Correlation energy of nuclear matter and neutron star masses

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(Received 3 October 2009; published 16 April 2010)

We consider nuclear matter in the frames of the sigma model and find the role of correlation energy in the determination of the parameters of neutron stars. The response-function formalism is used for calculations within the Hartree-Fock approach and beyond. When electrons and muons are present in the neutron-rich matter, the maximal mass of the star is $M_* = 1.64$ (in the unit of the solar mass $M_\odot$). The correlation energy becomes very important for the stars with $M_* \sim 0.7 \div 1.5 M_\odot$ and its effect is estimated as $0.3 \div 0.4 M_\odot$ extracted from the relevant values obtained in the frames of the Hartree-Fock approximation. On the whole, the nuclear equation of state is definitely "softened."

DOI: 10.1103/PhysRevC.81.048801 PACS number(s): 21.65.−f, 26.60.Kp

The nuclear matter calculations have various applications in astrophysics [1–3]. First, it concerns the neutron stars. They are associated with remnants of supernovae explosions. The most famous of them are the Crab and Vela X-1. A typical neutron star has the mass M_* of around 1–2 masses of the Sun $M_\odot$ and the radius $r \sim 10$ km. The central density $\sim 10^{17}$ g/cm^3 is so high that there is no atomic structure inside the star, but it is nuclear matter, composed of neutrons with a small fraction of protons and electrons. The gravitational field is very strong and the metric tensor is defined by the Schwarzschild solution for a spherical-symmetric body. The latter implies that the pressure P, the energy density ε and the mass M distribution along radius r obey the Oppenheimer-Volkoff equations:

$$\frac{dP}{dr} = -(\varepsilon + P)\frac{M + 4\pi r^3 P}{r(r-2M)} \tag{1}$$

$$\frac{dM}{dr} = 4\pi r^2 \varepsilon \tag{2}$$

(here the universal system of units $c = G = 1$ is used). The initial condition $M(0) = 0$ and the boundary condition $P(r_*) = 0$ determines the outer radius of the star r_*. Thus, it is possible to find the total mass of the star $M_* = M(r_*)$. The result depends on the central density $\varepsilon(0) = \varepsilon_*$, and the particular choice of the functional dependence $P[\varepsilon]$ known as the equation of state (EOS).

The nuclear matter of neutron stars is something between the "stiff" matter with $P = \varepsilon$ and the ideal Fermi-gas whose pressure obeys the general law $P \sim \varepsilon^\alpha$ and whose energy density at zero temperature is determined as

$$T_0 = \frac{\gamma}{8\pi^2}k_F^3 E_F + \frac{1}{4}n_s m, \tag{3}$$

where $k_F = [6\pi^2 n/\gamma]^{1/3}$ is the Fermi momentum and $E_F = \sqrt{m^2 + k_F^2}$ is the Fermi energy, while m is the particle mass, n is the particle number density, and

$$n_s = \frac{\gamma}{4\pi^2}m\left(k_F E_F - m^2 \ln\frac{k_F + E_F}{m}\right) \tag{4}$$

is the scalar density. The degeneracy factor of symmetric nuclear matter $\gamma = 4$ (for pure neutron matter $\gamma = 2$). The interaction between the particles of nuclear matter results in the interaction term W included in the energy density of the whole system

$$\varepsilon = T_0 + W. \tag{5}$$

The the general parameters of nuclear matter is the binding energy per nucleon $E = \varepsilon/n$ and the compression modulus

$$K = k_F^2 \frac{d^2E}{dk_F^2} = 9n^2\frac{d^2E}{dn^2}. \tag{6}$$

Knowledge of the energy functional $\varepsilon[n]$ or $E[n]$ is equivalent to knowledge of the pressure $P = -n^2(dE/dn)$ that, together with the material density $\rho = mn$, will be substituted in Eqs. (1) and (2) to obtain the stellar parameters. The stronger interaction implies "stiffness" of the EOS and admits the most massive neutron stars. Modern estimations of the maximal mass are in the range from approximately 1.5 to 3.0 solar masses [1–3]. The uncertainty in the value reflects the fact that the EOS of very dense matter is not known well at high accuracy and that the neutron star mass is very sensible to the smallest changes in the EOS parameters.

The researchers never stop their attempts in making better approximation to the interaction W and getting the most realistic EOS of nuclear matter. For many calculations the interaction is approximated by the Hartree term W_0 [4]. The more complicated Hartree-Fock approximation allows to evaluate the additional exchange contribution W_x [5–7]. Many calculations are truncated at this level, while the exact value (5) includes the contribution of the *correlation energy*:

$$W = W_0 + W_x + W_c. \tag{7}$$

Its estimation is always very complicated [8], and it is usually omitted in the practice of nuclear matter research. The difficulties are the result of the nonlinear terms for meson fields that are also present in the well-known σ-ω model [4,6]. The calculations require much effort and are mostly performed within the random-phase approximation [9], that, however, can evaluate only about 40% of the total correlation energy value [10]. Another difficulty concerns the massive scalar σ meson (which is necessary for description of nucleon density distributions in finite nuclei and which does not exist in nature) that leads to the change of nucleon mass m and implies necessity to take into account the vacuum corrections.

Nevertheless, it is highly desirable to find the neutron star parameters with exact EOS and reveal the essence of correlation energy. The recent exact accounts of the correlation energy [11] were performed for the σ model with the Lagrangian

$$L = \bar{\psi}(i\gamma^{\nu}\partial_{\nu} - g\sigma - ig\gamma_5\vec{\tau}\cdot\vec{\pi})\psi + \tfrac{1}{2}\partial_{\nu}\sigma\partial^{\nu}\sigma + \tfrac{1}{2}\partial_{\nu}\vec{\pi}\cdot\partial^{\nu}\vec{\pi}. \quad (8)$$

The complicated method of linear response functions [12,13] have definite advantages. There is no problem with the origin of σ-meson field which is ultimately expressed through the pion variables that obey the constraint $\sigma^2 + \vec{\pi}^2 = f_{\pi}^2$. The repulsion produced by the vector ω meson is incorporated in the zero-point vibration, and there is no vacuum corrections. As a result, the equations operate with the proper nucleon mass m. Particularly, the simple Hartree term

$$W_0 = -\frac{g^2}{2m^2}n_s^2 \quad (9)$$

is extended to the exchange interaction which is a sum $W_x = W_{x1} + W_{x2}$ of two terms:

$$W_{x1} = -2g^2\int\frac{d^3pd^4q}{(2\pi)^6}\frac{m^2-q^2}{E_pE_{p+q}}\delta[q_0+E_p-E_{p+q}] \times V(q)N_pN_{p+q} \quad (10)$$

$$W_{x2} = 4g^2\int\frac{d^3pd^4q}{(2\pi)^6}\frac{m^2-q^2}{E_pE_{p+q}}\delta[q_0-E_p-E_{p+q}] \times[V(q,\lambda)-V(q,\lambda_0)]N_p, \quad (11)$$

where

$$V(q) = -\frac{1}{\lambda^2+\vec{q}^2-q_0^2} + \frac{2q_0^2}{\left(\lambda^2+\vec{q}^2-q_0^2\right)^2} \quad (12)$$

and $\lambda^2 > \lambda_0^2 = g^2n_s/(2m)$, $N_p = \Theta(k_F - p)$ and $E_p = \sqrt{m^2+p^2}$. The parameter λ is calculated self-consistently

$$\frac{2m\lambda^2}{g^2} = n_s - \frac{\lambda^4}{n_s}\int\frac{dk^4}{(2\pi)^4}\frac{1}{\lambda^2+\vec{k}^2-k_0^2} \times\left[\frac{\chi_{\sigma0}^2(k)}{1-\chi_{\sigma0}(k)R_{\sigma}(k)} + 3\frac{\chi_{\pi0}^2(k)}{1-\chi_{\pi0}(k)R_{\pi}(k)}\right], \quad (13)$$

where n_s is the scalar density (4), and exact calculation of λ requires numeric solution of complicated equations:

$$R_{\sigma}(k) = g^2\left[V + n_s\frac{\partial V}{\partial n_s} + \frac{1}{2}n_s^2\frac{\partial^2 V}{\partial n_s^2}\right] + g^2\frac{\partial^2}{\partial n_s^2}[J_1 + 2J_2 + Z_{\sigma}], \quad (14)$$

$$R_{\pi}(k) = g^2\left[V + n_s\frac{\partial V}{\partial n_s} + \frac{1}{2}n_s^2\frac{\partial^2 V}{\partial n_s^2}\right] + g^2\frac{\partial^2}{\partial n_s^2}\left[\frac{J_1}{2} + J_2 + Z_{\pi}\right], \quad (15)$$

and

$$J_1 = -\int\frac{d^3pd^4q}{(2\pi)^6}\frac{2m^2-q^2}{\varepsilon_p\varepsilon_{p+q}}\{\delta(q_0+E_p-E_{p+q}) + \delta(q_0-E_p+E_{p+q})\}V(q-k)N_pN_{p+q}, \quad (16)$$

$$J_2 = \int\frac{d^3pd^4q}{(2\pi)^6}\frac{2m^2-q^2}{\varepsilon_p\varepsilon_{p+q}}\{\delta(q_0-E_p-E_{p+q}) + \delta(q_0+E_p+E_{p+q})\}[V(q-k,\lambda)-V(q-k,\lambda_0)]N_p, \quad (17)$$

$$Z_{\sigma} = -\frac{1}{2}\int_0^1 d\xi\int\frac{dq^4}{(2\pi)^4}V(q-k) \times\left\{\frac{\chi_{\sigma0}^2(q)R_{\sigma\xi}(q)}{1-\chi_{\sigma0}(q)R_{\sigma\xi}(q)} + 3\frac{\chi_{\pi0}^2(q)R_{\pi\xi}(q)}{1-\chi_{\pi0}(q)R_{\pi\xi}(q)}\right\}, \quad (18)$$

$$Z_{\pi} = -\frac{1}{2}\int_0^1 d\xi\int\frac{dq^4}{(2\pi)^4}V(q-k) \times\left\{\frac{\chi_{\sigma0}^2(q)R_{\sigma\xi}(q)}{1-\chi_{\sigma0}(q)R_{\sigma\xi}(q)} + \frac{\chi_{\pi0}^2(q)R_{\pi\xi}(q)}{1-\chi_{\pi0}(q)R_{\pi\xi}(q)}\right\}, \quad (19)$$

where $\chi_{\sigma0}$ and $\chi_{\pi0}$ are the σ meson and pion noninteracting response functions, respectively; while $R_{\sigma\xi}$ and $R_{\pi\xi}$ are the Fourier images of effective interaction [13].

Inclusion of (18)–(19) in (14)–(15) gives rise to the correlation energy [12]:

$$W_c = -\frac{g}{2}\int_0^1 d\xi\int\frac{dq^4}{(2\pi)^4}V(q) \times\left\{\frac{\chi_{\sigma0}^2(q)R_{\sigma\xi}(q)}{1-\chi_{\sigma0}(q)R_{\sigma\xi}(q)} + 3\frac{\chi_{\pi0}^2(q)R_{\pi\xi}(q)}{1-\chi_{\pi0}(q)R_{\pi\xi}(q)}\right\} \quad (20)$$

in addition to the exchange terms (10)–(11). The total energy density of the system ε is determined by Eqs. (5), (7), (9), (10), and (11), so we immediately get the binding energy per nucleon $E = \varepsilon/n$.

The only free parameter of this model is the coupling constant g. Its choice depends on the saturation value of binding energy E_B. Or it can be adjusted to achieve saturation at definite density n_0. Starting with initial $\lambda^2 = g^2n_s/(2m)$ in (13), Eqs. (12)–(19) yield high accurate solution after a series of iterations. In our previous simulation [11] we requested saturation at $k_F = 1.36\ \mathrm{fm}^{-1}$, that yielded the binding energy $E_B = -12.8$ MeV per nucleon and the compression modulus $K = 314$ MeV. Now we have adjusted the coupling constant $g = 13.96$ in order to get the empirical saturation value of $E_B = -15.75$ MeV. The saturation density corresponds to $k_F = 1.39\ \mathrm{fm}^{-1}$ and the compression modulus is $K = 285$ MeV. The resulting equation of state, with the effect of correlation energy, is given on Fig. 1 (solid line). The present calculation deserves more reliability because the compression modulus is closer to the empirical value $K = 250$ MeV, while the theory, in principle, may predict K varying from 110 MeV to 470 MeV [4].

We have also calculated the EOS without the correlation energy contribution (20). The accuracy of this approximation is corresponding to the level of Hartree-Fock approach, and it is working when we switch of the integral terms (18)–(19)

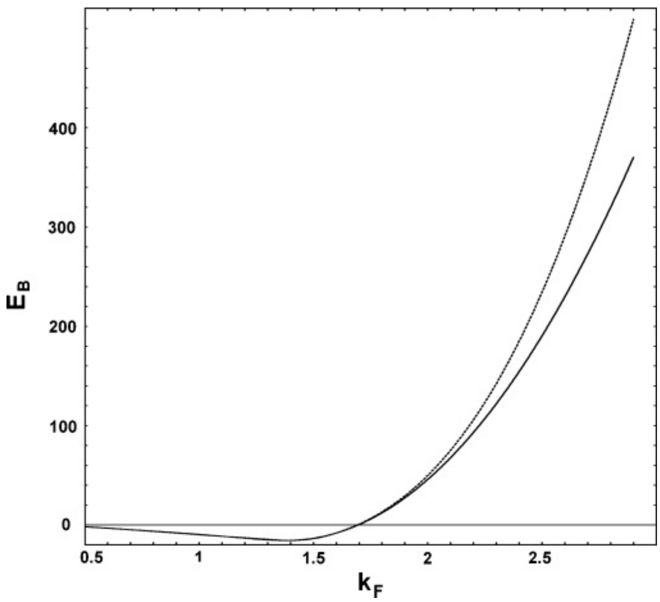

FIG. 1. Binding energy per nucleon E_B (MeV) vs. Fermi momentum (fm^{-1}).

in the effective interaction (14)–(15), while the self-consistent equation (13) remains valid. Now the binding energy E_{HF} comes to the saturation value of $E_B = -15.75$ MeV at $k_F = 1.41\ \mathrm{fm}^{-1}$ and the compression modulus is high $K = 377$ MeV. The saturation occurs at a bit higher density, while the matter becomes stiffer. It is demonstrated by dashed line on Fig. 1.

When the nuclear matter consists of arbitrary number of neutrons and protons $n = n_n + n_p$, then the number of electrons must satisfy the requirement of charge neutrality $n_e = n_p$. Appearance of negative-charged muons will imply

$$n_p = n_e + n_\mu \tag{21}$$

and the fraction of protons in the β equilibrium is determined by the constraint imposed on the chemical potentials

$$\mu_n = \mu_p + \mu_e \quad \mu_\mu = \mu_e. \tag{22}$$

The energy of neutron-rich matter with arbitrary fraction of protons $x = n_p/n$ is interpolated by formula [1–3]

$$\begin{aligned} E[n,x] &= E[n] + (1-2x)^2 E_{\mathrm{sym}}[n] \\ E_{\mathrm{sym}}[n] &= E[n,0] - E[n], \end{aligned} \tag{23}$$

where $E[n,0]$ is the energy of pure neutron matter, and $E[n] = E[n,0.5]$ is the energy of symmetric nuclear matter. The energy of symmetry is approximated as [14,15]:

$$E_{\mathrm{sym}}[n] = Ck_F^3 + \frac{1}{6}\frac{k_F^2}{\sqrt{k_F^2 + m^2}} \tag{24}$$

and the constant C is responsible for the empirical value $E_{\mathrm{sym}}[n_0] = \frac{1}{8}\partial^2 E/\partial x^2|_{x=0.5} = 30$ MeV at the saturation density n_0. Note that we should put in Eq. (24) the proper nucleon mass m with whom we have operated above in the frames of the linear response function approach [12] to the Lagrangian (8), since no massive meson field is provided in this model. For a more precise interpolation we can choose the formula [16]

$$\begin{aligned} E[n,x] = {} & E[n] - E_F[n] + E_F[n,x] + (1-2x)^2 \\ & \times \{E[n,0] - E[n] + E_F[n,x] - E_F[n]\}, \end{aligned} \tag{25}$$

where $E_F[n,x] = E_F[nx] + E_F[n - nx]$ and the Fermi levels are calculated at the proton/neutron degeneracy factor $\gamma = 2$.

Appearance of electrons and muons implies that the total energy density (5) will be extended by the additional contribution

$$T^{(e)} + T^{(\mu)} + W_{xc}^{(e)} + W_{xc}^{(\mu)} + W_{xc}^{(p)} \tag{26}$$

that includes the energy of ideal electron and muon gas $T^{(e)}$ and $T^{(\mu)}$, calculated by the standard formula (3) with $m = m_e$ and $m = m_\mu$, the degeneracy factor $\gamma = 2$, and the Fermi momentum $k_F^{(e)} = [3\pi^2 n_e]^{1/3}$ and $k_F^{(\mu)} = [3\pi^2 n_\mu]^{1/3}$. In the light of the charge neutrality (21) there is no Coulomb energy contribution. The electromagnetic exchange-correlation energy in (26) has the same standard form for electrons, muons, and protons [17]:

$$\begin{aligned} W_{xc} = {} & \frac{e^2}{4\pi^3}k_F^3\left[1 - \frac{3}{2}\frac{E_F}{k_F} + \frac{3}{2}\frac{m^2}{E_F^2}\ln\frac{k_F + E_F}{m}\right] \\ & + \frac{e^4}{12\pi^4}k_F^4\Phi\left(\frac{k_F}{m}\right), \end{aligned} \tag{27}$$

where $\Phi(z)$ is a tabulated function, and for m, k_F and E_F the electron, muon, and proton values are substituted respectively (we omit their labels in this formula for simplicity). The exchange and correlation interaction (27) gives rise to a bit nonideality of the electron and muon gas, but there is no strong effect, especially in the ultrarelativistic electron gas, on account of the small constant of electromagnetic interaction

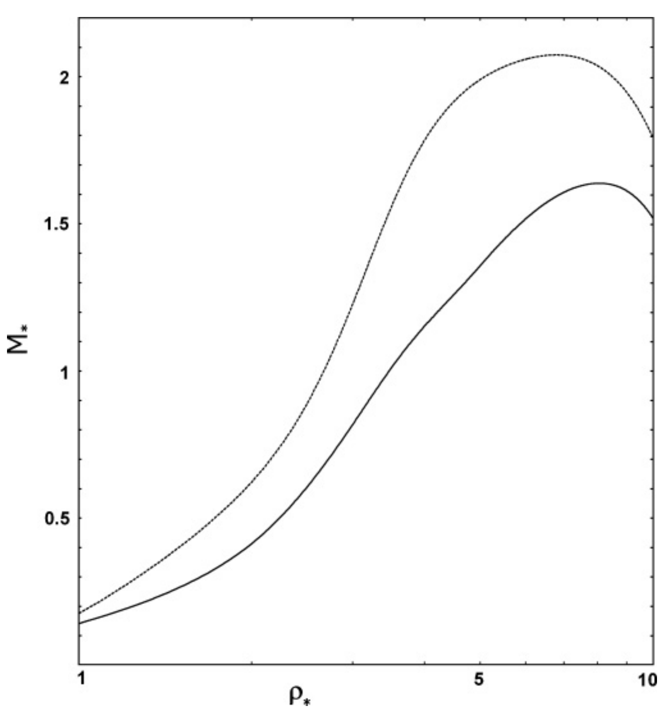

FIG. 2. Neutron star mass M_* (in the unit of solar mass $M_\odot$) vs. central density ρ_* (in the unit of $\rho_0 = 2.8 \times 10^{17}\ \mathrm{g/cm}^3$).

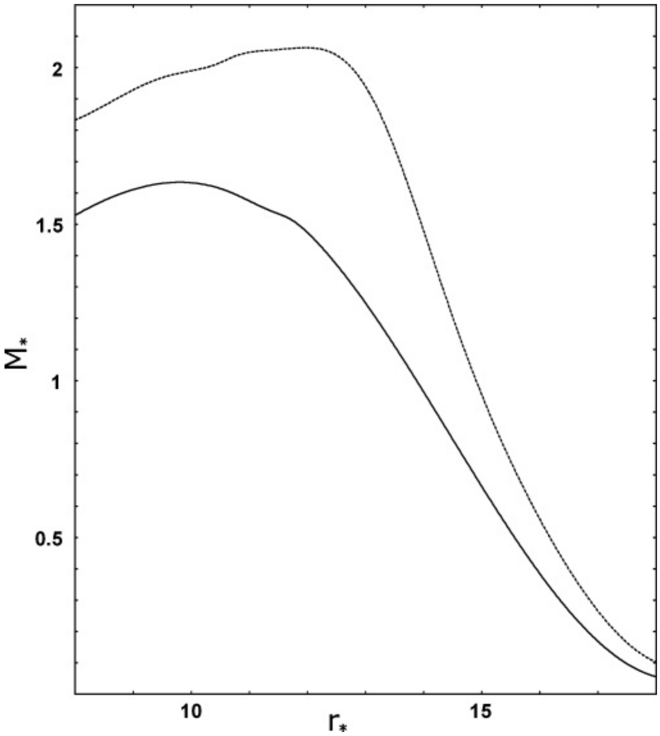

FIG. 3. Neutron star mass M_* (in the unit of solar mass $M_\odot$) vs. radius r_* (km).

$e^2 = 1/137$. In general, the presence of electrons and muons result in some softening of the EOS of stellar matter.

Thus, calculating the pressure and energy of the neutron-rich matter with electrons and muons by means of (21)–(27), and simulating the Oppenheimer-Volkoff equations (1)–(2), we obtain the results depicted on Fig. 2 and Fig. 3. The dependence of star mass M_* (per solar mass $M_\odot$) vs. central density ρ_* is given on Fig 2, solid line, the present EOS that includes the correlation energy, dashed line, the same EOS without the correlation energy. The mass M_* vs. radius r_* relationship is depicted on Fig. 3; solid and dashed lines correspond to the same conditions.

We have found that the maximal mass is $M_* = 1.64M_\odot$ and it corresponds to the central density of $7.87\rho_0$ (where $\rho_0 = 2.8 \times 10^{17}$ g/cm^3 is the normal nuclear density). The radius of this star is $r_* = 9.82$ km. If we apply the present calculation to the Crab pulsar (whose mass is $M_* = 1.44M_\odot$), we can estimate its radius as $r_* \cong 10.8$ km, and the central density $\rho_* \cong 5.2\rho_0$. The Hartree-Fock approximation predicts the star with maximal mass $M_* = 2.06M_\odot$ and radius $r_* = 12.07$ km when its central density is $\rho_* = 6.79\rho_0$. Particularly, the Crab pulsar will have the radius $r_* \cong 14.1$ km and $\rho_* \cong 3.4\rho_0$. In general, the profiles of mass and radius are very sensitive to smallest changes of density when $\rho_* = 3 \div 6\rho_0$ that corresponds, in the frames of our calculation, to the stellar masses $0.8M_\odot \lesssim M_* \lesssim 1.5M_\odot$.

Our results are within the range of values discovered in earlier research [1–3,14,15]. The nuclear matter may have very "soft" EOS and the compression modulus $K = 225$ MeV that corresponds to the maximal mass of neutron star $M_* = 1.45M_\odot$ [14]. The "stiff" nuclear matter with the compression modulus $K = 300$ MeV [15] gives the greater maximal mass $M_* = 2.1M_\odot$ [15]. Our Lagrangian (8) corresponds to an intermediate situation ($K = 285$ MeV and $M_* = 1.64M_\odot$). However, the main advantage of (8) concerns the possibility of development in terms of response-functions and the possibility to evaluate the correlation energy (20) [11–13]. It is well-known from solid-state physics that the correlation energy softens the EOS [17], now it is proved in the direct nuclear matter calculation. The Hartree-Fock approximation has a tendency to overestimate the stellar masses, and inclusion of the correlation energy implies a weighty correction of $-0.3 \div 0.4M_\odot$. Most careful estimation is required when the mass of the star is expected to be $M_* = 1.1 \pm 0.4M_\odot$. Can we warrant that $M_* = 1.6 \div 1.7M_\odot$ is the upper limit of neutron star mass? We have described the nuclear matter by the Lagrangian (8), more sophisticated interaction models may bring wider alternatives, but each particular Lagrangian requires special analysis and construction of solution by means of a response functions technique [12,13]. The most massive stars have complex structure, containing exotic particles (besides neutrons, protons, and leptons) and a quark core in the central part. Nevertheless, we can conclude, at the qualitative level, that the neutron star masses are regularly overestimated when the correlation energy is not taken into account. It may give fresh ideas for further research in this area.

The authors are grateful to Victor Khodel and Leonid Menshikov for inspiring discussions.

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