

## Remarks on the origin of Castillejo-Dalitz-Dyson poles

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Castillejo-Dalitz-Dyson (CDD) poles are known to be connected with bound states and resonances. We discuss a new type of CDD pole associated with primitives i.e., poles of the  $P$  matrix that correspond to zeros of the  $D$  function on the unitary cut. The Low scattering equation is generalized for amplitudes with primitives. The relationship between the CDD poles and the primitives is illustrated by a description of the  $S$ -wave nucleon-nucleon phase shifts.

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The poles introduced by Castillejo *et al.* in Ref. [1] are known as the Castillejo-Dalitz-Dyson (CDD) poles. They describe ambiguities in solutions to the Low scattering equation [2] for amplitudes that satisfy correct analytical properties and unitarity. To clarify the physical meaning of the CDD poles, Dyson constructed a model [3] that demonstrates the relation of the CDD poles to bound states and resonances.

Some time ago, Jaffe and Low [4] proposed a method for identifying exotic multi-quark states with primitives that appear as poles of the  $P$  matrix rather than the  $S$  matrix. The analysis, performed for scalar mesons [4] and nucleon-nucleon scattering [5], revealed primitives, in agreement with expectations from the Massachusetts Institute of Technology bag model. A dynamical model of the  $P$  matrix was developed by Simonov [6] and was applied to the description of nucleon-nucleon scattering [6–11]. Other microscopic models of nucleon-nucleon forces have also been discussed [12–14].

The recent interest in the problem of nucleon-nucleon interactions is connected to new constraints on the equation of state (EOS) of nuclear matter, obtained from collective flow data and subthreshold kaon production in heavy-ion collisions [15,16] as well as astrophysical observations of massive neutron stars [17,18]. One-boson exchange models are in reasonable agreement with the laboratory data but predict surprisingly low masses for neutron stars in the  $\beta$  equilibrium [19,20]. Microscopic models can provide better insight into the short-range dynamics of nucleon-nucleon interactions and the high-density EOS.

In this Brief Report, we clarify the link between the CDD poles and the primitives, that can be useful for modeling the nucleon-nucleon interactions in the  $P$ -matrix formalism.

CDD poles arise when the interaction of particles includes intermediate states that are internally different from combined-particle states. These are discrete eigenstates of the system and basically form other channels in the scattering problem. They have also been referred to as elementary particle or compound states.

In the Dyson model, one starts from the scattering of two particles e.g., a nucleon and a pion. A nucleon can absorb a pion and can turn into an excited compound state  $N_\alpha$  of mass  $M_\alpha > \sqrt{s_0} = m + \mu$ , where  $m$  and  $\mu$  are the nucleon and the pion masses. The  $D$  function of the process can be

written as

$$D(s) = \Lambda(s) - \Pi(s), \quad (1)$$

where, in the relativistic notations,

$$\Lambda^{-1}(s) = \sum_{\alpha} \frac{g_{\alpha}^2}{s - M_{\alpha}^2}, \quad (2)$$

$$\Pi(s) = -\frac{1}{\pi} \int_{s_0}^{+\infty} \Phi_2(s') \frac{\mathcal{F}^2(s')}{s' - s} ds'. \quad (3)$$

Here,  $\Phi_2(s) = \pi k / \sqrt{s}$  is the relativistic two-body phase space,  $k$  is the center-of-mass momentum,  $g_{\alpha}$  is the coupling constant, and  $\mathcal{F}(s)$  is the form factor of the  $N_{\alpha} N \pi$  vertex. The  $S$  matrix has the form

$$S = e^{2i\delta(s)} = \frac{D(s - i0)}{D(s + i0)}. \quad (4)$$

The poles of  $\Lambda(s)$  are the CDD poles. They are located between the zeros of  $\Lambda(s)$  (i.e., between  $M_{\alpha}^2$  and  $M_{\alpha+1}^2$ ).

The  $D$  function constructed in such a way is the generalized  $R$  function [1]. It has no complex zeros on the first Riemann sheet of the complex  $s$  plane. It also has no zeros on the real half axis  $(-\infty, s_0)$ , which corresponds to bound states, provided  $D(s_0) < 0$  and  $s_0 < M_{\alpha}^2$ .

The simple roots of the equation

$$D(s) = 0, \quad (5)$$

located on the second Riemann sheet below the unitary cut, are identified as resonances. In the limit of small  $g_{\alpha}$ , roots of Eq. (5) are localized in the neighborhood of  $s = M_{\alpha}^2$ .  $\Re s$  gives the renormalized resonance mass, while  $\Im s$  determines the decay width  $\Gamma_{\alpha} = g_{\alpha}^2 \Im D(M_{\alpha}^2) / M_{\alpha}$ .

At the CDD poles  $\delta(s) = 0 \pmod{\pi}$ , the slope of the phase is positive. If  $s_{\gamma}$  is a CDD pole, then Eq. (2) gives  $\Lambda^{-1}(s_{\gamma}) = 0$  and  $\Lambda^{-1}(s_{\gamma}') < 0$ . By expanding the  $D$  function around  $s = s_{\gamma}$  and by using Eq. (4), one finds

$$\delta(s_{\gamma}') = -\Im D(s_{\gamma}) \Lambda^{-1}(s_{\gamma}') > 0.$$

Such behavior is in agreement with the Breit-Wigner formula according to which isolated resonances drive the phase shift up. In potential scattering, an increasing phase is associated with attraction.

The Dyson model, therefore, applies to systems with attraction where scattering phase shifts increase with increasing energy. The nucleon-nucleon phase shifts, conversely, decrease with increasing energy and provide evidence for repulsion. In Refs. [1–3],  $\Im D(s)$  is strictly positive. Softening this constraint to  $\Im D(s) \geq 0$  allows extension of the Dyson model to systems with repulsion:

Let us consider the scattering of two nucleons through compound states, dibaryons, with form factors  $\mathcal{F}(s)$  that have a simple zero at  $s = s_p > s_0 = 4m^2$ . Consequently,  $\Im D(s) \sim (s - s_p)^2$ . Such behavior is presupposed in the quark compound bag (QCB) model developed by Simonov for the description of nucleon-nucleon interactions [6]. In the QCB model, the separable potential generated by the compound six-quark bags is restricted to the bag surfaces. The  $S$ -wave form factor has the form  $\mathcal{F}(s) = \sin(kb)/kb$ , where  $k$  is the center-of-mass momentum and  $b$  is the effective interaction radius. In relativistic notation, the  $D$  function of the model has the form of Eq. (1), while  $\Lambda(s)$  and the self-energy operator  $\Pi(s)$  are equivalent to Eqs. (2) and (3), respectively.

This analogy allows the techniques developed in Ref. [1] to be used to parametrize nucleon-nucleon scattering amplitudes with functions that have the correct analytical properties.

If  $\Re D(s_p) \neq 0$ , the phase touches, at  $s = s_p$ , one of the  $\delta(s) = 0 \pmod{\pi}$  levels without crossing. However, if Eq. (5) holds at  $s = s_p$  for both the real and the imaginary parts, the phase crosses one of the levels  $\delta(s) = 0 \pmod{\pi}$  with a negative slope. This can be verified by expanding  $D(s)$  around  $s = s_p$ . By taking Eq. (4) and the conditions  $\Im D(s_p)'' > 0$  and  $\Re D(s_p)' > 0$  into account, one gets

$$\delta(s_p)' = -\frac{\Im D(s_p)''}{2\Re D(s_p)'} < 0.$$

In potential scattering, a negative slope of the phase shift is associated with repulsion.

The Low scattering equation [2] is modified in the presence of primitives. In the systems with  $\Pi(s)$  given by Eq. (3), the scattering amplitude  $A(s) = e^{i\delta(s)} \sin \delta(s)$  can be represented as follows:

$$A(s) = -\frac{\Phi_2(s)\mathcal{F}^2(s)}{D(s)}. \quad (6)$$

This amplitude obeys the generalized Low scattering equation

$$\frac{A(s)}{\Phi_2(s)\mathcal{F}^2(s)} = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{|A(s')|^2}{\Phi_2(s')\mathcal{F}^2(s')} \frac{ds'}{s' - s} - \sum_b \frac{C_b}{s - s_b} - \sum_p \frac{C_p}{s - s_p} - C, \quad (7)$$

which is essentially the dispersion integral representation for the inverse denominator function  $D(s)$  that accounts for the poles, which correspond to the bound states and primitives.  $A(s)$  and  $\mathcal{F}(s)$  have simple zeros at  $s = s_p$ , so the integrand in Eq. (7) is a regular function at  $s' = s_p$ . The bound states and the primitives generate poles at  $s_b < s_0$  and  $s_p > s_0$  on the real

axis, the coefficient

$$C_p = -\frac{2A(s_p)'}{\Phi_2(s_p)\mathcal{F}^2(s_p)'}$$

is positive.

In the QCB model, the  $P$  matrix takes the form

$$P = P_{\text{free}} + \kappa^{-1} \Lambda^{-1}, \quad (8)$$

where  $P_{\text{free}}$  is the free  $P$  matrix. For the  $S$  wave,  $P_{\text{free}} = kb \cot(kb)$  and  $P = kb \cot[kb + \delta(s)]$ . The value of  $\kappa$  is fixed by the normalization of  $D(s)$ . The compound states of masses  $M_\alpha$  show up as poles of the  $P$  matrix. The poles of the  $P$  matrix split into two groups according to their physical nature:

The first group is related to the bound states and resonances.

One bound state always exists at  $D(s_0) > 0$ . Additional bound states can be generated by compound states with masses  $M_\alpha < \sqrt{s_0}$ .

A characteristic feature of a resonance is the condition  $\mathcal{F}(s) \neq 0$  in the neighborhood of  $s = M_\alpha^2$ . Equation (5) can then be used to find a simple pole of the  $S$  matrix. The roots of Eq. (5) that lie on the real half axis  $(-\infty, s_0)$  of the second Riemann sheet are virtual states and can be influenced by the compound states.

The poles of the second group are related to roots of Eq. (5) on the unitary cut in the neighborhood of  $s = M_\alpha^2$ . Such poles do not show up as  $S$ -matrix poles and cannot be treated as resonances. They are called primitives according to Jaffe and Low [4]. If a resonance moves from the second Riemann sheet to the unitary cut, its singular effect on the  $S$  matrix cancels out. As distinct from resonances, primitives drive the phase shift down and mimic repulsion.

In the Dyson model, there exist at most one bound state and at most one resonance, which are not associated with the CDD poles. In the QCB model, there are CDD poles related to primitives that do not give rise to bound states or resonances. The neighboring CDD poles squeeze masses of compound states that become bound states, resonances, or primitives when the interaction is switched on. This is illustrated in Fig. 1.

The  $S$ -wave nucleon-nucleon scattering can be considered as an example of dynamics influenced by the CDD poles that are connected to primitives.

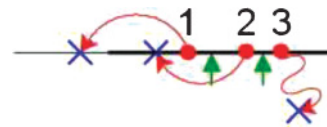


FIG. 1. (Color online)  $D$ -function zeros in the complex  $s$  plane. The unitary cut is shown by a bold solid line. Compound states 1, 2, and 3 are eigenstates of the free Hamiltonian. Upon switching on the  $s$ -channel interaction, zeros move to new positions shown by crosses. Compound states 1, 2, and 3 become bound state, primitive, and resonance, respectively. A pair of the CDD poles that squeezes compound state 2 of the primitive type is shown by arrows.

The model we discuss is the relativistic extension of the QCB model. The  $P$ -matrix formalism is recovered with

$$\mathcal{F}(s) = \left(\frac{s}{s_0}\right)^{1/4} \frac{\sin(kb)}{kb}. \quad (9)$$

Equation (3) for  $\kappa = 2mb/\pi$  gives

$$\kappa\Pi(s+i0) = -\frac{\sin(kb)}{kb} e^{ikb}. \quad (10)$$

In addition to the interaction through compound states, we introduce a contact interaction. This amounts to a redefinition of  $\Lambda^{-1}(s)$  as compared to Eq. (2). In the case of one CDD pole, the most general expression for  $\Lambda^{-1}(s)$  becomes

$$\kappa^{-1}\Lambda^{-1}(s) = c_p \left( \frac{r_p}{s-s_p} - \frac{r_p}{s_d-s_p} \right) - \frac{1}{\gamma}, \quad (11)$$

where  $c_p$  is a free parameter such that  $\kappa c_p r_p = g_\alpha^2$ ,  $r_p = 8(\pi/b)^2$  is the residue of  $P_{\text{free}}$  in the  $S$  wave, and  $s_d = M_d^2$  is the deuteron pole. The  $D$  function with the contact interaction remains the generalized  $R$  function.

In the  ${}^3S_1$  channel, the phase shift vanishes at  $T_{\text{lab}} = 354$  MeV. The  $S$  matrix according to Eq. (4) is unit in two cases:  $\Lambda(s) = \infty$  and  $\Im D(s) \equiv -\Im \Pi(s) = 0$ . The poles of  $\Lambda(s)$  are the CDD poles. At the CDD poles, the slope of the phase is positive, which corresponds to attraction. The second case  $\Im \Pi(s) = 0$  gives repulsion.

$T_{\text{lab}} = 354$  MeV is equivalent to  $k = 408$  MeV. The equation  $\mathcal{F}(s) = 0$  gives  $kb = \pi$ . Thus, we determine  $b = 1.52$  fm. Since  $\mathcal{F}(s_p) = 0$  if and only if  $\Re \Pi(s_p) = 0$ , Eq. (5) simplifies to  $\Lambda(s_p) = 0$ .  $\Lambda(s)$  vanishes when  $s = M_\alpha^2$ . The compound state shows up as the primitive of mass  $M_\alpha = 2\sqrt{k^2 + m^2} = 2047$  MeV.

The parametrization ensures the existence of the deuteron pole for

$$\gamma = -\kappa\Pi(s_d) > 0. \quad (12)$$

Unphysical zeros of the  $D$  function are eliminated by constraining  $c_p$ . One can easily show that  $\Im D(s) \sim \Im s$  and that the coefficient of proportionality is positive for positive  $c_p$ . In this case,  $D(s)$  has no zeros for  $\Im s \neq 0$ . The real half axis  $(-\infty, s_0)$  remains to be checked. The derivative  $D(s)'$  is positive below the threshold.  $D(s)$  crosses the real axis at  $s = s_d < s_0$ . This is the unique zero of the  $D$  function, provided  $\Lambda(s)$  has no poles for  $s < s_0$ . Let us investigate the zeros of  $\Lambda^{-1}(s)$ . Since  $\kappa^{-1}\Lambda^{-1}(s_d) = -1/\gamma < 0$ ,  $\Lambda^{-1}(s) < 0$  and  $\Lambda^{-1}(s)$  has no poles for  $s < s_p$  by construction, the condition  $\Lambda^{-1}(-\infty) < 0$  is sufficient to exclude unphysical zeros. Finally,  $c_p$  satisfies the constraint:

$$0 < c_p < c_p^{\text{max}} = \frac{s_p - s_d}{\gamma r_p}. \quad (13)$$

In Fig. 2(a), we show the  ${}^3S_1$  phase shift versus the proton kinetic energy for  $c_p = 0.9c_p^{\text{max}}$ . This is compared to the partial-wave analysis data provided by Ref. [21]. For the  $pn$  system,  $s = s_0 + 2m_n T_{\text{lab}}$ , where  $m_n$  is neutron mass. The CDD pole is located at  $M = 3203$  MeV. The pion production threshold is at  $T_{\text{lab}} = 280$  MeV, and the inelasticity is small up to  $\sim 350$  MeV.

In Fig. 2(c), we show  $\Re D(s)$  and  $\Im D(s)$  as functions of  $T_{\text{lab}}$ .  $\Re D(s)$  has one zero below  $s_0$ , which corresponds to the deuteron. The second zero at  $s > s_0$  with  $\Im D(s) \neq 0$  and a negative slope of  $\Re D(s)$  ensures the crossing of the level  $\delta(s) = \pi$ . The third zero corresponds to the primitive. In the  ${}^1S_0$  channel, the phase shift vanishes at  $T_{\text{lab}} = 265$  MeV.

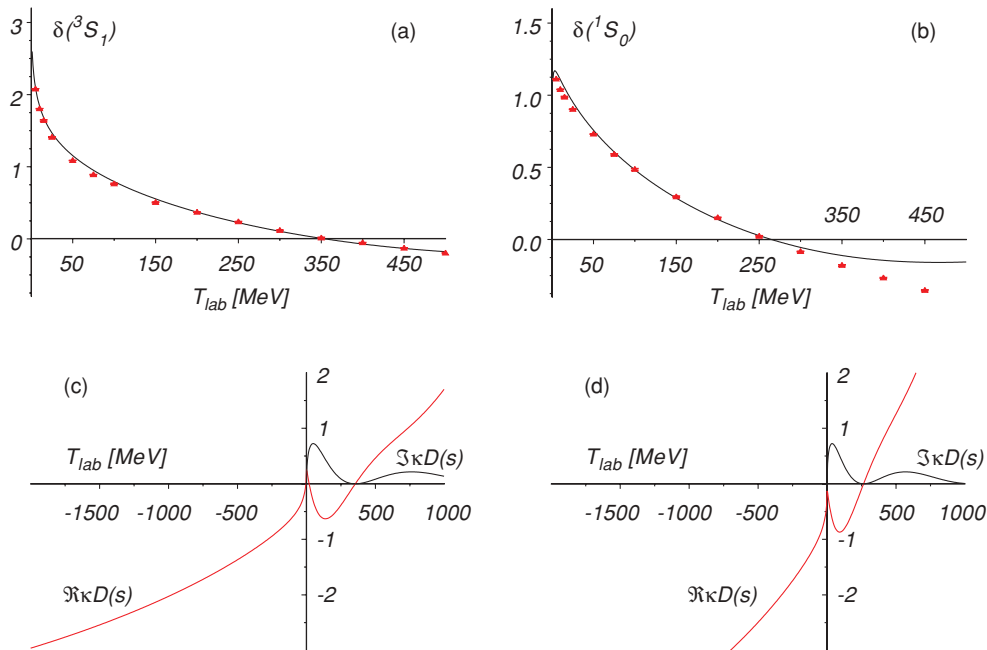


FIG. 2. (Color online)  ${}^3S_1$  and  ${}^1S_0$  scattering phase shifts in radians (upper panel) and real and imaginary parts of the  $D$  functions (lower panel) versus the proton kinetic energy. The solid curves are parametrizations within the relativistic QCB model. The experimental phase shifts [21] are shown by triangles.

The same arguments as before give  $b = 1.76$  fm and  $M_\alpha = 2006$  MeV.

$\Lambda^{-1}(s)$  has the form of Eq. (11), with  $s_d$  replaced by  $s_0 = 4m^2$ . Near the threshold,  $\kappa D(s) = -\gamma + 1 + ikb + \dots$ . From the other side,  $D(s) \sim 1 - i\delta(k) + \dots = 1 - ika + \dots$ , where  $a = 23.56$  fm is the scattering length. One has to require

$$\gamma = 1 + \frac{b}{a}. \quad (14)$$

$D(s)$  has no zeros for complex values of  $s$ . Its derivative is positive for real  $s < s_0$ . To avoid unphysical zeros, it is sufficient to require

$$\kappa D(-\infty) = \frac{1}{-\frac{c_p r_p}{s_0 - s_p} - \frac{1}{\gamma}} < \kappa D(s_0) = 1 - \gamma < 0.$$

The second inequality is fulfilled, and the first one gives  $c_p < \min[c_p^{\max}, c_p^{\max}/(\gamma - 1)]$ . Since  $b \ll a$ , this reduces to Eq. (13) with  $s_d$  replaced by  $s_0$ .

In Fig. 2(b), we show our fit of the  $pn$   $^1S_0$  phase shift with  $c_p = 0.9c_p^{\max}$  compared to the experimental data [21]. The CDD pole occurs at  $M = 2916$  MeV. Shown in Fig. 2(d) are the real and the imaginary parts of the  $D$  function versus the proton kinetic energy. In Figs. 2(c) and 2(d), the real and the imaginary parts of the  $D$  functions vanish at  $s = s_p$ . These are signatures of the primitives, along with crossing the levels  $\delta(s) = 0$  with negative slopes on Figs. 2(a) and 2(b). The values of  $b$  and  $M_\alpha$  are close to those obtained in Refs. [5,6].

Benjamins and van Dijk [13] used the hybrid Lee model with one compound state in each channel to describe the nucleon-nucleon  $S$ -wave phase shifts below  $T_{\text{lab}} = 500$  MeV and to reproduce parameters related to the deuteron and the virtual  $^1S_0$  state. The model does not have explicit CDD poles

and primitives. However, it can be reformulated in terms of the QCB model with one CDD pole and two compound states that correspond to the primitive and a high-mass resonance [22].

Resonances and primitives do not exist as asymptotic states. In Feynman diagrams, propagators of primitives  $1/(s - M_\alpha^2)$  are multiplied by form factors  $\mathcal{F}(s)$ . Such combinations do not have poles at  $s = M_\alpha^2$ . Primitives, thus, do not propagate, although they influence the dynamics.

To summarize, the physical meaning of the CDD poles was revisited. In the general case, the neighboring CDD poles squeeze masses of compound states related to bound states, resonances, or primitives. The primitives are  $P$ -matrix poles associated with zeros of the  $D$  function on the unitary cut, which do not show up as poles of the  $S$  matrix. The Low scattering equation was generalized for amplitudes with primitives. The primitive-type CDD poles occur in systems with repulsion. In the  $^3S_1$  and  $^1S_0$  nucleon-nucleon channels, the CDD poles at  $M = 3203$  MeV and  $M = 2916$  MeV are associated with the primitives at  $M_\alpha = 2047$  MeV and  $M_\alpha = 2006$  MeV, respectively. The model we used ensures that the  $D$  function has the correct analytical properties on the first Riemann sheet of the complex  $s$  plane and provides the partial-wave amplitudes that satisfy the generalized Low scattering equation.

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