Hyperspherical effective interaction for nonlocal potentials

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The effective interaction hyperspherical-harmonics method, formulated for local forces, is generalized to accommodate nonlocal interactions. As for local potentials this formulation retains the separation of the hyperradial part leading solely to a hyperspherical effective interaction. By applying the method to study ground-state properties of ^{4}He with a modern effective-field-theory nucleon-nucleon potential model (Idaho-N3LO), one finds a substantial acceleration in the convergence rate of the hyperspherical-harmonics series. Also studied are the binding energies of the six-body nuclei ^{6}He and ^{6}Li with the JISP16 nuclear force. Again an excellent convergence is observed.

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I. INTRODUCTION

The effective interaction method is one of the pillars of contemporary nuclear many-body theory. Its roots can be traced back to the early theoretical effort to bridge the gap between the hard-core bare nucleon-nucleon (NN) interaction of the time and the soft phenomenological potential models that were successfully employed in shell-model calculations to reproduce nuclear properties. Over the years different approaches have been developed to derive the effective interaction. The most notable are the Bloch-Horowitz equation, the Brückner G matrix, the Kuo-Brown folded diagram method, and the Lee-Suzuki similarity transformation. In contrast with the many-body community, until the middle of the 1990s effective interactions were rarely used, if at all, in nuclear few-body calculations. Into this field the effective interaction was introduced by Navratil and Barrett [1] who have performed a no-core shell-model (NCSM) calculation in a harmonic-oscillator (HO) basis. By using the Lee-Suzuki similarity transformation [2], Navratil and Barrett have shown that an effective interaction leads to substantial acceleration in the convergence of the HO expansion, and that reliable convergent results can be obtained with this method.

By following the NCSM example, we have formulated an effective interaction for the hyperspherical-harmonics (HH) expansion [3]. The HH basis is not obtained from a confining potential, which is different from the HO case, and therefore leads to a better description of the asymptotic tail of the wave function and, consequently, of low-energy observables. By constructing an effective interaction for the HH one aims to keep this advantage intact. The Hilbert space for the HH basis is an outer product of hyper-radial and hyperspherical sections. The hyperspherical part exhibits a very slow convergence for potentials with a strong short-range repulsion. Therefore, in such cases it is desirable to construct an effective interaction that accelerates the convergence of the hyperspherical expansion. Similar problems do not exist for the hyper-radial part which thus should be left untouched. Consequently, one is led into a solution where the model space is defined through the hyperspherical (grand) angular momentum quantum number K while the hyper-radial coordinate ρ appears in the HH effective interaction as a *parameter*. The starting point in constructing this effective interaction is a pseudo-two-body Hamiltonian that contains the hyperspherical kinetic energy but no hyper-radial derivatives. The HH effective interaction (EIHH) derived that way is by construction state dependent and, in addition, it depends on the collective hyper-radial coordinate ρ. As has been shown in various applications (see, e.g., [4–9]) the EIHH technique leads to a tremendous acceleration of the HH convergence rate.

The introduction of modern potential models, such as the CD-Bonn [10] or the effective-field-theory (EFT) $NN + NNN$ potentials [11–13], poses a new challenge to the EIHH method. In contrast with local realistic $NN + NNN$ potentials, such as the Argonne AV18 NN force [14] and the Urbana IX three-nucleon force [15], the CD-Bonn and the EFT potentials are nonlocal, and therefore the existing EIHH formalism is not suitable to deal with them.

In this work we describe a generalization of the EIHH method to also accommodate nonlocal forces. In this formulation we aim to retain the advantages of the HH expansion, in particular the previously mentioned good long-range behavior. For simplicity we shall restrict the discussion to two-body interactions. A generalization of our results to the full nuclear Hamiltonian, also including three-body forces, is straightforward.

The paper is organized as follows. In Sec. I we briefly review the original formulation of the EIHH method for local potentials. In Sec. II we describe its generalization to nonlocal forces. Numerical results presenting the merits of the EIHH method for nonlocal forces are given in Sec. III.

II. OUTLINE OF THE EIHH METHOD—LOCAL POTENTIALS

In the effective interaction approach [3,16,17] the lowest eigenvalues of an A-body Hamiltonian,

$$H^{[A]} = H_0 + V, \tag{1}$$

are treated in the following way. The Hilbert space of $H^{[A]}$ is divided into a model space and a residual space through the use of the eigenprojectors P and Q of H_0, which satisfy the

relations

$$[H_0, P] = [H_0, Q] = 0, \quad QH_0P = PH_0Q = 0, \\ P + Q = 1. \tag{2}$$

The Hamiltonian $H^{[A]}$ is then replaced by the effective model space Hamiltonian

$$H^{[A]\text{eff}} = PH_0P + PV^{[A]\text{eff}}P, \tag{3}$$

which by construction has the same energy levels as the low-lying states of $H^{[A]}$. In general the effective interaction appearing in Eq. (3) is an A-body interaction. Its construction is as difficult as finding the full-space solutions. Therefore, one has to approximate $V^{[A]\text{eff}}$. However, one must build the approximate effective potential in such a way that it coincides with the bare one for $P \longrightarrow 1$, so that an enlargement of the model space leads to a convergence of the eigenenergies to the *true* values. The NCSM and the EIHH methods are developed along these lines.

In the HH formalism a Hamiltonian

$$H^{[A]} = \sum_{i=1}^{A} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i<j}^{A} v_{ij} \tag{4}$$

for A particles with particle momenta $\boldsymbol{p}_i$ and equal masses m is written as

$$H^{[A]} = T_\rho + T_K(\rho) + V^{[A]}(\rho, \Omega_A), \tag{5}$$

where

$$V^{[A]}(\rho, \Omega_A) \equiv \sum_{i<j}^{A} v_{ij} \tag{6}$$

denotes the bare two-body potential and

$$T_\rho = -\frac{1}{2m}\Delta_\rho, \quad T_K(\rho) = \frac{1}{2m}\frac{\hat{K}_A^2}{\rho^2} \tag{7}$$

are the hyper-radial and hypercentrifugal kinetic energies, respectively. In the previous equation Δ_ρ is the Laplace operator with respect to the hyper-radial coordinate

$$\rho = \sqrt{\sum_{j=1}^{N} \eta_j^2}, \tag{8}$$

where the various $\boldsymbol{\eta}_j$ denote the $N = A - 1$ Jacobi vectors, while $\hat{K}_A$ is the hyperspherical grand angular momentum operator and Ω_A the $(3A - 4)$-dimensional hyperangle. The wave function is expanded into the HH series,

$$\Psi = \sum_{[K_A]n} C_{n[K_A]} R_n(\rho) \mathcal{Y}_{[K_A]}(\Omega_A), \tag{9}$$

where $R_n(\rho)$ are the hyper-radial basis functions and $\mathcal{Y}_{[K_A]}(\Omega_A)$ are the hyperspherical-harmonics functions coupled with the internal degrees of freedom to yield a completely antisymmetric wave function with definite angular momentum and isospin ($[K_A]$ stands for a set of quantum numbers, see Ref. [3]). The shortcoming of the HH expansion is the slow convergence of the HH series for realistic nuclear forces. On the contrary, usually only a small number of hyper-radial basis states, R_n, is needed. It is therefore desirable to design an effective interaction that accelerates the hyperspherical convergence. These considerations lead to the natural choice of $H_0 = T_K(\rho)$ for the unperturbed Hamiltonian [3]. Accordingly, the model space P is defined as the complete set of HH basis functions with generalized angular momentum quantum number $K_A \leqslant K_P$, and the Q space as the complete set of HH basis functions with $K_A > K_P$. The states will be denoted by $\{|p\rangle,\ p = 1, 2, \ldots, n_P\}$ for the P space and $\{|q\rangle,\ q = n_{p+1}, n_{p+2}, \ldots, n_Q\}$ for the Q space. Of course, in principal one has $n_Q \longrightarrow \infty$, but for actual calculations one has to consider a finite Q space but with a sufficiently large n_Q.

For each value ρ of the hyper-radius an effective adiabatic Hamiltonian is constructed,

$$\mathcal{H}^{[A]\text{eff}}(\rho, \Omega_A) = PT_K(\rho)P + P\,V^{[A]\text{eff}}(\rho, \Omega_A)\,P. \tag{10}$$

We would like to emphasize that in Eq. (10) ρ has to be regarded as a parameter, while the dynamics is described by the variable Ω_A.

As already pointed out, the effective potential would be a complicated A-body interaction, therefore $V^{[A]\text{eff}}$ is approximated by a sum of *two-body* terms

$$V^{[A]\text{eff}} \simeq \sum_{i<j}^{A} v_{i,j}^{[2]\text{eff}}. \tag{11}$$

Owing to the use of antisymmetric wave functions one only needs to calculate the effective interaction operator relative to one pair, since

$$\langle V^{[A]\text{eff}} \rangle \simeq \left\langle \sum_{i<j}^{A} v_{i,j}^{[2]\text{eff}} \right\rangle = \frac{A(A-1)}{2} \langle v_{A,A-1}^{[2]\text{eff}} \rangle. \tag{12}$$

The two-body effective potential $v_{A,A-1}^{[2]\text{eff}}$ is determined as follows. First, for each value ρ of the hyper-radial coordinate one defines a *quasi-two-body* adiabatic Hamiltonian containing the hypercentrifugal kinetic energy and the bare potential between the last two particles

$$\mathcal{H}^{[2]}(\rho\,; \Omega_{A,A-1}) = T_K(\rho) + v_{A,A-1}(\rho, \Omega_{A,A-1}), \tag{13}$$

where the hyperangle $\Omega_{A,A-1} = (\theta_{A,A-1}, \hat{\eta}_{A,A-1})$ is related to the Jacobi vector

$$\boldsymbol{\eta}_{A,A-1} = \sqrt{\frac{1}{2}}(\boldsymbol{r}_A - \boldsymbol{r}_{A-1}) \tag{14}$$

through the relation

$$\boldsymbol{\eta}_{A,A-1} = \rho \sin\theta_{A,A-1} \hat{\eta}_{A,A-1}. \tag{15}$$

The Hamiltonian of Eq. (13) is then transformed into the A-body HH basis,

$$\mathcal{H}^{[2]}_{[K_A],[K'_A]}(\rho) = \delta_{[K_A],[K'_A]} K_A(K_A + 3N - 2)\frac{1}{2m}\frac{1}{\rho^2} \\ + v^{[2]}_{[K_A],[K'_A]}(\rho), \tag{16}$$

and diagonalized. Here

$$v^{[2]}_{[K_A],[K'_A]}(\rho) = \int d\Omega_A\, \mathcal{Y}^*_{[K_A]}(\Omega_A) v_{A,A-1}(\rho, \Omega_{A,A-1}) \\ \times \mathcal{Y}_{[K'_A]}(\Omega_A). \tag{17}$$

Such a diagonalization is easily performed because ρ is only a parameter in $\mathcal{H}^{[2]}$ (there are no derivatives with respect to ρ) and for each value of ρ the Hamiltonian $\mathcal{H}^{[2]}(\rho\,;\Omega_{A,A-1})$ depends only on three variables. This is just due to our choice of the A-$(A-1)$ pair. The obtained eigenstates will be denoted by $|\varphi_j(\rho)\rangle$ as they are continuous functions of ρ.

One proceeds with applying the Lee-Suzuki [2] similarity transformation to $\mathcal{H}^{[2]}$ in order to get the corresponding Hermitian effective Hamiltonian

$$\mathcal{H}^{[2]\text{eff}}(\rho) = \mathcal{U}^{\dagger}(\rho)\,\mathcal{H}^{[2]}(\rho)\mathcal{U}(\rho), \tag{18}$$

where

$$\mathcal{U}(\rho) = [P + \omega(\rho)]\,\frac{1}{\sqrt{P[1+\omega^{\dagger}(\rho)\omega(\rho)]P}}\,. \tag{19}$$

The operator $\omega(\rho)$ is obtained using the following property [2]:

$$\omega(\rho) = Q\,\omega(\rho)\,P. \tag{20}$$

The matrix $\omega(\rho)$ is calculated for each value of ρ taking the n_P states $|\varphi_j(\rho)\rangle$ with the lowest eigenvalues. Each of these states leads to the following system of $(n_Q - n_P)$ equations:

$$\langle q|\varphi_j(\rho)\rangle = \sum_p \langle q|\omega(\rho)|p\rangle\,\langle p|\varphi_j(\rho)\rangle. \tag{21}$$

The $n_P(n_Q - n_P)$ matrix elements $\langle q|\omega(\rho)|p\rangle$ are obtained by solving this equation system (21). Once the effective quasi-two-body Hamiltonian $\mathcal{H}^{[2]\text{eff}}$ is constructed, the effective potential is obtained by a subtraction of the hypercentrifugal kinetic energy

$$v^{[2]\text{eff}}_{A,A-1} = \mathcal{H}^{[2]\text{eff}}(\rho) - T_K(\rho). \tag{22}$$

Using this ρ-dependent effective potential and taking into account Eqs. (10)–(12) one solves the A-body problem with the effective Hamiltonian

$$H^{[A]\text{eff}} = T_\rho + \mathcal{H}^{[A]\text{eff}} = T_\rho + T_K + \sum_{i<j} v^{[2]\text{eff}}_{ij} \tag{23}$$

in the P space. One repeats the procedure, enlarging the P space up to a convergence of the low-lying energies of the A-body system.

We would like to emphasize the following points: (i) The hyper-radius is a parameter rather than a coordinate, and $v^{[2]\text{eff}}$ is determined for various fixed ρ values; therefore, while being a two-body interaction, it depends on the whole A-body system via this collective coordinate. (ii) There is an additional dependence of $v^{[2]\text{eff}}$ on the quantum number K_{A-2} of the residual system [see Eqs. (19) and (20) of Ref. [3]]. (iii) It is evident that $\mathcal{U}(\rho) \longrightarrow 1$ for $P \longrightarrow 1$ and thus $v^{[2]\text{eff}}$ converges to the bare $v_{A,A-1}$; therefore the energy spectrum converges to the exact one. (iv) Via the operator $\mathcal{U}(\rho)$ the effective potential $v^{[2]\text{eff}}$ contains information about a large part of the PQ space interaction, hence the convergence to the exact eigenvalues of $H^{[A]}$ is accelerated with respect to the normal HH expansion.

III. HYPERSPHERICAL EFFECTIVE INTERACTION FOR NONLOCAL POTENTIALS

In the HO effective interaction approach there is no principal difference in the treatment of local and nonlocal forces. Therefore, in the HH case one might be tempted to follow the same steps as in the preceding section, that is, to define $T_K(\rho)$ as H_0, to keep ρ as a parameter, and to substitute the potential

$$v_{A,A-1}(\rho, \Omega_{A,A-1}; \rho', \Omega'_{A,A-1}) = v_{A,A-1}(\boldsymbol{r}, \boldsymbol{r}')$$

in Eq. (13), in order to get the following quasi-two-body Hamiltonian:

$$\mathcal{H}^{[2]}(\rho\,\Omega_{A,A-1}; \rho'\Omega'_{A,A-1}) = T_K(\rho) + v_{A,A-1}(\rho, \Omega_{A,A-1}; \rho', \Omega'_{A,A-1}). \tag{24}$$

By doing so one immediately realizes that such a $\mathcal{H}^{[2]}$ is non-Hermitian since for a given parameter set $\rho \neq \rho'$ one has

$$v_{A,A-1}(\rho, \Omega_{A,A-1}; \rho', \Omega'_{A,A-1}) \neq v^{*}_{A,A-1}(\rho, \Omega'_{A,A-1}; \rho', \Omega_{A,A-1}).$$

A possible remedy to this problem is to include the hyper-radial kinetic energy term in the quasi-two-body Hamiltonian so that ρ will become a dynamic variable. Such a step forces an enlargement of the model space to include not only HH states but also hyper-radial states. In the following we will point out that an effective interaction derived from a quasi-two-body Hamiltonian containing T_ρ will be either useless or will ruin the asymptotic behavior of the wave function, which is one of the most important advantages of the HH method.

To illustrate this point we include T_ρ into H_0 [$H_0 = T_\rho + T_K(\rho)$]. The eigenfunctions of H_0 are then given by $\varphi_E = 1/(q\rho)^{3N/2-1} J_{K+(3N-2)/2}(q\rho)\mathcal{Y}(\Omega_A)$ that have eigenvalues $E = q^2/(2m)$ [here $J_\nu(x)$ are the Bessel functions]. It is now evident that the spectrum becomes independent of the HH quantum numbers and for each HH state there is an infinite number of hyper-radial states. This leads to important consequences for the construction of the corresponding effective interaction from the quasi-two-body $\mathcal{H}^{[2]} = H_0 + v_{A,A-1}$ with a realistic (i.e., nonconfining), two-nucleon potential $v_{i,j}$. In fact, at a sufficiently high energy the kinetic energy term becomes dominant, leading to eigenfunctions similar to the functions φ_E discussed previously. Consequently, at some point the effective interaction constructed that way will be completely ineffective in folding HH states into the P space. One could change this picture by including a confining potential V_0 to H_0, which, of course, eventually will be subtracted. This would be very similar to the effective interaction in the HO case and would lead to an asymptotic behavior of the hyper-radial basis states R_n which is determined by the confining V_0. Thus one would obtain nonphysical basis states. Therefore, we limit our attention to the choice $H_0 = T_K(\rho)$, and seek a way to construct an HH effective interaction based on the same P and Q spaces as in the preceding section. It is evident that for $\rho = \rho'$ the quasi-two-body Hamiltonian $\mathcal{H}^{[2]}$, Eq. (24), is a Hermitian operator in the HH space. In that specific case $\mathcal{H}^{[2]}$ is local and $v^{[2]\text{eff}}_{[K_A],[K'_A]}(\rho, \rho)$ can be constructed following the

steps outlined in Sec. I. Using this result we can approximate the nonlocal effective interaction through the relation

$$v^{[2]\text{eff}}_{[K_A],[K'_A]}(\rho,\rho') = v^{[2]}_{[K_A],[K'_A]}(\rho,\rho') + \delta(\rho-\rho') \times \left[v^{[2]\text{eff}}_{[K_A],[K'_A]}(\rho,\rho) - v^{[2]}_{[K_A],[K'_A]}(\rho,\rho)\right]. \tag{25}$$

In this expression all nonlocal effects regarding the hyperspherical coordinates are incorporated, while the hyperradial part of the nonlocality remains excluded. Because the hyper-radius is a so-called slow variable and, in addition, because NN forces show only moderate nonlocalities this formulation of the HH effective interaction should be rather satisfactory.

Equation (25) represents a nonlocal HH effective interaction derived from the diagonal hyper-radial matrix element of $\mathcal{H}^{[2]}$ in the position representation. This derivation can be generalized to an arbitrary hyper-radial basis. Consider a complete set of hyper-radial states $\{R_n(\rho); n = 1,\ldots\}$. The matrix elements of the quasi-two-body Hamiltonian between basis states of the form $R_n(\rho)\mathcal{Y}_{[K_A]}$ are given by

$$\mathcal{H}^{[2]}_{n[K_A],n'[K'_A]} = \delta_{[K_A],[K'_A]}K_A(K_A+3N-2) \times \frac{1}{2m}\langle n|\frac{1}{\rho^2}|n'\rangle + v^{[2]}_{n[K_A],n'[K'_A]}, \tag{26}$$

where

$$v^{[2]}_{n[K_A],n'[K'_A]} = \int dV\, dV'\, R^*_n(\rho)\mathcal{Y}^*_{[K_A]}(\Omega_A)v_{A,A-1} \times (\rho,\Omega_{A,A-1};\rho',\Omega'_{A,A-1})R_{n'}(\rho')\mathcal{Y}_{[K'_A]}(\Omega'_A) \tag{27}$$

and

$$\langle n|\frac{1}{\rho^2}|n'\rangle = \int \rho^{3A-4}\,d\rho\, R^*_n(\rho)\frac{1}{\rho^2}R_{n'}(\rho). \tag{28}$$

In this representation, the parameters ρ, ρ' are replaced by the indices n, n', and $\mathcal{H}^{[2]}$ is Hermitian in the HH sector only if $n = n'$. Setting $n = n'$, Eq. (26) can be used as a starting point for deriving an effective interaction $v^{[2]\text{eff}}_{n[K_A],n[K'_A]}$, following the procedure in Sec. II for the Hamiltonian $\mathcal{H}^{[2]}_{[K_A],[K'_A]}(\rho)$. The resulting n-diagonal effective interaction $v^{[2]\text{eff}}_{n[K_A],n[K'_A]}$ can then be used to define an effective interaction of the form

$$v^{[2]\text{eff}}_{n[K_A]n',[K'_A]} = v^{[2]}_{n[K_A],n'[K'_A]} + \delta_{n,n'}\left(v^{[2]\text{eff}}_{n[K_A],n[K'_A]} - v^{[2]}_{n[K_A],n[K'_A]}\right). \tag{29}$$

This procedure is valid for an arbitrary choice of hyper-radial basis functions. The question is which of this choices will lead to a better effective interaction (i.e., to a faster convergence of the HH expansion). Before answering this question through numerical experiments, we would like to put it in a different form. Assume that $\{R_n(\rho); n = 1,\ldots,N\}$ is our hyper-radial basis of choice, truncated at some $n = N$. This basis can be used to diagonalize any Hermitian hyper-radial operator $\hat{\mathcal{O}}(\rho,\rho')$,

$$\hat{\mathcal{O}}R^{\hat{\mathcal{O}}}_\nu(\rho) = \lambda_\nu R^{\hat{\mathcal{O}}}_\nu(\rho), \tag{30}$$

where

$$R^{\hat{\mathcal{O}}}_\nu(\rho) = \sum_{n=1}^{N} U_{\nu,n}R_n(\rho) \tag{31}$$

and $U_{\nu,n}$ are the eigenvectors of the matrix $\hat{\mathcal{O}}_{n,n'} = \langle R_n|\hat{\mathcal{O}}|R_{n'}\rangle$ that we approximate through Gaussian integration. Obviously the basis $\{R^{\hat{\mathcal{O}}}_\nu\}$ is defined by the operator $\hat{\mathcal{O}}$, and we can replace the quest for the best basis set by a quest for the best operator $\hat{\mathcal{O}}$. Using this transformation the effective interaction takes the form

$$\begin{aligned} v^{[2]\text{eff}}_{n[K_A]n',[K'_A]} &= \sum_{\nu,\nu'} U^{-1}_{n,\nu}\left[v^{[2]}_{\nu[K_A],\nu'[K'_A]} + \delta_{\nu,\nu'}\left(v^{[2]\text{eff}}_{\nu[K_A],\nu[K'_A]} - v^{[2]}_{\nu[K_A],\nu[K'_A]}\right)\right]U_{\nu',n'} \\ &= v^{[2]}_{n[K_A],n'[K'_A]} + \sum_{\nu} U^{-1}_{n,\nu}\left(v^{[2]\text{eff}}_{\nu[K_A],\nu[K'_A]} - v^{[2]}_{\nu[K_A],\nu[K'_A]}\right)U_{\nu,n'}. \end{aligned} \tag{32}$$

Possible choices for $\hat{\mathcal{O}}$ may be $\hat{\mathcal{O}} = 1$, $\hat{\mathcal{O}} = \hat{\rho}$, or in general $\hat{\mathcal{O}} = \hat{\rho}^m$. In the limit $N \longrightarrow \infty$ and with accurate numerical integration, the operators of the form $\hat{\mathcal{O}} = \hat{\rho}^m$, $m \neq 0$ lead to the position representation regardless of the value of m. For a finite basis, however, this equivalence does not hold and some values of m might be better than others. The choice $\hat{\mathcal{O}} = 1$, the identity operator, is equivalent to retaining our original arbitrary basis. These various possibilities are explored in Sec. IV.

To conclude this section we would like to discuss different possibilities of extending the nonlocal effective interaction to off-diagonal hyper-radial potential matrix elements. It is evident that one can Hermitize the interaction by considering $v^{[2]}_{n[K_A],n'[K'_A]} + v^{[2]}_{n'[K_A],n[K'_A]}$, which then leads to an effective potential $(v^{[2]}_{n[K_A],n'[K'_A]} + v^{[2]}_{n'[K_A],n[K'_A]})^{\text{eff}}$. This can then be used for the following approximation of the off-diagonal matrix elements:

$$v^{[2]\text{eff}}_{n[K_A],n'[K'_A]} = \tfrac{1}{2}\left[\left(v^{[2]}_{n[K_A],n'[K'_A]} + v^{[2]}_{n'[K_A],n[K'_A]}\right)^{\text{eff}} + \left(v^{[2]}_{n[K_A],n'[K'_A]} - v^{[2]}_{n'[K_A],n[K'_A]}\right)\right]. \tag{33}$$

Another possibility is given by the introduction of a new hyper-radial state

$$|n+n'\rangle = |n\rangle + |n'\rangle \tag{34}$$

with the subsequent calculation of the effective interaction $v^{[2]\text{eff}}_{n+n'[K_A],n+n'[K'_A]}$. In addition, if we assume the approximations

$$v^{[2]\text{eff}}_{n+n'[K_A],n+n'[K'_A]} = v^{[2]\text{eff}}_{n[K_A],n[K'_A]} + v^{[2]\text{eff}}_{n'[K_A],n'[K'_A]} + v^{[2]\text{eff}}_{n[K_A],n'[K'_A]} + v^{[2]\text{eff}}_{n'[K_A],n[K'_A]}, \tag{35}$$

$$v^{[2]\text{eff}}_{n[K_A],n'[K'_A]} - v^{[2]\text{eff}}_{n'[K_A],n[K'_A]} = v^{[2]}_{n[K_A],n'[K'_A]} - v^{[2]}_{n'[K_A],n[K'_A]}, \tag{36}$$

one finds

$$v^{[2]\text{eff}}_{n[K_A],n'[K'_A]} = \tfrac{1}{2}\big(v^{[2]\text{eff}}_{n+n'[K_A],n+n'[K'_A]} - v^{[2]\text{eff}}_{n[K_A],n[K'_A]} - v^{[2]\text{eff}}_{n'[K_A],n'[K'_A]} + v^{[2]}_{n[K_A],n'[K'_A]} - v^{[2]}_{n'[K_A],n[K'_A]}\big). \quad (37)$$

We tested the effective interactions from Eqs. (33)–(37), but they did not lead to an improvement of the HH convergence. This is probably due to the mixing of bare and effective interactions in Eqs. (33)–(37).

A more promising possibility, which we did not yet explore numerically, can be obtained in the following way. In addition to Eq. (34), one considers

$$|n - n'\rangle = |n\rangle - i|n'\rangle \quad (38)$$

and calculates the diagonal effective interaction for each of the hyper-radial states $|n \pm n'\rangle$. Besides using Eq. (35), one also uses the approximation

$$v^{[2]\text{eff}}_{n-n'[K_A],n-n'[K'_A]} = v^{[2]\text{eff}}_{n[K_A],n[K'_A]} + v^{[2]\text{eff}}_{n'[K_A],n'[K'_A]} - i v^{[2]\text{eff}}_{n[K_A],n'[K'_A]} + i v^{[2]\text{eff}}_{n'[K_A],n[K'_A]}. \quad (39)$$

The off-diagonal matrix elements $v^{[2]\text{eff}}_{n[K_A],n'[K'_A]}$ can then be deduced from Eqs. (35) and (39).

TABLE I. Convergence of the HH expansion for the ^{4}He ground-state energy (in MeV) and the ^{4}He ground-state energy for the ^{4}He radius root-mean-square radius (in fm) with the bare nonlocal Idaho-N3LO potential and with the corresponding effective interaction using $\mathcal{O} = 1/\rho^2$. Corresponding results with other methods are also given.

$K_{\max}$	Bare		Effective	
	$\langle H\rangle$	$\sqrt{\langle r^2\rangle}$	$\langle H\rangle$	$\sqrt{\langle r^2\rangle}$
2	−3.507	1.935	−17.773	1.620
4	−13.356	1.523	−22.188	1.533
6	−20.135	1.446	−24.228	1.496
8	−23.721	1.451	−25.445	1.498
10	−24.617	1.470	−25.363	1.506
12	−25.115	1.491	−25.439	1.515
14	−25.259	1.501	−25.398	1.516
16	−25.310	1.509	−25.390	1.518
18	−25.359	1.513	−5.385	1.518
20	−25.370	1.515	−25.381	1.518
	−25.37(2)	1.515(4)	−25.38(1)	1.518(1)
HH [20]	−25.38	1.516		
FY [20,21]	−25.37	–		
NCSM [22]	−25.39(1)	1.515(2)		

IV. RESULTS AND DISCUSSION

The Idaho-N3LO NN force of Entem and Machleidt [12] is a typical example of a modern EFT potential model. In this section we study the binding energy and radius of ^{4}He with this interaction as a first test case for the quality of the nonlocal EIHH method presented in Sec. III. As second test case we will then consider the ground-state energies of ^{6}He and ^{6}Li with the nonlocal JISP16 nuclear force [18]. In our calculation we take the symmetrized HH basis functions of [19] and the hyper-radial functions

$$R_n(\rho) = \sqrt{\frac{n!}{(n+\alpha_L)!}} b^{-3(A-1)/2} \left(\frac{\rho}{b}\right)^{(\alpha_L - 3A + 4)/2} \times L_n^{\alpha_L}(\rho/b)\exp[-\rho/(2b)], \quad (40)$$

where $L_n^{\alpha_L}(x)$ are the associated Laguerre polynomials, $b = 0.3$ fm is a range parameter, and the Laguerre parameter α_L is taken to be 5.

In order to work with the nonlocal Idaho-N3LO force we use a representation of the potential model on a HO basis ($\Omega_{\text{HO}} = 30$ MeV, $N_{\max} = 20$, $J_{\max} = 4$). With the bare potential we obtain a ^{4}He ground-state energy of −25.37(2) MeV and a ^{4}He radius of 1.515(4) fm. By comparing with other high-precision results for ^{4}He ground-state energy [−25.38 MeV [20], −25.37 MeV [20,21], −25.39(1) MeV [22], and radius (1.516 fm [20], 1.515(2) fm [22]], one finds excellent agreement (see also Table I). Thus, it is evident that our treatment of the Idaho-N3LO NN potential leads to sufficiently accurate results.

Now we turn to the numerical tests for the nonlocal effective interaction. In Fig. 1 we present a comparison of results for different effective interactions (i.e., calculated for different operators $\hat{\mathcal{O}}$ according to the prescription given in the preceding section). In these calculations we have have used $N = 30$ hyper-radial basis functions. From the results in the figure one can infer that $\hat{\mathcal{O}} = 1/\rho^m$, with $m > 1$, leads to the best results. It means that in spite of the principle equivalence of $\hat{\mathcal{O}} = \rho$ and $\hat{\mathcal{O}} = 1/\rho^2$, in practice a stronger decreasing function of ρ has an apparent advantage. By comparing the results for bare and effective interaction one sees that with the effective interaction a very good convergence is already obtained with $K = 8$, whereas for the bare potential one needs to go up to $K = 14$ for a similarly good convergence.

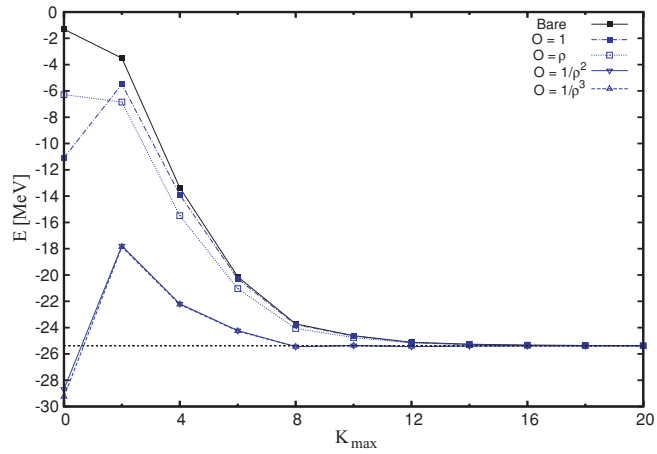

FIG. 1. (Color online) The ground-state energy of the ^{4}He nucleus with the Idaho N3LO NN force [12]. A comparison between different nonlocal HH effective interactions that correspond to different choices of the operator $\mathcal{O}$ (see text).

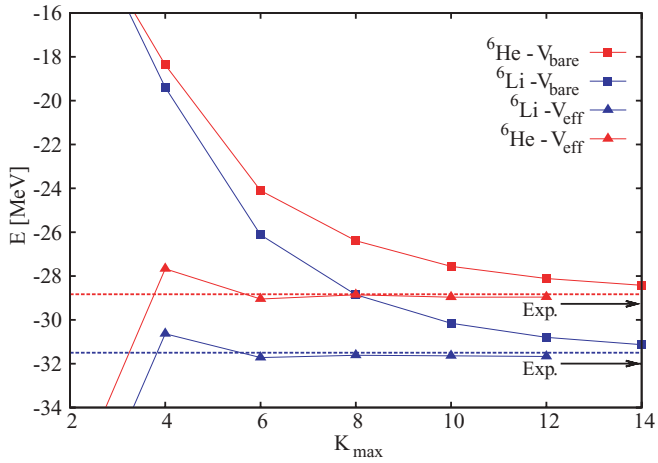

FIG. 2. (Color online) The ground-state energies of ^{6}He and ^{6}Li with the bare JISP16 nuclear force [18] and with the corresponding nonlocal effective interaction taking $\hat{\mathcal{O}} = 1/\rho^2$; dashed lines: extrapolated final results for the bare JISP16 force.

In Table I we illustrate the convergence patterns for effective ($\hat{\mathcal{O}} = 1/\rho^2$) and bare interactions in greater detail. Besides the results for the ^{4}He ground-state energy we also list those for the ^{4}He radius. One sees from the inspection of the table that for the bare interactions one needs to go up to $K_{\max} = 16$ and 18 for the ground-state energy and radius, respectively, if one requests a convergence precision of 0.3%. For the effective interaction one has a completely different scenario. In fact, $K_{\max} = 8$ (ground-state energy) and 12 (radius) are already sufficient for such high-precision results. As the convergence pattern of the EIHH method is usually unknown we make the following estimates. In the case of a monotonic convergence for at least the four highest $K_{\max}$ values we take twice the difference of the last two calculated $K_{\max}$ points. Otherwise, we try to give an estimate by considering the overall picture.

In Fig. 2 we show the results for the ground-state energies of the six-body nuclei ^{6}He and ^{6}Li with the JISP16 nuclear force. It is readily seen that one has essentially converged results with the effective interaction for both nuclei with a very low $K_{\max}$ value, namely $K_{\max} = 6$. The convergence of the bare JISP16 potential (taken from [23]) is by comparison much slower. Even for the highest considered $K_{\max}$ value, $K_{\max} = 14$, converged results are not yet obtained. However, the very regular convergence patterns allow us to make extrapolations to determine approximate final results (dashed lines in Fig. 2). In Table II we present the ^{6}He, ^{6}Li ground-state energies as a function of $K_{\max}$. From the table it can be seen that the effective interaction results predicts an extra binding of 200–300 keV with respect to the extrapolated HH bare calculations [23] as well as with the extrapolated NCSM calculations [24] that agree with each other. At first sight, this discrepancy seems to be too large in view of the estimated extrapolation errors of [23,24] (see [24] for a detailed analysis). However, without a theorem for the convergence pattern of the ground-state energy, the estimated extrapolation error is at best an educated guess. Further investigations would be needed before one can conclude whether there is a real contradiction between the results of the various methods.

We summarize our results as follows. We have extended the formulation of the effective interaction hyperspherical-harmonics approach to nonlocal two-body potentials. In the derived effective interaction all nonlocal effects regarding the hyperspherical coordinates are incorporated, while the hyper-radial part of the nonlocality remains excluded. Because the hyper-radius is a so-called slow variable and because NN forces show only moderate nonlocalities, this formulation of the HH effective interaction turns out to be rather satisfactory. In fact, we have tested this effective interaction in applications with four- and six-body nuclei. In the case of ^{4}He we have used the Idaho N3LO NN potential and calculated ^{4}He ground-state energy and radius with bare and effective interactions. For both observables a much faster HH convergence is obtained with the effective interaction. For the ^{6}He and ^{6}Li we calculated the ground-state energies with the JISP16 nuclear force. With the present computer resources it was not possible to obtain in [23] converged HH results for the bare interaction, but the regular convergence pattern allowed us to extrapolate the final results.

TABLE II. Convergence of the HH expansion for the ^{6}He, ^{6}Li ground-state energies (in MeV) with the bare nonlocal JISP16 potential and with the corresponding effective interaction using $\mathcal{O} = 1/\rho^2$. Corresponding results with the NCSM method are also given.

$K_{\max}$	^{6}He		^{6}Li	
	Bare	Effective	Bare	Effective
2	−11.043	−37.792	−10.422	−40.622
4	−18.366	−27.670	−19.392	−30.635
6	−24.103	−29.049	−26.124	−31.718
8	−26.391	−28.858	−28.854	−31.614
10	−27.560	−28.964	−30.155	−31.643
12	−28.112	−28.963	−30.797	−31.671
14	−28.424	–	−31.132	–
		−28.96(3)		−31.67(3)
HH (extrap.) [23]	−28.70(13)		−31.46(5)	–
NCSM (extrap. *A*) [24]	−8.68(12)		−31.43(12)	
NCSM (extrap. *B*) [24]	−28.69(5)		−31.44(5)	

On the contrary, with the effective interaction convergence was reached with a rather small effective interaction model space. In addition these results agree quite well with the extrapolated results for the bare interaction. All this shows that use of the nonlocal EIHH leads to very reliable results. For the future the possibility now exists of using the EIHH approach for nonlocal NN potentials for microscopic calculation of continuum reactions with six-body nuclei via the Lorentz integral transform method [25,26].

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