

Synchrotron radiation by fast fermions in heavy-ion collisions

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We study the synchrotron radiation of gluons by fast quarks in strong magnetic field produced by colliding relativistic heavy ions. We argue that due to high electric conductivity of plasma, the magnetic field is almost constant during the entire plasma lifetime. We calculate the energy loss due to synchrotron radiation of gluons by fast quarks. We find that the typical energy loss per unit length for a light quark at the Large Hadron Collider is a few GeV per fm. This effect alone predicts quenching of jets with p_\perp up to about 20 GeV. We also show that the spin-flip transition effect accompanying the synchrotron radiation leads to a strong polarization of quarks and leptons with respect to the direction of the magnetic field. Observation of the lepton polarization may provide a direct evidence of existence of strong magnetic field in heavy-ion collisions.

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I. INTRODUCTION

It has been recently argued [1] that the magnetic field B produced by colliding relativistic heavy ions can be as large as $eB \approx m_\pi^2/\hbar$ at the Relativistic Heavy Ion Collider (RHIC) and $eB \approx 15 m_\pi^2/\hbar$ at Large Hadron Collider (LHC) [1,2]. This is comparable to the Schwinger critical value $eB_c = m^2/\hbar$ for a quark of mass m . In such strong fields many interesting perturbative and nonperturbative phenomena are expected to take place (see, e.g., Refs. [1,3–7]). In this letter we discuss the energy loss and polarization of fast light quarks moving in external magnetic field [8,9]. In QED these phenomena have been studied in detail due to their significance for collider physics, see, e.g., Refs. [10,11]. In turn, energy loss of fast particles in heavy-ion collisions is one of the most important probes of the hot nuclear medium [12,13]. Synchrotron radiation in chromomagnetic fields was discussed in Refs. [14–16].

A typical diagram contributing to the synchrotron radiation, i.e., radiation in an external magnetic field, by a quark is shown in Fig. 1. This diagram is proportional to $(eB)^n$, where n is the number of external field lines. In strong field, powers of eB must be summed up, which may be accomplished by exactly solving the Dirac equation for the relativistic fermion and then calculating the matrix element for the transition $q \rightarrow q + g$. Such a calculation has been done in QED for some special cases including the homogeneous constant field and can be readily generalized for gluon radiation.

II. TIME DEPENDENCE OF MAGNETIC FIELD

At first we would like to determine whether the constant field approximation is applicable to magnetic fields created in relativistic heavy-ion collisions. For periodic fields, the measure of how fast the field varies is the Keldysh “adiabaticity” parameter γ defined as [17,18]

$$\gamma = \frac{m\omega_B}{eB_m}, \quad (1)$$

where ω_B is frequency and B_m is amplitude of the magnetic field. We can adopt γ for adiabatically changing fields as

well. In this case $\omega_B = |\dot{B}/B|$ is the rate of the field change and B_m is the average field over time $1/\omega$. The constant field approximation corresponds to $\gamma \ll 1$.

Let us now estimate γ . At first, we assume that B is determined only by the valence charges of initial nuclei and neglect the magnetic response of the produced nuclear matter. In this case B decreases with time t according to the power law, implying that $\omega_B \sim 1/t$ [1], where t is the time measured in the center-of-mass frame. Obviously, at early times ω_B is largest. Therefore, to set the upper bound on γ we need to estimate the time t_0 after a heavy-ion collision when the quarks are released from the nuclear wave functions. If Q_s is the saturation momentum, this time is $t_0 \sim 1/Q_s$. For a semiperipheral collisions of Gold nuclei at RHIC we obtain $\omega_B = Q_s \approx 1$ GeV. Thus, the adiabaticity for the u quark is $\gamma = 0.1$ – 0.2 , while for the d quark $\gamma = 0.5$ – 0.8 , which marginally justifies the constant field approximation. Here we used $m_u = 1.5$ – 3.3 MeV and $m_d = 3.5$ – 6 MeV [19]. For heavier quarks $\gamma \gg 1$ and it seems that we cannot apply the constant field approximation. Energy dependence of the magnetic field follows from its transformation properties under boosts $eB \propto e^Y$, where Y is the rapidity. On the other hand, $Q_s \propto e^{\lambda Y/2}$, with $\lambda \ll 1$ [20,21]. Therefore, we expect that γ will decrease with energy improving the applicability of the constant field approximation.

So far we have neglected the magnetic response of the quark-gluon plasma. The plasma seems to be strongly coupled and we expect that its dynamics in strong external magnetic field is highly nontrivial. Solution of the problem of plasma magnetic response requires extensive numerical simulations of the relativistic hydrodynamic equations. Bearing this in mind, we, however, can use semiclassical arguments to derive a simple estimate of how the time dependence is affected by the plasma magnetic response. We will denote the magnetic field due to the valence quarks as \vec{B}_0 . According to the Faraday law, decreasing magnetic field \vec{B}_0 induces electric field \vec{E} circulating around the direction of magnetic field. The electric field generates electric current that in turn produces the magnetic field \vec{B}_i pointing in the positive z direction according to the Lenz rule. In the adiabatic approximation the resulting

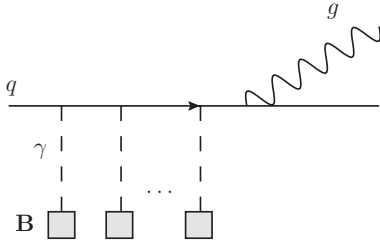


FIG. 1. A typical diagram contributing to the synchrotron radiation by a quark.

total magnetic field \vec{B} satisfies the following equations [22]

$$\nabla^2 \vec{B} = \mu\sigma \partial_t \mathbf{B}, \quad \vec{\nabla} \cdot \mathbf{B} = 0. \quad (2)$$

Here μ and σ are electric permeability and conductivity of plasma. The initial condition at $t = t_0$ reads

$$\vec{B}(\vec{r}, t_0) = \vec{B}_m e^{-\frac{\rho^2}{R^2}}, \quad (3)$$

where we used the cylindrical coordinates $\{z, \rho, \phi\}$ and R is of the order of nuclear radius. We neglect the external magnetic field at $t > t_0$.

The solution to the problem (2) and (3) is

$$\vec{B}(\vec{r}, t) = \int dV' \vec{B}(\vec{r}', t_0) G(\vec{r} - \vec{r}', t - t_0), \quad (4)$$

where

$$G(\vec{r}, t) = \frac{1}{(4\pi t/\sigma)^{3/2}} \exp\left[-\frac{\vec{r}^2}{4t/\sigma}\right], \quad (5)$$

is the Green's function and we assume $\mu = 1$. Equations (4) and (5) describe evolution of the initial field (3) in time. Integrating over the entire volume V' we derive

$$\vec{B}(\vec{r}, t) = \vec{B}_m \frac{R^2}{R^2 + 4(t - t_0)/\sigma} \exp\left[-\frac{\rho^2}{R^2 + 4(t - t_0)/\sigma}\right]. \quad (6)$$

It follows from (6) that as long as $t - t_0 \ll \tau$, where τ is a characteristic time

$$\tau = \frac{R^2 \sigma}{4}. \quad (7)$$

the magnetic field \vec{B} is approximately time independent.

To estimate τ we use the value of the electric conductivity found in the lattice calculations $\sigma \approx 6T^2/T_c$ [23]. At $T \approx 2T_c$ we have $\sigma \approx 25 \text{ fm}^{-1}$.¹ With the conservative estimate of the medium size $R \approx 5 \text{ fm}$ we have $\tau \approx 160 \text{ fm}$. This time is longer than any other time describing the plasma evolution. Moreover, recent lattice calculations show that the electric conductivity increases in the presence of a strong magnetic field [25]. As plasma expands it cools down; at $T < T_c$ electric conductivity rapidly decreases and magnetic field vanishes. We thereby conclude—keeping in mind the qualitative

nature of our argument—that the magnetic field can be considered as approximately stationary during the plasma lifetime.

We could have concluded that the magnetic field is quasistationary by mere examining the diffusion equation (2). Indeed, on dimensional grounds magnetic field significantly varies over the time $\tau \sim R^2 \sigma$, cp. (7). This argument is arguably qualitative. A quantitative analysis must include a more realistic contribution of external sources, effect of plasma expansion, and realistic transverse geometry. The contribution of external sources would appear in (4) as an additional term involving integration over space and time (from t_0 to t) of $\nabla \times \vec{j}_e$, where \vec{j}_e is the current density of valence quarks of receding nuclear remnants. Such corrections will certainly introduce time variations of the magnetic field. However, since τ is more than an order of magnitude larger than the plasma lifetime, we expect that adiabatic approximation $\gamma \ll 1$ still holds in a more accurate treatment. Of course, as has been already pointed out, a comprehensive numerical simulation is required in order to obtain the precise time and space dependence of the magnetic field.

In light of our discussion in this section, we will treat magnetic field as slowly varying in time. Formulas for energy loss in Sec. III should be understood as time average, and B as a mean magnetic field. On the other hand, polarization effects discussed in Sec. IV are sensitive only to the initial value of the magnetic field $B(t_0)$.

III. ENERGY LOSS DUE TO SYNCHROTRON RADIATION

Quark propagating in the external magnetic field radiates gluons as depicted in Fig. 1. The intensity of the radiation can be expressed via the invariant parameter χ defined as

$$\chi^2 = -\frac{\alpha_{\text{em}} Z_q^2 \hbar^3}{m^6} (F_{\mu\nu} p^\nu)^2 = \frac{\alpha_{\text{em}} Z_q^2 \hbar^3}{m^6} (\mathbf{p} \times \mathbf{B})^2, \quad (8)$$

where the initial quark four-momentum is $p^\mu = (\varepsilon, \mathbf{p})$, Z_q is the quark charge in units of the absolute value of the electron charge e . At high energies,

$$\chi \approx \frac{Z_q B \varepsilon}{B_c m}. \quad (9)$$

The regime of weak fields corresponds to $\chi \ll 1$, while in strong fields $\chi \gg 1$. In our case, $eB/eB_c \approx (m_\pi/m_u)^2 \gg 1$ (at RHIC) and therefore $\chi \gg 1$. In terms of χ , spectrum of radiated gluons of frequency ω can be written as [8]

$$\frac{dI}{d\omega} = -\alpha_s C_F \frac{m^2 \omega}{\varepsilon^2} \times \left\{ \int_x^\infty \text{Ai}(\xi) d\xi + \left(\frac{2}{x} + \frac{\omega}{\varepsilon} x^{1/2} \right) \text{Ai}'(x) \right\}, \quad (10)$$

where I is the intensity,

$$x = \left(\frac{\hbar \omega}{\varepsilon' \chi} \right)^{2/3},$$

and ε' is the quark's energy in the final state. Ai is the Ayri function. Equation (10) is valid under the assumption that the initial quark remains ultrarelativistic, which implies that the

¹Electric conductivity of quark-gluon plasma in the perturbative regime was calculated in Ref. [24] and is given by $\sigma \approx 6T/e^2$.

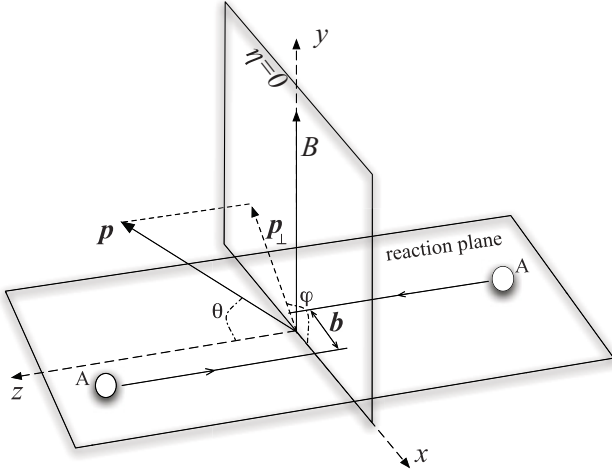


FIG. 2. Geometry of a heavy-ion collision. \mathbf{p} denotes the quark momentum. Two orthogonal planes are the “reaction plane” in which the initial heavy-ion momenta lie, and the midrapidity plane $\theta = \pi/2$, which is labeled as $\eta = 0$.

energy loss due to the synchrotron radiation $\Delta\epsilon$ should be small compared to the quark energy itself $\Delta\epsilon \ll \epsilon$.

Energy loss by a relativistic quark per unit length is given by [11]

$$\begin{aligned} \frac{d\epsilon}{dl} &= - \int_0^\infty d\omega \frac{dI}{d\omega} \\ &= \alpha_s C_F \frac{m^2 \chi^2}{2} \int_0^\infty \frac{4 + 5\chi x^{3/2} + 4\chi^2 x^3}{(1 + \chi x^{3/2})^4} \text{Ai}'(x) x dx. \end{aligned} \quad (11)$$

In two interesting limits, energy loss behaves quite differently. At $\eta = \varphi = 0$ we have [11]

$$\frac{d\epsilon}{dl} = - \frac{2\alpha_s \hbar C_F (Z_q e B)^2 \epsilon^2}{3m^4}, \quad \chi \ll 1, \quad (12a)$$

$$\frac{d\epsilon}{dl} = -0.37 \alpha_s \hbar^{-1/3} C_F (Z_q e B \epsilon)^{2/3}, \quad \chi \gg 1. \quad (12b)$$

In the strong field limit energy loss is independent of the quark mass, whereas in the weak field case it decreases as m^{-4} . Since $\chi \propto \hbar$, limit of $\chi \ll 1$ corresponds to the classical energy loss.

To apply this result to heavy-ion collisions we need to write down the invariant χ in a suitable kinematic variables. The geometry of a heavy-ion collision is depicted in Fig. 2. We can see that the vector of magnetic field \vec{B} is orthogonal to the “reaction plane,” which is spanned by the impact parameter vector \vec{b} and the collision axis (z axes). For a quark of momentum \vec{p} we define the polar angle θ with respect to the z axis and azimuthal angle φ with respect to the reaction plane. In this notation, $\mathbf{B} = B \hat{\mathbf{y}}$ and $\mathbf{p} = p_z \hat{\mathbf{z}} + p_\perp (\hat{\mathbf{x}} \cos \varphi + \hat{\mathbf{y}} \sin \varphi)$, where $p_\perp = |\mathbf{p}| \sin \theta \approx \epsilon \sin \theta$. Thus, $(\mathbf{B} \times \mathbf{p})^2 = B^2 (p_z^2 + p_\perp^2 \cos^2 \varphi)$. Conventionally, one expresses the longitudinal momentum and energy using the rapidity η as $\epsilon = m_\perp \cosh \eta$ and $p_z = m_\perp \sinh \eta$, where $m_\perp^2 =$

$m^2 + p_\perp^2$. We have

$$\chi^2 = \frac{\hbar^2 (eB)^2}{m^6} p_\perp^2 (\sinh^2 \eta + \cos^2 \varphi). \quad (13)$$

In Fig. 3 we show the calculation of the energy loss *per unit length* in a constant magnetic field using (11) and (13). We see that energy loss of a u quark with $p_\perp = 10$ GeV is about 0.2 GeV/fm at RHIC and 1.3 GeV/fm at LHC. This corresponds to the loss of 10% and 65% of its initial energy after traveling 5 fm at RHIC and LHC respectively. Therefore, energy loss due to the synchrotron radiation at LHC gives a phenomenologically important contribution to the total energy loss.

Energy loss due to the synchrotron radiation has a very nontrivial azimuthal angle and rapidity dependence that comes from the corresponding dependence of the χ parameter (13). As can be seen in Fig. 3(c), energy loss has a minimum at $\varphi = \pi/2$ that corresponds to quark's transverse momentum \mathbf{p}_\perp being parallel (or antiparallel) to the field direction. It has a maximum at $\varphi = 0, \pi$ when \mathbf{p}_\perp is perpendicular to the field direction and thus lying in the reaction plane. It is obvious from (13) that at midrapidity $\eta = 0$ the azimuthal angle dependence is much stronger pronounced than in the forward/backward direction. Let me emphasize that the energy loss (11) divided by m^2 , i.e., $d\epsilon/(dl m^2)$ scales with χ . In turn, χ is a function of magnetic field, quark mass, rapidity, and azimuthal angle. Therefore, all the features seen in Fig. 3 follow from this scaling behavior.

IV. POLARIZATION OF LIGHT QUARKS

Synchrotron radiation leads to polarization of electrically charged fermions; this is known as the Sokolov-Ternov effect [9]. Unlike energy loss that we discussed so far, this is a purely quantum effect. It arises because the probability of the spin-flip transition depends on the orientation of the quark spin with respect to the direction of the magnetic field and on the sign of fermion's electric charge. The spin-flip probability per unit time reads [9]

$$\begin{aligned} w &= \frac{5\sqrt{3}\alpha_s C_F}{16} \frac{\hbar^2}{m^2} \left(\frac{\epsilon}{m}\right)^5 \left(\frac{Z_q e |\mathbf{v} \times \mathbf{B}|}{\epsilon}\right)^3 \\ &\times \left[1 - \frac{2}{9} (\zeta \cdot \mathbf{v})^2 - \frac{8\sqrt{3}}{15} \text{sign}(e_q) (\zeta \cdot \hat{\mathbf{B}}) \right], \end{aligned} \quad (14)$$

where $\vec{\zeta}$ is a unit axial vector that coincides with the direction of quark spin in its rest frame, $\mathbf{v} = \mathbf{p}/\epsilon$ is the initial fermion velocity.

The nature of this spin flip transition is transparent in the nonrelativistic case, where it is induced by the interaction [10]

$$H = -\mu \cdot \mathbf{B} = -\left(\frac{geZ_q \hbar}{2m}\right) \vec{s} \cdot \vec{B}. \quad (15)$$

It is seen that negatively charged quarks and antiquarks (e.g. \bar{u} and \bar{d}) tend to align against the field, while the positively charged ones (e.g., u and d) align parallel to the field.

Let $n_{\uparrow(\downarrow)}$ be the number of fermions or antifermions with given momentum and spin direction parallel (antiparallel) to

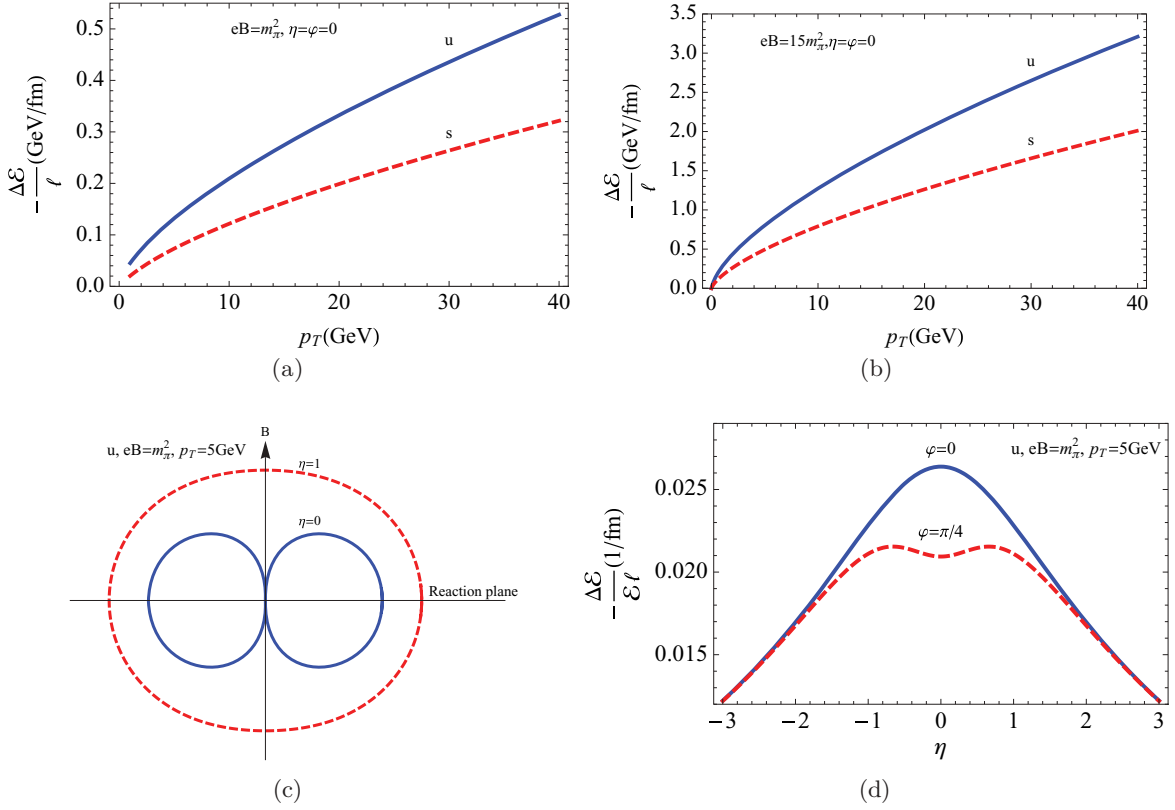


FIG. 3. (Color online) Energy loss *per unit length* by quarks moving in constant external magnetic field as a function of (a) p_T at RHIC at $\eta = \varphi = 0$, (b) p_T at LHC at $\eta = \varphi = 0$, and (c) azimuthal angle φ with respect to the reaction plane for $p_T = 5$ GeV and $\eta = 0, 1$. (d) Fractional energy loss vs. rapidity η at $p_T = 5$ GeV and $\varphi = 0, \pi/4$.

the field produced in a given event. At initial time $t = t_0$ the spin-asymmetry defined as

$$A = \frac{n_{\uparrow} - n_{\downarrow}}{n_{\uparrow} + n_{\downarrow}} \quad (16)$$

vanishes. Equation (14) implies that at later times, a beam of nonpolarized fermions develops a finite asymmetry given by [9]

$$A = \frac{8\sqrt{3}}{15} (1 - e^{-\frac{t-t_0}{\tau}}), \quad (17)$$

where

$$\tau = \frac{8}{5\sqrt{3} m \alpha_s C_F} \left(\frac{m}{\varepsilon} \right)^2 \left(\frac{m^2}{Z_q e |\mathbf{v} \times \mathbf{B}|} \right)^3 \quad (18)$$

is the characteristic time [do not confuse with Eq. (7)!] over which the maximal possible asymmetry is achieved. This time is extremely small on the scale of t_0 . For example, it takes only $\tau = 6 \times 10^{-8}$ fm for a d quark of momentum $p_{\perp} = 1$ GeV at $\eta = \varphi = 0$ at RHIC to achieve the maximal asymmetry of $A_m = \frac{8}{5\sqrt{3}} = 92$. Therefore, quarks and antiquarks are polarized almost instantaneously after being released from their wave functions. Note, that for quarks moving nearly parallel to the field direction, τ can be large. However, the corresponding azimuthal angle and rapidity are always much smaller than their experimental resolution.

As the quarks propagate through the hot medium their polarization is washed-out. This, however, does not seem to be an important effect because (i) the quark-gluon plasma is chirally symmetric phase of QCD and (ii) the spin-field interaction energy, estimated for, say, d quarks from (15) is of order 1.5 GeV, while the temperature in the plasma is of order 0.2 GeV. Nevertheless, we do expect that the polarization of quarks is completely washed out in the fragmentation process at $T < T_c$ because the chiral symmetry is broken in the QCD vacuum.

The Sokolov-Ternov effect can be observed by studying the polarization of light charged *leptons*. The characteristic time τ is still very small. Indeed, in (18) one replaces $\alpha_s C_F$ by α_{em} which—in the case of electron—is compensated by smaller mass. In my opinion, observation of such a lepton polarization asymmetry would be the most definitive proof of existence of the strong magnetic field at early times after a heavy-ion collision regardless of its later time dependence.

V. CONCLUSIONS

In this article we argued that the magnetic field created by fast heavy ions can be considered as approximately constant due to high electric conductivity of the quark-gluon plasma. We then used some well-known QED results to estimate the energy loss suffered by fast quarks in external magnetic field in heavy-ion collisions. We found that the energy loss

per unit length for a light quark with $p_T = 15$ GeV is about 0.27 GeV/fm at RHIC and 1.7 GeV/fm at LHC, which is comparable to the losses due to interaction with the plasma. Thus, the synchrotron radiation alone is able to account for quenching of jets at LHC with p_\perp as large as 20 GeV. Synchrotron radiation seems to be one of the missing pieces in the puzzle of the jet energy loss in heavy-ion collisions.

We also pointed out that quarks and leptons are expected to be strongly polarized in plasma in the direction parallel or antiparallel to the magnetic field depending on the sign of their electric charge. While polarization of quarks is expected to be washed out during the fragmentation, that of leptons should survive to present a direct experimental evidence for the existence of the magnetic field. Finally, we

would like to mention a possible deep connection between the “tunneling through the horizon” thermalization mechanism that we discussed before [26] and the polarization of fermions in external magnetic field that can be viewed as an Unruh effect [27–29].

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- [1] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, *Nucl. Phys. A* **803**, 227 (2008).
 - [2] V. Skokov, A. Y. Illarionov, and V. Toneev, *Int. J. Mod. Phys. A* **24**, 5925 (2009).
 - [3] D. Kharzeev, *Phys. Lett. B* **633**, 260 (2006).
 - [4] D. Kharzeev and A. Zhitnitsky, *Nucl. Phys. A* **797**, 67 (2007).
 - [5] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, *Phys. Rev. D* **78**, 074033 (2008).
 - [6] G. Basar, G. V. Dunne, and D. E. Kharzeev, [arXiv:1003.3464](#) [hep-ph].
 - [7] M. Asakawa, A. Majumder, and B. Muller, [arXiv:1003.2436](#) [hep-ph].
 - [8] A. I. Nikishov and V. I. Ritus, *Sov. Phys. JETP* **19**, 529 (1964) [*Zh. Eksp. Teor. Fiz.* **46**, 776 (1964)].
 - [9] A. A. Sokolov and I. M. Ternov, *Sov. Phys. Dokl.* **8**, 1203 (1964) [*Dokl. Akad. Nauk Ser. Fiz.* **153**, 1052 (1964 PHDOE, 8, 1203–1205.1964)].
 - [10] J. D. Jackson, *Rev. Mod. Phys.* **48**, 417 (1976).
 - [11] V. B. Berestetsky, E. M. Lifshitz, and L. P. Pitaevsky, *Course of Theoretical Physics: Quantum Electrodynamics* (Pergamon, Oxford, UK, 1982), Vol. 4, p. 652.
 - [12] M. Gyulassy and X. N. Wang, *Nucl. Phys. B* **420**, 583 (1994).
 - [13] R. Baier, Y. L. Dokshitzer, S. Peigne, and D. Schiff, *Phys. Lett. B* **345**, 277 (1995).
 - [14] E. V. Shuryak and I. Zahed, *Phys. Rev. D* **67**, 054025 (2003).
 - [15] D. E. Kharzeev, [arXiv:0806.0358](#) [hep-ph].
 - [16] B. G. Zakharov, *JETP Lett.* **88**, 475 (2008).
 - [17] L. V. Keldysh, *Sov. Phys. JETP* **20**, 1307 (1965).
 - [18] V. I. Ritus, *J. Sov. Laser Res.* **6**, 497 (1985).
 - [19] C. Amsler *et al.* (Particle Data Group), *Phys. Lett. B* **667**, 1 (2008).
 - [20] L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Phys. Rept.* **100**, 1 (1983).
 - [21] K. J. Golec-Biernat and M. Wusthoff, *Phys. Rev. D* **60**, 114023 (1999).
 - [22] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, MA, 1984), Sect. 58.
 - [23] S. Gupta, *Phys. Lett. B* **597**, 57 (2004).
 - [24] P. B. Arnold, G. D. Moore, and L. G. Yaffe, *J. High Energy Phys.* **05** (2003) 051.
 - [25] P. V. Buividovich, M. N. Chernodub, D. E. Kharzeev, T. Kalaydzhyan, E. V. Luschevskaya, and M. I. Polikarpov, [arXiv:1003.2180](#) [hep-lat].
 - [26] D. Kharzeev and K. Tuchin, *Nucl. Phys. A* **753**, 316 (2005).
 - [27] J. S. Bell and J. M. Leinaas, *Nucl. Phys. B* **284**, 488 (1987).
 - [28] J. M. Leinaas, [arXiv:hep-th/9804179](#).
 - [29] E. T. Akhmedov and D. Singleton, *Int. J. Mod. Phys. A* **22**, 4797 (2007).