Effect of single-particle splitting in the exact wave function of the isovectorial pairing Hamiltonian

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The structure of the exact wave function of the isovectorial pairing Hamiltonian with nondegenerate single-particle levels is discussed. The way that the single-particle splittings break the quartet condensate solution found for $N = Z$ nuclei in a single degenerate level is established. After a brief review of the exact solution, the structure of the wave function is analyzed and some particular cases are considered where a clear interpretation of the wave function emerges. An expression for the exact wave function in terms of the isospin triplet of pair creators is given. The ground-state wave function is analyzed as a function of pairing strength, for a system of four protons and four neutrons. For small and large values of the pairing strength a dominance of two-pair (quartets) scalar couplings is found, whereas for intermediate values enhancements of the nonscalar couplings are obtained. A correlation of these enhancements with the creation of Cooper-like pairs is observed.

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I. INTRODUCTION

Pairing is an important piece of nuclear structure studies. Traditional approaches to pairing in nuclei followed those developed in studying metal superconductivity (i.e., BCS theory), a valid approximation in the thermodynamical limit ($A \to \infty$). Finite-size corrections, relevant in the nuclei case, can be incorporated by projecting to a subspace with a fixed number of particles [1]. Alternatively, for some particular (but relevant) cases the pairing Hamiltonian with nondegenerate single-particle levels can be diagonalized in a space with a fixed number of particles. This solution, found by Richardson [2] five years after the BCS proposal, received a recent renewal attention with the experimental advent of ultrasmall superconducting grains [3]. The Richardson solution for the so-called SU(2) pairing considers only correlations between like particles, electron-electron for metal superconductors, and proton-proton and neutron-neutron for nuclear matter. Contrary to metal superconductors, the nuclei isospin symmetry demands pairing between unlike particles (proton-neutron), which gives a richer phenomenology. The matrix elements of this Hamiltonian were calculated years ago by Hecht [4] and the simplest cases were studied by diagonalizing numerically the Hamiltonian matrix. Other traditional approaches consider pairing in a single degenerate level [5,6]; in this case the pairing Hamiltonian exhibits a dynamical symmetry and complete expression for energies and wave functions have been given; in the $N = Z$ case the wave function is a condensate of four-particle bosons. It is the purpose of this paper to study the effect of single-particle splittings in the wave function of pairing Hamiltonians, and evaluate to what extent these splittings break the boson condensate of quartets. The exact solution for the so-called isovectorial pairing Hamiltonian including single-particle levels was more challenging than in the SU(2) case, and first attempts to find it [7] were shown to be valid only when the number of pairs is less than or equal to two [8]. More refined mathematical tools to deal with the complexity of the exact solution had to be developed to include proton-neutron correlations. In this context, Links *et al.* [9] succeeded in demonstrating the integrability of the Hamiltonian with proton-neutron correlations; moreover, they derived the set of nonlinear equations whose solutions give the exact wave functions and energies. Their derivation made use of the so-called inverse scattering method, the SO(5) structure underlying the isovectorial pairing Hamiltonian and an appropriate Bethe ansatz. Numerical results and physical interpretation of the solutions were reported in Ref. [10]. In this contribution a more detailed study of the wave function is presented to determine its structure in the region where neither single-particle levels nor pairing dominate. As discussed in the last section this situation can be present in real nuclei in the *fpg* and *sdgh* shells.

It is known [11,12] that the BCS solution applied to isovectorial and/or isoscalar pairing Hamiltonians fails to describe quartet correlations and quartet condensation. The results presented in this paper could be useful to develop approximative methods to adequately describe the correlations of isovectorial and isoscalar pairing Hamiltonians. Additionally, the exact solution can be used as a testing ground to these approximative methods, which eventually could be useful in more general contexts. Condensation and clusterization of α-like particles and the determination of α-transfer probabilities in $N = Z$ nuclei are issues where the present exact solution could shed some light.

II. THE EXACT SOLUTION

We briefly review the exact solutions for the SU(2) and SO(5) pairing Hamiltonians in the seniority zero case (non-unpaired particles), and compare the respective numerical results, particularly the dependence of pairing energy on isospin T. The reduced pairing Hamiltonian for like particles reads

$$\widehat{\mathbf{H}}_{\mathrm{SU}(2)} = \sum_i \epsilon_i (\widehat{\mathbf{N}}_{pi} + \widehat{\mathbf{N}}_{ni}) - g \sum_{ij} (\widehat{\mathbf{b}}^\dagger_{1i} \widehat{\mathbf{b}}_{1j} + \widehat{\mathbf{b}}^\dagger_{-1i} \widehat{\mathbf{b}}_{-1j}), \tag{1}$$

where the operators in the first term are, respectively, number operators for protons and neutrons in different single-particle levels. The interaction part is written in terms of time-reversed pair operators $\widehat{\mathbf{b}}_{1i}^{\dagger} = \widehat{\mathbf{p}}_{im}^{\dagger}\widehat{\mathbf{p}}_{i\bar{m}}^{\dagger}$ and $\widehat{\mathbf{b}}_{-1i}^{\dagger} = \widehat{\mathbf{n}}_{im}^{\dagger}\widehat{\mathbf{n}}_{i\bar{m}}^{\dagger}$, where $\widehat{\mathbf{p}}_{im}^{\dagger}$ and $\widehat{\mathbf{n}}_{im}^{\dagger}$ are, respectively, proton and neutron creation operators in the ith single-particle level. The first index in the pair operators is the isospin projection (τ). The Bethe ansatz suitable to solve the previous Hamiltonian in the case of non-unpaired particles is (for the most general case see Ref. [13]):

$$|\psi\rangle = \prod_{s}^{M_p} \widehat{\mathbf{b}}_1^{\dagger}(e_s) \prod_{q}^{M_n} \widehat{\mathbf{b}}_{-1}^{\dagger}(e_q)|O\rangle, \tag{2}$$

where M_p and M_n are, respectively, the number of proton-proton and neutron-neutron pairs. The pair operators $\widehat{\mathbf{b}}_1^{\dagger}(e_s)$ are a linear combination of the pairs in each single-particle level:

$$\widehat{\mathbf{b}}_{\tau}^{\dagger}(e_s) = \sum_i \frac{\widehat{\mathbf{b}}_{\tau i}^{\dagger}}{2\epsilon_i - e_s} \quad (\tau = -1, 0, 1).$$

By applying the Hamiltonian (1) to Eq. (2), we get a term proportional to the original ansatz and terms perpendicular to it. By letting the factors multiplying the nonproportional terms be zero, we guarantee that the ansatz will be an eigenvector, then by reading the term multiplying the ansatz we get the respective eigenvalue (for details see, e.g., Ref. [14]). Here, we present the results. The energies are given:

$$E = \sum_s e_s, \tag{3}$$

where the parameters e_s (as many as nucleon pairs) have to satisfy the following set of nonlinear equations:

$$\sum_{p\neq s}^{M_\tau} \frac{2}{e_s - e_p} + \sum_j \frac{1}{2\epsilon_j - e_s} = \frac{1}{g},$$

with $\tau = n, p$, and $s = 1, \ldots, M_\tau$.

The inclusion of proton-neutron correlations coupled to isospin $T = 1$ changes the Hamiltonian to:

$$\widehat{\mathbf{H}}_{\mathrm{SO}(5)} = \sum_i \epsilon_i(\widehat{\mathbf{N}}_{pi} + \widehat{\mathbf{N}}_{ni}) - g\sum_{ij}(\widehat{\mathbf{b}}_{1i}^{\dagger}\widehat{\mathbf{b}}_{1j} + \widehat{\mathbf{b}}_{0i}^{\dagger}\widehat{\mathbf{b}}_{0j} + \widehat{\mathbf{b}}_{-1i}^{\dagger}\widehat{\mathbf{b}}_{-1j}). \tag{4}$$

The middle term in the interaction part is written in terms of the pairs $\widehat{\mathbf{b}}_{0i}^{\dagger} = \frac{1}{\sqrt{2}}(\widehat{\mathbf{n}}_{im}^{\dagger}\widehat{\mathbf{p}}_{i\bar{m}}^{\dagger} + \widehat{\mathbf{p}}_{im}^{\dagger}\widehat{\mathbf{n}}_{i\bar{m}}^{\dagger})$. The Bethe ansatz to diagonalize this Hamiltonian is more involved and includes a new set of parameters (ω_p). For a state of isospin T and M nucleon-nucleon pairs, in the seniority zero case, the ansatz reads (for the general case with unpaired particles see Ref. [10]):

$$|\Psi\rangle = \left(\prod_s^M \widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)\right) \prod_p^{M_\omega} \left(\sum_s^M \frac{\overleftarrow{\mathbf{I}}_{s+}}{e_s - \omega_p}\right)|O\rangle. \tag{5}$$

The number of pairs (M) is equal to the number of pair energies (e_s parameters), whereas the number of ω_p parameters is given by $M_\omega = M - T$. The operators $\overleftarrow{\mathbf{I}}_{s+}$ act upon the $\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)$ pair creators labeled with the same index s, increasing the isospin projection of the pair operator: $\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)\overleftarrow{\mathbf{I}}_{s+} = \widehat{\mathbf{b}}_0^{\dagger}(e_s)$, $\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)(\overleftarrow{\mathbf{I}}_{s+})^2 = \widehat{\mathbf{b}}_1^{\dagger}(e_s)$, and $\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)(\overleftarrow{\mathbf{I}}_{s+})^3 = 0$.

The ω_p parameters determine the isospin couplings of the wave function. The eigenvalues of the Hamiltonian can be obtained from the expression (3), but the nonlinear equations are modified and involve both the e_s and ω_p parameters:

$$\sum_{q\neq s}^{M} \frac{2}{e_s - e_q} + \sum_q^{M_\omega} \frac{1}{\omega_q - e_s} + \sum_j \frac{1}{2\epsilon_j - e_s} = \frac{1}{g},$$

$$\sum_q^{M} \frac{2}{e_q - \omega_p} + \sum_{q\neq p}^{M_\omega} \frac{2}{\omega_p - \omega_q} = 0,$$

with $s = 1, \ldots, M$ and $p = 1, \ldots, M_\omega$.

We solved the nonlinear equations by using a standard Newton's method, as described in Ref. [15]. To illustrate the effect of proton-neutron correlations, we calculated the binding energy from proton-neutron pairing, defined:

$$BE_{PN} \equiv \Delta E_{\mathrm{PairSU(2)}} - \Delta E_{\mathrm{PairSO(5)}}, \tag{6}$$

where $\Delta E_{\mathrm{PairSU(2)}}$ and $\Delta E_{\mathrm{PairSO(5)}}$ are obtained from the ground-state energies of Hamiltonians (1) and (4), by subtracting the contribution of the single-particle levels $\Delta E_{\mathrm{Pair}} \equiv _g\langle\Psi|H_g|\Psi\rangle_g - _o\langle\Psi|H_{g=0}|\Psi\rangle_o$.

For concreteness, a schematic system is considered, which consists of $A = 48$ nucleons moving in a space of 100 fourfold degenerate (two spin and two isospin projections) and equally spaced singe-particle levels [$\epsilon_i = \epsilon_o(i-1)/2$].

In Fig. 1, proton-neutron binding energy is plotted as a function of proton number (Z) for $g/(\overline{\Delta\epsilon}) = 0.34$, where $\overline{\Delta\epsilon}$ is the mean spacing among the single-particle levels. As expected, the proton-neutron correlations increase the binding energy by a factor that grows as T approaches to zero. Pairing

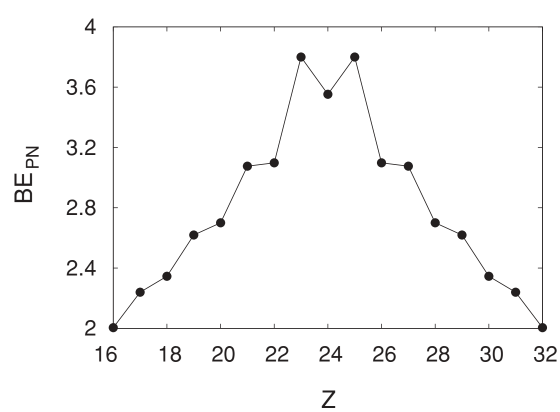

FIG. 1. Binding energy from proton-neutron pairing as a function of proton number (Z), for a system of $A = 48$ nucleons moving in 100 fourfold degenerate and equally spaced ($\epsilon_0/2$) single-particle levels. The ratio between the coupling constant and mean single-particle spacings is $g/(\overline{\Delta\epsilon}) = 0.34$.

energy in the SU(2) case is not very sensitive to the number of protons, whereas, in the isovectorial case, it increases as we approach the symmetric case $N = Z$. The figure makes evident the so-called Wigner energy (i.e., the increase of binding energy as the number of protons equals that of neutrons).

III. THE WAVE FUNCTION

Contrary to the Hamiltonian eigenvalues, which depend only on the pair energies e_s, the wave function depends also on the ω_p parameters. The numerical solution of the nonlinear equations allows us to get insight about the structure of the many-body wave function. In $g \sim 0$, the pair energies are real and close to twice the single particle energies ($2\epsilon_i$), this corresponds to a simple filling of the single-particle levels up to the Fermi energies [16]; we call it a normal state. For a critical g two of the pair energies become complex, signaling the simultaneous creation of two Cooper-like pairs [17], and marking the crossing to a superconducting state. As the coupling increases more and more pairs leave the real axis to form two arcs in the complex plane.

In Fig. 2 the values of the e_s and ω_p parameters are plotted in the complex plane for the same pairing strength as Fig. 1 $[g/(\overline{\Delta\epsilon}) = 0.34]$ and for two cases: $N > Z$ in the top panel and $N = Z$ in the bottom panel. In the $N > Z$ case, the upper arc is formed by four isolated e_s parameters, whereas the lower arc is composed by four e_s parameters closely surrounded each by two ω_p parameters. The rest of the pair energies are real, which implies these pairs are still in the normal state, occupying the single-particle levels. To determine the nature of the Cooper pairs created, let us consider the wave function (5) in the limit where eight ω_p parameters (say $p = 1, \ldots, 8$) approach in pairs to four complex e_s parameters (say $s = 1, \ldots, 4$); we label the rest of the complex pairs (those of the upper arc) by $s = 5, \ldots, 8$. The limit, $\lim_{\omega_p \to e_s} \sum_q \frac{\overleftarrow{\mathbf{I}}_{q+}}{e_q - \omega_p} = \frac{\overleftarrow{\mathbf{I}}_{s+}}{e_s - \omega_p}$, allows us to approximate the exact wave function (5) as

$$\lim_{(\omega_{2s-1},\omega_{2s}) \to e_s} |\Psi\rangle = \widehat{\mathbf{P}}_{\text{normal}} \prod_{s=1}^{4} \left(\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s) \frac{\overleftarrow{\mathbf{I}}_{s+}}{e_s - \omega_{2s-1}} \frac{\overleftarrow{\mathbf{I}}_{s+}}{e_s - \omega_{2s}} \right) \times \prod_{s=5}^{8} \widehat{\mathbf{b}}_{-1}^{\dagger}(e_s) |O\rangle,$$

where $\widehat{\mathbf{P}}_{\text{normal}}$ represents the wave function factor depending on real e_s close to twice the single-particle energies, with leading term [16] $\widehat{\mathbf{P}}_{\text{normal}} \approx c_1 \prod_i \widehat{\mathbf{b}}_{1i} \prod_j \widehat{\mathbf{b}}_{-1j}$. Then, by applying the $\overleftarrow{\mathbf{I}}_{s+}$ operators we get, finally,

$$|\Psi\rangle \approx C\, \widehat{\mathbf{P}}_{\text{normal}} \prod_{s=1}^{4} \widehat{\mathbf{b}}_{1}^{\dagger}(e_s) \prod_{s=5}^{8} \widehat{\mathbf{b}}_{-1}^{\dagger}(e_s) |O\rangle,$$

where C is an irrelevant constant. The wave function has, in this limit, the same structure as the exact solution in the SU(2) case, that is, a many-body wave function where a certain number of pairs are occupying the deepest single-particle levels, a group of pairs (those of the upper arc) is forming a superfluid state of neutron-neutron pairs $[\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)]$, and another set (those of the lower arc) is forming a superconducting state of proton-proton pairs $[\widehat{\mathbf{b}}_{1}^{\dagger}(e_s)]$. The previous description is valid only as a leading approximation to the exact solution; corrections, which additionally restore the isospin symmetry, come from the fact that the two ω_p parameters are not exactly equal to the nearest e_s parameter. A different picture emerges when the number of neutrons equals that of protons (Fig. 2, bottom panel); in this case the upper and lower arcs approach to give rise to several sets formed by two e_s and two ω_p parameters. The previous simple picture of two different like-particle superfluids disappears and a superfluid state of quartets emerges. This quartet structure, already anticipated by other authors (see Refs. [6] and [18], e.g.), is the simplest one that can accommodate simultaneously like particles and proton-neutron correlations.

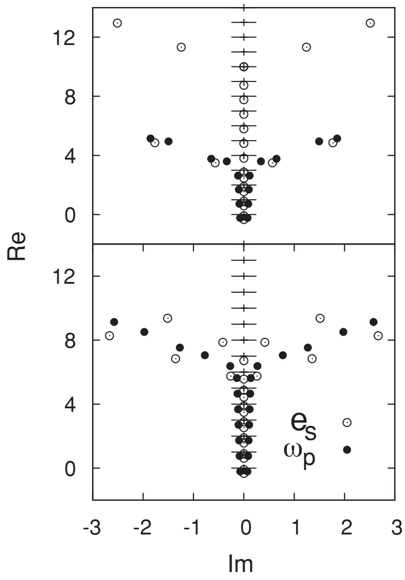

FIG. 2. e_s (open circles) and ω_p (solid circles) parameters in the complex plane for $N > Z$ ($N = 32$ and $Z = 16$) in top panel, and $N = Z = 24$ in bottom panel. The system is the same as that of Fig. 1. The tiny horizontal lines are the values of twice the single-particle energies ($2\epsilon_i$).

The discussion of the previous paragraphs allowed us to get a qualitative general overview of the wave function in different cases, nevertheless, quantitative results require a more detailed analysis of the Bethe ansatz (5). In a first stage we will express the wave function entirely in terms of pair creators; a numerical analysis in the $N = Z$ case will be presented in the next section. To express the Bethe ansatz in terms of pair creators, we apply the $\overleftarrow{\mathbf{I}}_{s+}$ operators to the pair creators $\widehat{\mathbf{b}}_{-1}^{\dagger}(e_s)$. After a straightforward calculation, the resulting wave function is a sum of products of the three different types of pair creators ($\tau = -1, 0, 1$):

$$|\Psi\rangle = \sum_{G \in \mathcal{G}} \frac{1}{2^{|E_1|}} \text{Perma}(M_G) \times \prod_{e_s \in E_1} \widehat{\mathbf{b}}_{1}^{\dagger}(e_s) \prod_{e_p \in E_0} \widehat{\mathbf{b}}_{0}^{\dagger}(e_p) \prod_{e_q \in E_{-1}} \widehat{\mathbf{b}}_{-1}^{\dagger}(e_q) |O\rangle, \quad (7)$$

where G is a partition in three subsets (E_{-1}, E_0, E_1) of the set of pair energies $\{e_1, \ldots, e_M\}$, $|E_\tau|$ is the cardinality of subset E_τ, and the sum runs over the set ($\mathcal{G}$) of three-subset partitions restricted to the conditions: $0 \leqslant |E_1| \leqslant r_{\max}$, $|E_0| = M - T - 2|E_1|$, and $|E_{-1}| = T + |E_1|$, with $r_{\max} = (M-T)/2$ if $M - T$ is even and $r_{\max} = (M - T - 1)/2$ otherwise. Perma(M_G) is the permanent of matrix M_G, which depends, for each partition, on the elements of the subsets E_1 and E_0, and on the ω_p parameters. Let $E_1 = \{e_{s_1}, e_{s_2} \ldots e_{s_{|E_1|}}\}$ and $E_0 = \{e_{q_1}, e_{q_2} \ldots e_{q_{|E_0|}}\}$ be the elements of respective subsets for a given partition G, the matrix M_G is defined:

$$M_G = \begin{pmatrix} (e_{s_1} - \omega_1)^{-1} & \cdots & (e_{s_1} - \omega_{M_\omega})^{-1} \\ (e_{s_1} - \omega_1)^{-1} & \cdots & (e_{s_1} - \omega_{M_\omega})^{-1} \\ (e_{s_2} - \omega_1)^{-1} & \cdots & (e_{s_2} - \omega_{M_\omega})^{-1} \\ (e_{s_2} - \omega_1)^{-1} & \cdots & (e_{s_2} - \omega_{M_\omega})^{-1} \\ \vdots & \ddots & \vdots \\ (e_{s_{|E_1|}} - \omega_1)^{-1} & \cdots & (e_{s_{|E_1|}} - \omega_{M_\omega})^{-1} \\ (e_{s_{|E_1|}} - \omega_1)^{-1} & \cdots & (e_{s_{|E_1|}} - \omega_{M_\omega})^{-1} \\ (e_{q_1} - \omega_1)^{-1} & \cdots & (e_{q_1} - \omega_{M_\omega})^{-1} \\ (e_{q_2} - \omega_1)^{-1} & \cdots & (e_{q_2} - \omega_{M_\omega})^{-1} \\ \vdots & \ddots & \vdots \\ (e_{q_{|E_0|}} - \omega_1)^{-1} & \cdots & (e_{q_{|E_0|}} - \omega_{M_\omega})^{-1} \end{pmatrix}.$$

Note that each element of subset E_1 appears in two consecutive rows, then the dimension of the matrix is $(M - T) \times (M - T)$. The expression (7) is the Bethe ansatz written entirely in terms of pair creators. Compare this expression with the equivalent ansatz for the SU(2) pairing Hamiltonian (2); its complexity explains why the first attempts to derive the exact solution of the isovectorial pairing Hamiltonian beginning from a Bethe ansatz written entirely in terms of pair creators, were unable to go further than three pairs.

IV. APPLICATION TO THE $N = Z$ CASE

In Ref. [6] the authors studied the isovectorial pairing Hamiltonian in the case of a completely degenerate single-particle level. This situation can be reached from the present model in the limit $g \to \infty$, where the pairing interaction dominates and, consequently, the single-particle details are completely diluted. In this limit, the ground-state wave function in the even-even symmetric $N = Z$ case, can be written:

$$(\widehat{\mathbf{b}}^\dagger \cdot \widehat{\mathbf{b}}^\dagger)^{M/2} |O\rangle, \tag{8}$$

where the central dot represents an isospin scalar product of the isovector-pair operators $\widehat{\mathbf{b}}_\tau^\dagger \equiv \sum_i \widehat{\mathbf{b}}_{i\tau}^\dagger$, with index i running over all the (degenerate) single-particle levels. This wave function corresponds to a condensate of quartets [19]. How does the system cross from a normal state of pairs occupying single-particle levels ($g = 0$) to a boson condensate of quartets ($g \to \infty$)? The formulas presented in the previous section, allow us to answer this question. The first step is to particularize the general expression (7) to the $N = Z$ case, however, the combinatorial problem involved in the determination of the wave-function norm prevented us from going further, and we decided to study the simplest even-even nontrivial system, which consists of four protons and four neutrons. Although arbitrary levels and degeneracies can be easily accommodated within the present formalism, for concreteness, six fourfold degenerate and equally spaced single-particle levels were considered. Knowing that the wave function can be written in terms of the operators $\widehat{\mathbf{b}}_\tau^\dagger(e_s)$, which are $T = 1$ tensor operators, the general form for a $T_o = 0$ state must be [8]:

$$|\Psi\rangle = \sum_{T=0,1,2} |\Psi_T\rangle = \mathcal{N} \sum_{T=0,1,2} A_T [[\widehat{\mathbf{b}}^\dagger(e_1) \times \widehat{\mathbf{b}}^\dagger(e_2)]^T \times [\widehat{\mathbf{b}}^\dagger(e_3) \times \widehat{\mathbf{b}}^\dagger(e_4)]^T]_0^{T_o=0} |O\rangle. \tag{9}$$

The coefficients A_T can be determined by comparing the previous expression to the exact wave function written in terms of pair creators [Eq. (7)]. Once we have determined the coefficients A_T, we have to calculate the norm of each component to obtain the probability of finding the state in each of these three different T-coupled components. This calculation is very cumbersome and prevents us from going further than the simplest nontrivial case (four protons and four neutrons). The result is shown in Fig. 3. The scalar ($T = 0$) coupling dominates completely the wave function for small ($g/(\overline{\Delta\epsilon}) < 0.36$) and large ($g/(\overline{\Delta\epsilon}) > 0.90$) values of the coupling constant, although the nature of the states in both intervals is very different: for small values the pair energies are real (see Fig. 3, bottom panel) and the wave function consists of particles occupying the first two single-particle levels; each level can accommodate two neutrons and two protons, and there is no correlation between particles in different levels. Particles in the first level couple to $T = 0$ and so do particles in the second one. The resulting wave function is a product of normal quartets, occupying, each, different single-particle levels. For large values of g all the pair energies are complex (Fig. 3, bottom panel) and the wave function is a product of two isoscalar quartets, consisting each of two Cooper-like pairs. Each quartet has the same structure

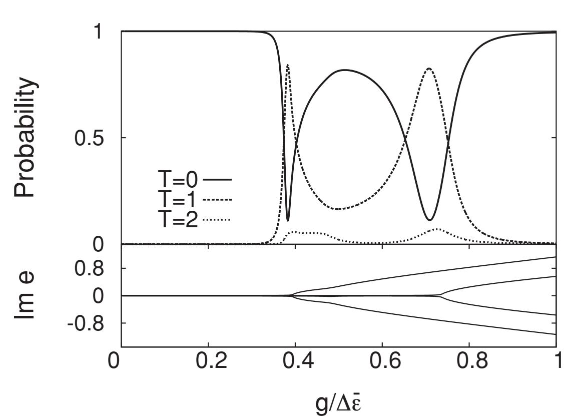

FIG. 3. Probability $(|\langle\Psi_T|\Psi_T\rangle|/|\langle\Psi|\Psi\rangle|)^2$ of finding the ground-state wave function in each of the three different T-coupled components (top) and Im part of the e_s parameters (bottom) as functions of the ratio between the coupling constant and the mean single-particle spacings $g/(\overline{\Delta\epsilon})$. The number of protons and neutrons is $N = Z = 4$, and the number of fourfold equally spaced single-particle levels is six.

in terms of single-particle pairs and contributes equal to the total energy; the resulting wave function is a finite-number version of a boson condensate. In this limit the wave function can be well approximated by the wave function of the totally degenerate solution [Eq. (8)]. For intermediate values of g the wave-function structure is more involved; for two critical values of the ratio $g/(\overline{\Delta\epsilon})$—($g/(\overline{\Delta\epsilon}) \approx 0.38$ and $g/(\overline{\Delta\epsilon}) \approx 0.71$)—two sudden enhancements of the $T = 1$ quartets are observed. These enhancements are related with the formation of two Cooper pairs, which are signaled by the appearance of an imaginary part in two of the pair energies (Fig. 3, bottom panel). In the first enhancement the probability of the $T = 1$ component increases up to 84%, whereas in the second one the percentage reaches 82.5. In this intermediate range the pairing energy is of the same order as the single-particle spacings, and the interplay between both terms in the Hamiltonian results in a complicated structure of the wave function, where scalar and $T = 1$ quartets play the dominant role. For all g, the contribution of $T = 2$ coupled quartets is very small; the greatest peak in the probability of this component reaches about 7%.

How do the given results compare with realistic values of pairing strength and mean single-particle spacings? A rough idea of the ratio $g/(\overline{\Delta\epsilon})$ for realistic cases can be obtained from a standard parametrization of the shell model [20]:

$$H = \hbar\omega_0(\eta + 3/2) - \kappa\hbar\omega_0(2L \cdot S + \mu L^2) - gH_P,$$

with $\hbar\omega_0 = 41A^{-1/3}[MeV]$, $g = g_p \approx g_n \approx g_{pn} = 19/A[MeV]$, and $\kappa = 0.08$, $\mu_\pi \approx \mu_\nu = 0.0$ for the *fpg* shell, and $\kappa = 0.0637$, $\mu_\pi \approx \mu_\nu = 0.51$ for the *sdgh* shell. In the $N = Z$ case we obtain a ratio ranging from $g/(\overline{\Delta\epsilon}) = 0.283$ ($N = Z = 50$) to 0.398 ($N = Z = 30$) in the *fpg* shell and a ratio from 0.700 ($N = Z = 82$) to 0.949 ($N = Z = 52$) in the *sdgh* shell. Even though these values for the ratio between the pairing strength and the mean level spacings of minor shells in major shells, are in the range where the nonscalar couplings are enhanced (compare to Fig. 3), the previous estimate can only give us an idea of the competition between pairing and single-particle splittings in real nuclei. A more refined study requires considering the details of the single-particle levels and not only the mean value of the level spacings (a large spacing below the Fermi energy is more effective to inhibit pairing effects than a small one, even if the mean spacings of the single-particle levels are equal). Additionally, it is worth mentioning that in a realistic calculation for nuclei far from closed shells, quadrupole-quadrupole interactions (not considered at all in this study) and Nilsson deformed levels have to be considered. The latter can be easily accommodated within the present approach; with an adequate parametrization for the deformed Nilsson potential a general study of the competition between single-particle splittings and pairing in $N = Z$ nuclei can be done. Work in this direction will be published elsewhere.

The observed enhancements of the nonisoscalar components of the wave function are of relevance in the determination of α-transfer probabilities [21], which must be affected by structural changes of the wave function. The enhancements of the nonscalar components must reduce these probabilities and sudden reductions of them in real nuclei could be related with the competition between singe-particle energies and pairing interactions. Work in this direction is currently underway.

V. CONCLUSIONS

A detailed study of the exact wave function for the isovectorial pairing Hamiltonian was presented; comparison with the like-particle pairing Hamiltonian, reproduces, from the exact solution, well-known results: the effect of proton-neutron correlation is enhanced when the number of protons equals that of neutrons, the like-particle pairing approximates the isovectorial pairing when $T \neq 0$, however, corrections and isospin symmetry restoration are obtained when proton-neutron correlations are incorporated. For the symmetric nuclear matter case ($T = 0$), the exact solution of the isovectorial pairing Hamiltonian yields a richer phenomenology for which the like-particle pairing is blind. A quartet structure of the wave function appears, which is transformed in a boson condensate of isoscalar quartets when the pairing energy is much greater than the single-particle energy splittings. For intermediate values of the coupling constant, a more involved structure appears, the wave function is a sum of isospin coupled quartets, and no factorial form of $T = 0$ quartets appears. For certain pairing strengths, the product of scalar quartets is even less probable than the $T = 1$ coupling of quartets. The enhancements of the nonscalar coupled quartets are related with the pairwise creation of Cooper pairs. Realistic values for the ratio $g/(\overline{\Delta\epsilon})$ in a spherical shell–nuclear model are of the order of magnitude of the range where neither single-particle levels nor pairing dominate. The obtained enhancements of the nonscalar quartets in the ground-state wave function of $N = Z$ nuclei could be of interest in the determination of α-transfer probabilities, which must be reduced when the nonscalar terms dominate the wave function. The present study can be extended to include more general situations such as unpaired particles, realistic single-particle energies and degeneracies, and different single-particle energies for protons and neutrons (breaking the isospin symmetry). Likewise, the present study can be extended to the SO(8) pairing Hamiltonian, which includes, additionally, proton-neutron correlation in the $T = 0$ channel. The combinatorial problem involved in the determination of the norm of the isospin-coupled components, has to be worked out to study systems with more than eight nucleons. The present exact solution can be used, as other exactly solvable models have, as a testing ground for approximative methods to study pairing phenomenology. Finally, although this presentation was focused on nuclear matter, the presented exact solution is of interest as a mathematical problem (integrable and exactly solvable models) and in the study of mesoscopic atomic systems.

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