_1' p_1'', E_{q_1'}) \, G_0(p_1'', q_1') G_{\alpha_1' \alpha_2'}(q_1' q_2' x)$$
$$\times \int dp_2' p_2'^2 \, \frac{\delta(p_2' - \pi_2(q_1' q_2' x))}{p_2'^2} \, \langle p_2' q_2' \alpha_2' | U_{2,a}. \quad \text{(B1)}$$

The two δ functions are then changed according to

$$\delta(p_1'' - \pi_1(q_1' q_2' x)) = \frac{2 p_1''}{2\alpha q_1' q_2'} \, \delta(x - x_0) \, \Theta(1 - |x_0|),$$
$$\delta(p_2' - \pi_2(q_1' q_2' x)) = \delta\left(p_2' - \sqrt{\gamma q_1'^2 + \frac{\beta}{\alpha} p_1''^2 - \frac{\beta\gamma}{\alpha} q_2'^2} \right)$$
$$\times \Theta\left(\gamma q_1'^2 + \frac{\beta}{\alpha} p_1''^2 - \frac{\beta\gamma}{\alpha} q_2'^2 \right), \quad \text{(B2)}$$

with

$$x_0 = \frac{p_1''^2 - \alpha^2 q_1'^2 - q_2'^2}{2\alpha q_1' q_2'}. \quad \text{(B3)}$$

Inserting this into Eq. (B1) leads to

$$\int dx \int dq_2' q_2'^2 \sum_{\alpha_2'} \int dp_1'' p_1''^2 \, \frac{2 p_1''}{2\alpha q_1' q_2'} \, \delta(x - x_0)$$
$$\times \Theta(1 - |x_0|) \, t_{\alpha_1'}(p_1' p_1'', E_{q_1'}) G_0(p_1'', q_1') \, G_{\alpha_1' \alpha_2'}(q_1' q_2' x)$$
$$\times \int dp_2' p_2'^2 \, \delta\left(p_2' - \sqrt{\gamma q_1'^2 + \frac{\beta}{\alpha} p_1'^2 - \frac{\beta\gamma}{\alpha} q_2'^2} \right)$$
$$\times \Theta\left(\gamma q_1'^2 + \frac{\beta}{\alpha} p_1'^2 - \frac{\beta\gamma}{\alpha} q_2'^2 \right) \langle p_2' q_2' \alpha_2' | U_{2,a}$$
$$= \frac{1}{\alpha q_1'} \int dq_2' q_2' \sum_{\alpha_2'} \int dp_1'' p_1'' \, t_{\alpha_1}(p_1' p_1'', E_{q_1'}) \, G_0(p_1'', q_1')$$
$$\times G_{\alpha_1' \alpha_2'}(q_1' q_2' x_0) \left\langle \sqrt{\gamma q_1'^2 + \frac{\beta}{\alpha} p_1'^2 - \frac{\beta\gamma}{\alpha} q_2'^2} \; q_2' \alpha_2' \right| U_{2,a}$$
$$\times \Theta\left(1 - \left| \frac{p_1''^2 - \alpha^2 q_1'^2 - q_2'^2}{2\alpha q_1' q_2'} \right| \right)$$
$$\times \Theta\left(\gamma q_1'^2 + \frac{\beta}{\alpha} p_1''^2 - \frac{\beta\gamma}{\alpha} q_2'^2 \right). \quad \text{(B4)}$$

The two Θ functions define the domain D for the integrations over p_1'' and q_2'. Thus we end up with

$$\frac{1}{\alpha q_1'}\int dp_1'' p_1''\, t_{\alpha_1}(p_1' p_1'', E_{q_1'})\,\frac{1}{E+i\epsilon-\frac{p_1''^2}{m}-\frac{q_1'^2}{2M_1}} \times \int_{|p_1''-\alpha q_1'|}^{p_1''+\alpha q_1'} dq_2' q_2' \sum_{\alpha_2'} G_{\alpha_1'\alpha_2'}(q_1' q_2' x_0) \times \left\langle \sqrt{\gamma q_1'^2+\frac{\beta}{\alpha}p_1'^2-\frac{\beta\gamma}{\alpha}q_2'^2}\; q_2'\alpha_2' \right| U_{2,a}. \qquad \text{(B5)}$$

The singularity in $G_0(p_1'', q_1')$ is now a single pole in p_1'' for a given q_1'. This type of singularity does not pose any numerical problem and can be implemented with standard techniques [50]. Note that for $q_1' \geqslant \sqrt{2M_1E}$ there is no pole and one might as well keep the original form.

APPENDIX C: PARTIAL WAVE DECOMPOSED TRANSITION AMPLITUDE

The definition of the partial wave decomposed transition amplitude is

$$\langle \Phi_{\mathbf{p}_1'\mathbf{q}_1'm_2'm_3'}|U^{00}|\Phi_{\mathbf{p}_1\mathbf{q}_1m_2m_3}\rangle \equiv \sum_{\tilde{\alpha}_1}\int d\tilde{p}_1\tilde{p}_1{}^2 d\tilde{q}_1\tilde{q}_1{}^2 \langle \Phi_{\mathbf{p}_1'\mathbf{q}_1'm_2'm_3'}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle \times \langle \tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'|U^{00}|\tilde{p}_1\tilde{q}_1\tilde{\alpha}_1\rangle\langle \tilde{p}_1\tilde{q}_1\tilde{\alpha}_1|\Phi_{\mathbf{p}_1\mathbf{q}_1m_2m_3}\rangle. \qquad \text{(C1)}$$

This inserted into Eq. (6.15) yields

$$\delta(E-E')\sum_{\tilde{\alpha}_1'}\int\sum_{\tilde{\alpha}_1}\int \langle \tilde{p}_1\tilde{q}_1\tilde{\alpha}_1|\Phi_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle \times \langle \Phi_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle \Big[i\,\langle \tilde{p}_1\tilde{q}_1\tilde{\alpha}_1|U^{00}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle^* - i\langle \tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'|U^{00}|\tilde{p}_1\tilde{q}_1\tilde{\alpha}_1\rangle + 2\pi\sum_{m_2''m_3''}\int d^3p_1'' d^3q_1'' \times \delta(E''-E)\sum_{\tilde{\alpha}_1''}\int \langle \Phi''|\tilde{p}_1''\tilde{q}_1''\tilde{\alpha}_1''\rangle^*\langle \tilde{p}_1''\tilde{q}_1''\tilde{\alpha}_1''| \times U^{00}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle^* \sum_{\tilde{\alpha}_1'''}\int \langle \Phi''|\tilde{p}_1'''\tilde{q}_1'''\tilde{\alpha}_1'''\rangle \times \langle \tilde{p}_1'''\tilde{q}_1'''\tilde{\alpha}_1'''|U^{00}|\tilde{p}_1\tilde{q}_1\tilde{\alpha}_1\rangle \Big] = 0, \qquad \text{(C2)}$$

which, using the completeness relation,

$$\sum_{m_2''m_3''}\int d^3p_1'' d^3q_1''\delta(E''-E)\langle \Phi''|\tilde{p}_1''\tilde{q}_1''\tilde{\alpha}_1''\rangle^*\langle \Phi''|\tilde{p}_1'''\tilde{q}_1'''\tilde{\alpha}_1'''\rangle = \frac{\delta(\tilde{p}_1''-\tilde{p}_1''')}{(\tilde{p}_1'')^2}\frac{\delta(\tilde{q}_1''-\tilde{q}_1''')}{(\tilde{q}_1'')^2}\delta_{\tilde{\alpha}_1''\tilde{\alpha}_1'''}\,\delta(E''-E), \qquad \text{(C3)}$$

leads to

$$\delta(E-E')\sum_{\tilde{\alpha}_1'}\int\sum_{\tilde{\alpha}_1}\int \langle \tilde{p}_1\tilde{q}_1\tilde{\alpha}_1|\Phi_{\mathbf{p}_1\mathbf{q}_1,m_2m_3}\rangle \times \langle \Phi_{\mathbf{p}_1'\mathbf{q}_1',m_2'm_3'}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle \Big[i\langle \tilde{p}_1\tilde{q}_1\tilde{\alpha}_1|U^{00}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle^* - i\langle \tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'|U^{00}|\tilde{p}_1\tilde{q}_1\tilde{\alpha}_1\rangle + 2\pi\sum_{\tilde{\alpha}_1''}\int \delta(E''-E) \times \langle \tilde{p}_1''\tilde{q}_1''\tilde{\alpha}_1''|U^{00}|\tilde{p}_1'\tilde{q}_1'\tilde{\alpha}_1'\rangle^* \times \langle \tilde{p}_1''\tilde{q}_1''\tilde{\alpha}_1''|U^{00}|\tilde{p}_1\tilde{q}_1\tilde{\alpha}_1\rangle \Big] = 0. \qquad \text{(C4)}$$

Then using Eqs. (A1) and (6.16), the orthogonality of the spherical harmonics and of the Clebsch-Gordan coefficients, one can project onto the on s-shell unitarity relation of Eq. (6.17).

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