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Relation between equal-time and light-front wave functions

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The relation between equal-time and light-front wave functions is studied using models for which the four-dimensional solution of the Bethe-Salpeter wave function can be obtained. The popular prescription of defining the longitudinal momentum fraction using the instant-form free kinetic energy and third component of momentum is found to be incorrect except in the nonrelativistic limit. One may obtain light-front wave functions from rest-frame, instant-form wave functions by boosting the latter wave functions to the infinite momentum frame. Despite this difficulty, we prove a relation between certain integrals of the equal-time and light-front wave functions.

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I. INTRODUCTION

Light-front hadronic wave functions are used to interpret a variety of high-energy hadronic processes and experimentally observable quantities, including electromagnetic form factors [1–4], estimates of weak decay rates [5,6], quark recombination in heavy-ion collisions [7–9], coherent pion production of dijets [10–12], single-spin asymmetries in semi-inclusive deep inelastic scattering [13,14], computing various high-energy scattering amplitudes using the color dipole approach [15–18], computing the cross sections for electromagnetic production of vector mesons [19–21], and heavy-quark fragmentation in the quark-gluon plasma [22]. The common feature of all of these processes is that the observed matrix elements involve a correlation function in which a quark removed at a point is replaced by one separated from the first by a lightlike separation: $\Delta z + \Delta t = 0$. In this case, the front-form time $t + z = 0$ is a constant, and it is therefore natural to simplify a four-dimensional problem into a three-dimensional problem [involving the coordinates $(t - z, x, y)$]. Therefore, it is useful to understand how to obtain light-front wave functions from a fundamental point of view.

There is a large body of knowledge regarding techniques, models, and insights related to the equal-time rest-frame (ETRF) formalism. For example, spectroscopy is typically handled using this formalism. It is therefore natural to try to relate the ETRF wave function with the light-front wave function. One popular method uses a recipe to convert the spatial momenta of the constituents, $\boldsymbol{k}_i$, into light-front momenta $(x_i, \boldsymbol{k}_{i\perp})$. To be concrete, consider a bound state composed of two equal-mass constituents without spin. In this case, the ETRF wave function depends on the momentum $\boldsymbol{k}$ of one constituent. The recipe for converting the ETRF wave function to a light-front wave function is to introduce the longitudinal momentum fraction by the relation

$$x = \frac{k^+}{P^+} = \frac{E_{\boldsymbol{k}} + k^3}{2E_{\boldsymbol{k}}} = \frac{1}{2} + \frac{k^3}{2\sqrt{\boldsymbol{k}_\perp^2 + (k^3)^2 + m^2}}, \qquad (1)$$

where the single-particle energy is given by

$$E_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}, \qquad (2)$$

and P^+ is the plus component of the total momentum, P, of the bound state.[1] Using the recipe in Eq. (1) on a function of the single-particle energy invokes the change of variables

$$f(\boldsymbol{k}^2 + m^2) \longrightarrow f\left[\frac{\boldsymbol{k}_\perp^2 + m^2}{4x(1-x)}\right].$$

The latter form looks like the argument of a light-front wave function. The recipe for constructing a light-front wave function from an ETRF wave function often also includes a Jacobian factor, $\sqrt{J} = \sqrt{\partial k^3/\partial x}$, to preserve the wave-function normalization.

The relation in Eq. (1), however, appears to neglect any binding effect. While it is true in general that the plus momentum is additive [23], $P^+ = \sum_i k_i^+$, the energy of the bound state is not, $P^0 \neq \sum_i E_{\boldsymbol{k}_i}$. This leads one to suspect that there is nothing fundamental about making light-front wave functions by following the popular recipe. In fact, the issue can be resolved, because the formal relationship between the ETRF and the light-front wave functions has been known for a long time. Both involve energy integrals of the four-dimensional Bethe-Salpeter wave function, $\Psi(k, P)$: over k^0 in the case of the ETRF, and over k^- in the case of the light-front formulation. Given the covariant wave function

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[1]For any Lorentz four-vector A^μ, we define light-cone coordinates, $A^\pm$, by $A^\pm = A^0 \pm A^3$. Readers who employ a factor of $1/\sqrt{2}$ to define their light-cone coordinates should note that only one equation in this work depends on the choice of convention. This equation is an intermediate step appearing in Eq. (59).

Ψ, one can study the relationship between the ETRF and light-front wave functions. The purpose of this article is to provide such a study for a set of simple models. Although the treatment of particles with spin can be handled after suitable regularization [24,25], we consider only spin-zero systems made of two spinless constituents of equal mass throughout to simplify the presentation.

Here is an outline of our approach and summary of our findings. Section II is concerned with two-body bound states in covariant field theory and the Bethe-Salpeter equation. In particular, the explicit relation between the light-front (LF) and rest-frame instant-form wave functions (IF) and the solution of the Bethe-Salpeter equation is discussed. Next, in Sec. III an exactly soluble model involving pointlike coupling of a hadron to two scalar constituents is introduced to compare the light-cone and familiar instant-form wave functions. We find the simple transformation in Eq. (1) does not relate the IF wave function to the LF wave function, except in the nonrelativistic limit. Further, it is verified that boosting the ETRF wave function to infinite momentum produces the light-front wave function. Section IV investigates solutions of the Bethe-Salpeter wave function by means of the Nakanishi integral representation. Similarly, we find that the IF wave function is not related to the LF wave function by Eq. (1). For the general class of models of the Nakanishi type, we are able to show that the ETRF and light-front wave function agree in the nonrelativistic limit and that boosting the ETRF wave function to infinite momentum produces the light-front wave function. In Sec. V, we summarize our work and show that, despite the failure of the recipe to relate IF and LF wave functions, certain integrals of these wave functions are identical.

II. BETHE-SALPETER EQUATION AND BOUND STATES

We first discuss two-body bound states in covariant field theory. In terms of fully covariant operators, the Lippmann-Schwinger equation for the two-particle transition matrix T appears as

$$T = K + KGT. \tag{3}$$

In Eq. (3), K is the irreducible two-particle scattering kernel and G is the completely disconnected two-particle propagator, which is merely the product of two single-particle propagators. A pole in the T matrix (at some value of the total momentum-squared, $P^2 = M^2$, say) corresponds to a two-particle bound state of mass M. Investigation of the pole's residue gives an equation for the bound state vertex Γ,

$$\Gamma = KG\Gamma \tag{4}$$

(see Fig. 1). The bound-state amplitude Φ is defined as $G\Gamma$ and hence satisfies a similar equation, the Bethe-Salpeter equation

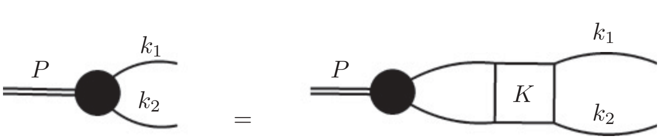

FIG. 1. Diagrammatic representation of the Bethe-Salpeter equation. The solid circle represents the vertex function Γ, and the total momentum is P.

(BSE) [26–28]:

$$\Phi = GK\Phi. \tag{5}$$

In the momentum representation and using the notation of Ref. [29], the BSE for two spinless particles reads

$$\Phi(k, P) = G\left(k + \frac{P}{2}, k - \frac{P}{2}\right)\int \frac{d^4k'}{(2\pi)^4} iK(k, k', P)\Phi(k', P). \tag{6}$$

The total momentum of the bound state is P, while the momenta of the constituents are $k_1 = k + \frac{1}{2}P$, and $k_2 = k - \frac{1}{2}P$. The relative momentum of the two constituents is then $k = \frac{1}{2}(k_1 - k_2)$. This form makes manifest the symmetry between the two particles. We also find it convenient to utilize a form of the BSE that is asymmetric. In this alternate form, we denote the bound-state amplitude by $\Psi(k_1, P)$, where k_1 is the momentum of one of the particles. The relation between the two amplitudes is

$$\Psi(k_1, P) = \Phi\left(k_1 - \frac{P}{2}, P\right). \tag{7}$$

We will often treat the subscript as implicit.

Armed with the Bethe-Salpeter amplitude $\Psi(k_1, P)$, one can calculate field-theoretic bound-state matrix elements by taking the appropriate residues of four-point Green's functions. These matrix elements may ultimately require knowledge of higher-point functions, which then must be solved for consistently in the same dynamics. The Bethe-Salpeter amplitude $\Psi(k_1, P)$ is in some ways the covariant analog of the Schrödinger wave function. While the features of relativistic field theory (in particular, particle creation and annihilation, retardation effects) make the exact analogy impossible, in the nonrelativistic limit, one can show that the BSE reduces to the Schrödinger equation.

The preceding discussion contains a graphical derivation of the BSE. It is useful to recall the field-theoretic coordinate-space definition of the Bethe-Salpeter wave function,

$$\Psi(x_1, x_2, P) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|P\rangle, \tag{8}$$

where the constituent fields are denoted by ϕ. One obtains the relation with $\Psi(k_1, P)$ by appealing to space-time translational invariance,

$$\Psi(x_1, x_2, P) = \Psi'(x_1 - x_2, P)\exp[-iP\cdot(x_1+x_2)/2], \tag{9}$$

and realizing that the Fourier transform is the amplitude $\Phi(k, P)$ mentioned previously, namely,

$$\Phi(k, P) = \int d^4z\Psi'(z, P)\exp(ik\cdot z). \tag{10}$$

Projecting the constituents onto states of definite four-momentum, we indeed find

$$\int d^4x_1 d^4x_2 \Psi(x_1, x_2, P)\exp(ik_1\cdot x_1 + ik_2\cdot x_2) = (2\pi)^4\delta^{(4)}(P - k_1 - k_2)\Psi(k_1, P). \tag{11}$$

The relation between three-dimensional wave functions and the Bethe-Salpeter wave function emerges from restricting the latter function to the corresponding initial boundary. In the case of light-front dynamics, the boundary surface is customarily defined on the plane $x^+ = 0$, while for instant-form dynamics, the boundary surface is specified by the origin of time, $x^0 = 0$. To carry out the projection onto the light front, one starts from an integral $I(k_1, k_2, P)$ that restricts the variation of the arguments of the latter function to the light-front plane. This plane is generally defined by the condition $\omega \cdot x = 0$, where ω is an arbitary four-vector with $\omega^2 = 0$ [30]. The light-front integral $I(k_1, k_2, P)$ is defined by the equation

$$I(k_1, k_2, P) \equiv \int d^4x_1 d^4x_2 \delta(x_1^+)\delta(x_2^+)\Psi(x_1, x_2, P) \times \exp(ik_1 \cdot x_1 + ik_2 \cdot x_2). \tag{12}$$

This integral does not produce the covariant momentum-space Bethe-Salpeter amplitude, rather the projection

$$I(k_1, k_2, P) = (2\pi)^3 \delta^{(+,\perp)}(P - k_1 - k_2) \int_{-\infty}^{\infty} \frac{dk_1^-}{2\pi} \Psi(k_1, P). \tag{13}$$

The preceding δ function is three-dimensional, $\delta^{(+,\perp)}(k) \equiv \delta(k^+)\delta(\boldsymbol{k}_\perp)$.

We can obtain another expression for $I(k_1, k_2, P)$ involving the light-front wave function and thereby deduce the relation with the covariant wave function. The valence light-front wave function is the coefficient of the valence state in the Fock-space expansion of $|P\rangle \equiv |P^+, \boldsymbol{P}_\perp\rangle$. On the light front, the bound state $|P\rangle$ is chosen to satisfy the covariant normalization condition, $\langle P'|P\rangle = 2P^+(2\pi)^3\delta^{(+,\perp)}(P' - P)$, and has the light-front Fock space expansion

$$|P\rangle = \frac{1}{\sqrt{2\mathcal{Q}}} \int \frac{dk_1^+ d\boldsymbol{k}_{1\perp}}{2k_1^+(2\pi)^3} \frac{dk_2^+ d\boldsymbol{k}_{2\perp}}{2k_2^+(2\pi)^3} \psi_{\text{LF}}(k_1, k_2, P) \times 2P^+(2\pi)^3\delta^{(+,\perp)}(P - k_1 - k_2)\, a_{k_1}^\dagger a_{k_2}^\dagger |0\rangle. \tag{14}$$

The light-front, Fock-space operator $a_{k_i}^\dagger$ creates an on-shell constituent, $a_{k_i}^\dagger|0\rangle = |k_i^+, \boldsymbol{k}_{i\perp}\rangle$. The light-front wave function $\psi_{\text{LF}}(k_1, k_2, P)$ is symmetric under the interchange of the constituent's momenta and by virtue of the momentum-conserving δ function always appears in the form $\psi_{\text{LF}}(k_1, P - k_1, P)$. We shall use schematic notation and write this simply as $\psi_{\text{LF}}(k_1, P)$, or even $\psi_{\text{LF}}(x_1, \boldsymbol{k}_{1\perp})$ in the hadron's rest frame, where $\boldsymbol{P}_\perp = \boldsymbol{0}$, with $x_1 = k_1^+/P^+$. While there are higher Fock-state contributions to the covariant bound-state wave function, we use a two-particle truncation throughout. The factor $\mathcal{Q}$ appearing in the Fock-space decomposition is the charge, which enters the normalization condition

$$\mathcal{Q} = \frac{1}{(2\pi)^3} \int \frac{dx d\boldsymbol{k}_\perp}{2x(1-x)} |\psi_{\text{LF}}(x, \boldsymbol{k}_\perp)|^2. \tag{15}$$

Using the number density operator, the natural choice for the total charge is $\mathcal{Q} = 2$.

Using light-front quantized fields, we can derive an expression for $I(k_1, k_2, P)$ using the Fock-space expansion of Eq. (8). This yields

$$I(k_1, k_2, P) = (2\pi)^3 \delta^{(+,\perp)}(P - k_1 - k_2) \frac{2P^+}{2k_1^+ 2k_2^+} \psi_{\text{LF}}(k_1, P). \tag{16}$$

Comparing with Eq. (13), we find

$$\psi_{\text{LF}}(k, P) = \frac{k^+(P^+ - k^+)}{\pi P^+} \int_{-\infty}^{\infty} dk^- \Psi(k, P). \tag{17}$$

The factors involving plus components of momentum arise from treating the phase space covariantly in the Fock-state expansion.

By contrast, the bound state $|\boldsymbol{P}\rangle$ in the instant-time formulation is chosen to satisfy the covariant normalization, $\langle \boldsymbol{P}'|\boldsymbol{P}\rangle = 2P^0(2\pi)^3\delta(\boldsymbol{P}' - \boldsymbol{P})$, and has the Fock-space expansion

$$|\boldsymbol{P}\rangle = \frac{1}{\sqrt{2\mathcal{Q}}} \int \frac{d\boldsymbol{k}_1}{2E_{\boldsymbol{k}_1}(2\pi)^3} \frac{d\boldsymbol{k}_2}{2E_{\boldsymbol{k}_2}(2\pi)^3} \psi_{\text{IF}}(k_1, k_2, P) \times 2P^0(2\pi)^3\delta(\boldsymbol{P} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, a_{\boldsymbol{k}_1}^\dagger a_{\boldsymbol{k}_2}^\dagger |0\rangle. \tag{18}$$

The instant-form, Fock-space operator $a_{\boldsymbol{k}_i}^\dagger$ creates an on-shell constituent $a_{\boldsymbol{k}_i}^\dagger = |\boldsymbol{k}_i\rangle$. Although we use a similar notation for Fock-space operators in the instant and light-front forms, they are not related by a finite Lorentz transformation (only by a boost to infinite momentum). The instant-form wave function, $\psi_{\text{IF}}(k_1, k_2, P)$, is symmetric under interchange of the constituent's momenta and by virtue of the momentum conserving δ function always appears in the form $\psi_{\text{IF}}(k_1, P - k_1, P)$. We shall use schematic notation and write this simply as $\psi_{\text{IF}}(\boldsymbol{k}_1, \boldsymbol{P})$, or $\psi_{\text{IF}}(\boldsymbol{k}_1)$ in the hadron's rest frame, $\boldsymbol{P} = \boldsymbol{0}$. The total charge $\mathcal{Q}$ enforces the rest-frame normalization condition

$$\mathcal{Q} = \frac{1}{(2\pi)^3} \int \frac{d\boldsymbol{k}}{2E_{\boldsymbol{k}}^2} |\psi_{\text{IF}}(\boldsymbol{k})|^2. \tag{19}$$

In general, the Fock-state expansion is expected to be much more complicated in the instant form because of the need to deal with vacuum fluctuations.

In the instant form of dynamics, the energy and Lorentz boosts are dynamical operators, and the initial conditions are sepcified on the boundary $x^0 = 0$. Thus, we define an instant form version, $I^0(k_1, k_2, P)$, of the integral $I(k_1, k_2, P)$:

$$I^0(k_1, k_2, P) \equiv \int d^4x_1 d^4x_2 \delta(x_1^0)\delta(x_2^0)\, \Psi(x_1, x_2, P) \times \exp(ik_1 \cdot x_1 + ik_2 \cdot x_2). \tag{20}$$

This integral produces a projection of the covariant Bethe-Salpeter wave function analogous to that in Eq. (13). Using the instant-form Fock state expansion [Eq. (18)], the instant-form wave function $\psi_{\text{IF}}(\boldsymbol{k}, \boldsymbol{P})$ is given by

$$\psi_{\text{IF}}(\boldsymbol{k}, \boldsymbol{P}) = \frac{E_{\boldsymbol{k}} E_{\boldsymbol{P}-\boldsymbol{k}}}{\pi P^0} \int_{-\infty}^{\infty} \Psi(k, P) dk^0. \tag{21}$$

Our aim is to elucidate the differences and connections between ψ_{LF} and ψ_{IF}.

III. TOY MODEL

We have discussed the covariant BSE for two-body bound states. In this section, we consider a toy model for the BSE that is exactly soluble. The solution will enable us to compare and contrast instant-form dynamics and light-front dynamics all while maintaining exact covariance.

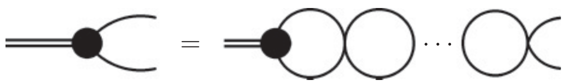

FIG. 2. Bethe-Salpeter equation for a point interaction. The state is bound by the infinite chain of bubbles.

One can obtain the simplest soluble BSE by choosing a pointlike interaction for the kernel $K(k, k'; P)$ in Eq. (6), namely, $K(k, k'; P) = g$, where g is a coupling constant. The two scalar particles that make up the scalar bound state thus interact infinitely many times according to the BSE to bind the state. For the pointlike interaction, a bubble chain is generated by the BSE and is shown in Fig. 2. With this choice of interaction, the bound-state equation simplifies tremendously. Because the kernel is independent of momentum, the only k' dependence that remains in Eq. (6) is in $\Psi(k', P)$, and this quantity is subsequently integrated over all k'. The integration merely produces a constant that can be absorbed into the overall normalization of the wave function. Thus, we are left with the solution

$$\Psi(k, P) = ig\, G(k, P - k), \tag{22}$$

where a proportionality constant is set to unity. The Bethe-Salpeter equation for the vertex $\Gamma(k, P)$ also determines the mass, $M^2 = P^2$, of the bound state via the consistency equation

$$1 = ig \int \frac{d^4k}{(2\pi)^4} G(k, P - k). \tag{23}$$

For simplicity, we do not discuss the necessary regularization and treat the coupling g as a renormalized parameter.

The single-particle propagator has the basic Klein-Gordon form, so the two-particle disconnected propagator is a product of these Klein-Gordon propagators. By virtue of Eq. (22), the covariant Bethe-Salpeter wave function is

$$\Psi(k, P) = -ig[k^2 - m^2 + i\varepsilon]^{-1}[(P - k)^2 - m^2 + i\varepsilon]^{-1}. \tag{24}$$

Here we have labeled the constituent mass by m. This is a four-dimensional analog of the usual Schrödinger wave function. There is, however, an important distinction. We also know the time dependence of the wave function—the time evolution governed by the Hamiltonian operator is automatically included because of the necessity of covariance. Moreover, we know from the Poincaré algebra that there are other dynamical operators besides the energy. As to which operators are kinematical depends upon the form of dynamics chosen.

A. Rest-frame wave functions

We shall next compute the instant-form wave function using Eq. (21) as evaluated in the rest frame. Given our solution to the BSE [Eq. (24)], we can carry out this projection onto the initial surface. The integration can be done using the residue theorem bearing in mind the four poles of the integrand: $k^0 = \pm E_{\boldsymbol{k}} \mp i\varepsilon$ and $M \pm E_{\boldsymbol{k}} \mp i\varepsilon$. We find

$$\psi_{\mathrm{IF}}(\boldsymbol{k}, \boldsymbol{0}) = -\frac{2g}{M} \frac{\sqrt{\boldsymbol{k}^2 + m^2}}{M^2 - 4(\boldsymbol{k}^2 + m^2)}. \tag{25}$$

Notice the wave function is manifestly rotationally invariant. This is indicative of the kinematic nature of the generators of rotations in the instant form.

In the front form of dynamics, one is interested in the properties of physical states along the advance of a wave front of light. The objects of front-form dynamics are the light-cone wave functions which are projections onto the initial surface $x^+ = 0$. In analogy with the instant form, one refers to x^+ as light-cone time and its Fourier conjugate k^- as light-front energy. In the front form, the energy is a dynamical operator along with two rotation operators corresponding to two independent rotations of the wave front of light. In contrast with the instant form, light-front Lorentz boosts are kinematical. We use Eq. (17) and work in the hadronic rest frame, $\boldsymbol{P}_\perp = \boldsymbol{0}$, to define $\psi_{\mathrm{LF}}(x, \boldsymbol{k}_\perp)$, with $x = k_1^+/P^+ = k^+/P^+$. The light-cone wave function corresponding to Eq. (24) is found by contour integration of Eq. (17) to be

$$\psi_{\mathrm{LF}}(x, \boldsymbol{k}_\perp) = -g \frac{\theta[x(1-x)]}{M^2 - \frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}}. \tag{26}$$

Note that the full rotational symmetry of the rest-frame wave function is not manifest.

We now inquire as to how the IF and LF wave functions are related to each other. In the literature, the rest frame IF wave function is converted into the rest frame the light-cone wave function by introducing an auxiliary variable, x, using Eq. (1). This variable has a physical interpretation as the fractional plus component of momentum in the center of mass system of two free particles. Inverted, this relation between x and k^3 reads [31]

$$k^3 = \left(x - \frac{1}{2}\right) \sqrt{\frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}}. \tag{27}$$

Simple algebra yields the relation

$$4(\boldsymbol{k}^2 + m^2) = \frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}, \tag{28}$$

from which we deduce

$$\psi_{\mathrm{IF}}(\boldsymbol{k}, 0) \to \psi_{\mathrm{IF}}(x, \boldsymbol{k}_\perp) = -\frac{g}{M} \sqrt{\frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}} \frac{1}{M^2 - \frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}}. \tag{29}$$

This bears a resemblance to the front-form wave function in the rest frame [Eq. (26)], but the instant-form wave function carries an additional factor of $E_{\boldsymbol{k}}/M$. This is a clear and major difference. One cannot interpolate between the instant form and the light-front form of the wave function.

One suspects that the two forms become equivalent in the nonrelativistic limit. This limit is defined by replacing $\sqrt{\boldsymbol{k}^2 + m^2}$ with m, so that Eq. (1) becomes

$$x \to \frac{1}{2} + \frac{k^3}{2m}. \tag{30}$$

In the nonrelativistic limit, we write the bound-state mass in terms of the constituent masses and a small binding energy

$B > 0$, namely, $M = 2m - B$. Expanding about $B = 0$ to linear order and replacing the factors $E_{\boldsymbol{k}}$ that appear in the relativistic phase space by m, Eq. (29) then becomes

$$\psi_{\rm IF}(x, \boldsymbol{k}_\perp) \rightarrow -g\frac{\theta[x(1-x)]}{M^2 - \frac{k_\perp^2 + m^2}{x(1-x)}}, \tag{31}$$

the same as Eq. (26). The θ function appears as a result of Eq. (28). We see that the wave functions of the two forms become identical only in the nonrelativistic limit. However, there is no reason to suspect that this limit should be valid because the wave functions fall off very slowly in momentum space. The only way to tell is to look at specific matrix elements.

It has been convenient to examine electromagnetic form factors. Truncating at the lowest Fock state, the expression for the electromagnetic form factor in terms of the front-form wave function is given by [1,2]

$$F_{\rm LF}(Q^2) = \frac{1}{(2\pi)^3}\int \psi_{\rm LF}(x, \boldsymbol{k}_\perp)\psi^*_{\rm LF}(x, \boldsymbol{k}_\perp + (1-x)\boldsymbol{Q}_\perp)\frac{dx d\boldsymbol{k}_\perp}{2x(1-x)}, \tag{32}$$

where the momentum transfer appears as $q^2 = -Q^2 = -\boldsymbol{Q}_\perp^2$, in a frame where $q^+ = 0$. A virtue of the light-front formulation is that the boost required between initial and final states in Eq. (32) is kinematical. The instant-form expression also requires a boosted wave function; however, instant-form boosts are dynamical. This complicates the interpretation of the form factor in terms of instant-form quanta. For example, it is well-known that boosting does not conserve particle number. With initial and final states differing in particle number, the instant-form form factor consequently cannot be the Fourier transform of a charge density. However, because of the kinematic nature of light-front boosts, the form factor has an interpretation in terms of the transverse charge density of quanta in the infinite momentum frame [32–37].

For our toy model (TM), we use Eq. (26) in the preceding expression to find

$$F^{\rm TM}_{\rm LF}(Q^2) = \frac{g^2}{(2\pi)^3}\int \frac{1}{M^2 - \frac{k_\perp^2+m^2}{x(1-x)}}\,\frac{1}{M^2 - \frac{[\boldsymbol{k}_\perp + (1-x)\boldsymbol{Q}_\perp]^2 + m^2}{x(1-x)}} \times \frac{dx d\boldsymbol{k}_\perp}{2x(1-x)}. \tag{33}$$

However, the use of the ersatz light-front wave function Eq. (29) in Eq. (32) would lead the appearance of a factor

$$\frac{1}{x(1-x)}\sqrt{(\boldsymbol{k}_\perp^2 + m^2)\{[\boldsymbol{k}_\perp + (1-x)\boldsymbol{Q}_\perp]^2 + m^2\}}$$

in the integrand of Eq. (33). This would lead to divergences in the integrals over both x and $d\boldsymbol{k}_\perp$. The form factor of this toy model was studied extensively for several different situations in Ref. [38]. There, it was shown that the equal-time wave function in the rest frame has no direct connection with the form factor, but the exact covariant evaluation of the form factor is indeed obtained using Eq. (33). In the nonrelativistic limit, the light-front and equal-time form factors do coalesce to the same result. However, this limit is satisfied for very limited kinematics, $B/M < 0.002$. Thus, the correspondence embodied by using the simple expression Eq. (1) does not work for the simplest possible toy model.

An additional ingredient common to the popular recipe for making a light-front wave function involves including a Jacobian factor to preserve the normalization of the wave function. The normalization of the ETRF wave function in Eq. (19) will pick up a Jacobian, $J = \partial k^3/\partial x$, if we view Eq. (27) as a change of variables. Taking into account the relativistic phase space factors, Eq. (19) will have exactly the form of Eq. (15) provided we make the identification

$$\psi_{\rm JIF}(x, \boldsymbol{k}_\perp) \equiv \sqrt{M}\left[\frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}\right]^{-1/4} \times \psi_{\rm IF}(x, \boldsymbol{k}_\perp) \longrightarrow \psi_{\rm LF}(x, \boldsymbol{k}_\perp). \tag{34}$$

For the toy model, however, the Jacobian modified instant-form wave function (JIF),

$$\psi_{\rm JIF}(x, \boldsymbol{k}_\perp) = -\frac{g}{\sqrt{M}}\left[\frac{\boldsymbol{k}_\perp^2 + m^2}{x(1-x)}\right]^{1/4}\frac{1}{M^2 - \frac{k_\perp^2+m^2}{x(1-x)}}, \tag{35}$$

is still not the light-front wave function $\psi_{\rm LF}(x, \boldsymbol{k}_\perp)$ in Eq. (26). A factor of the Jacobian squared, J^2, will produce the light-front wave function in this model; however, there is no justification to include two powers of the Jacobian.

To properly derive the instant-form expression for the form factor in the toy model, one starts from the covariant triangle diagram and performs the projection onto equal time by integrating over the loop energy, k^0. The time-ordered diagrams that result (see, for example, Ref. [39]) contain non-wave-function terms. The presence of such terms demonstrates that the form factor in the instant-form dynamics cannot be related to the Fourier transform of a charge density. In the toy model, the instant-form boost leads to nontrival effects, which nonetheless can be determined explicitly. In QCD, in contradistinction, the boost is too complicated to allow a general solution, although there has been progress for small momentum [40].

B. Boosting to the infinite momentum frame

One way to relate the IF and LF wave functions is by boosting the IF wave function to the infinite momentum frame. In that frame, it becomes the same as the LF wave function [41]. The way to see this is to obtain the IF wave function in a frame in which the 3 component of the momentum takes on an arbitrary value, and then let this value approach infinity.

To do this, we must first re-evaluate the expression Eq. (21) in a frame in which the system is moving with momentum $\boldsymbol{P}$ in a direction associated with the 3 axis. With the bound state energy P^0 given by $P^0 = \sqrt{\boldsymbol{P}^2 + M^2}$, evaluation of the contour integration of Eq. (21) using the toy model wave function $\Psi(k, P)$ in Eq. (24) yields the wave function

$$\psi_{\rm IF}(\boldsymbol{k}, \boldsymbol{P}) = -\frac{g}{2P^0}\left[\frac{1}{P^0 - E_{\boldsymbol{k}} - E_{\boldsymbol{P}-\boldsymbol{k}}} - \frac{1}{P^0 + E_{\boldsymbol{k}} + E_{\boldsymbol{P}-\boldsymbol{k}}}\right]. \tag{36}$$

The first term in Eq. (36) corresponds to a time-ordered graph with particle propagation, while the second term corresponds to particles propagating backward in time.

We wish to take the limit of $P \to \infty$. To this end, define the third component of $\boldsymbol{k}$ to be xP, so that the third component of of $\boldsymbol{P} - \boldsymbol{k}$ is $(1-x)P$. In the limit that $|\boldsymbol{P}|$ approaches infinity, the wave function of Eq. (36) vanishes unless $0 < x < 1$. In that case, the following limits hold:

$$\lim_{P\to\infty} E_{\boldsymbol{k}} = xP + \frac{\boldsymbol{k}_\perp^2 + m^2}{2xP}, \tag{37}$$

$$\lim_{P\to\infty} E_{\boldsymbol{P}-\boldsymbol{k}} = (1-x)P + \frac{\boldsymbol{k}_\perp^2 + m^2}{2(1-x)P}, \tag{38}$$

$$\lim_{P\to\infty} P^0 = P + \frac{M^2}{2P}. \tag{39}$$

For large values of P, only the first (or wave function) term of Eq. (36) is nonvanishing. Taking the limit of Eq. (36) as P approaches infinity leads immediately to the result

$$\lim_{P\to\infty} \psi_{\rm IF}(\boldsymbol{k}, \boldsymbol{P}) = \psi_{\rm LF}(x, \boldsymbol{k}_\perp). \tag{40}$$

While we have demonstrated this result using the toy model wave function, we remark that the instant-form Fock space expansion in Eq. (18) can be boosted to infinite momentum. One arrives at Eq. (14), which demonstrates the equivalence in Eq. (40) more generally.

IV. OTHER MODELS

We study more elaborate models defined by interactions other than pointlike coupling, using the formalism of Ref. [29]. In the BSE, the interaction kernel K is given by irreducible Feynman diagrams. Using any finite set of them is an approximation to the theory under consideration. If the kernel is given by a set of Feynman graphs [42,43], the Minkowski space BS amplitude Eq. (6) is found in terms of the Nakanishi integral representation [44]:

$$\Phi(k; P) = -\frac{i}{\sqrt{4\pi}}\int_{-1}^{1} dz \times \int_0^\infty d\gamma \frac{g(\gamma, z)}{\left[\gamma + m^2 - \frac{1}{4}M^2 - k^2 - P\cdot k\, z - i\varepsilon\right]^3}. \tag{41}$$

The weight function $g(\gamma, z)$ itself is not singular, whereas the singularities of the BS amplitude are fully reproduced by this integral. For example, if one sets $g(\gamma, z) = \sqrt{4\pi}g$ and calculates the integral, the result is the product of two free propagators appearing in Eq. (24).

The wave function in the ETRF is obtained by using Eq. (41) in Eq. (21), with the result

$$\psi_{\rm IF}(\boldsymbol{k}, \boldsymbol{0}) = -\frac{1}{\sqrt{4\pi}}\frac{3(\boldsymbol{k}^2 + m^2)}{8M}\int_{-1}^{1} dz \times \int_0^\infty d\gamma \frac{g(\gamma, z)}{\left[\gamma + \boldsymbol{k}^2 + m^2 - \frac{1}{4}M^2(1 - z^2)\right]^{5/2}}. \tag{42}$$

The light-front wave function $\psi_{\rm LF}(\boldsymbol{k}_\perp, x)$ is defined as before by an integration over k^-, as in Eq. (17). Substituting Eq. (41) into Eq. (17), the two-body light-front wave function is found to be [29]

$$\psi_{\rm LF}(\boldsymbol{k}_\perp, x) = -\frac{1}{\sqrt{4\pi}}\int_0^\infty \frac{x(1-x)g(\gamma, 1-2x)d\gamma}{[\gamma + \boldsymbol{k}_\perp^2 + m^2 - x(1-x)M^2]^2}. \tag{43}$$

Our next task is to compare the expressions in Eq. (42) and Eq. (43). It is possible to show in general that the nonrelativistic (NR) limit of these equations is the same, and boosting the ETRF wave function to the infinite momentum frame results in the light-front wave function. We handle this former first. Using the replacement Eq. (30) in the light-front wave function Eq. (43), and keeping terms linear in the binding energy, one obtains

$$\psi_{\rm LF}^{\rm NR}(\boldsymbol{k}) = -\frac{1}{4\sqrt{4\pi}}\int_0^\infty \frac{g(\gamma, 0)d\gamma}{\left(\gamma + \boldsymbol{k}^2 + m^2 - \frac{1}{4}M^2\right)^2}. \tag{44}$$

We next work with the instant-form wave function [Eq. (42)]. The mass-squared, $M^2 \approx 4m^2 - 4mB$, is a large quantity in the nonrelativistic limit. Thus, we may use $g(\gamma, z) \approx g(\gamma, 0)$ so that the integral over z can be performed. Note also that energies appearing in phase-space factors are replaced by constituent masses in the NR limit. Then we have

$$\psi_{\rm IF}^{\rm NR}(\boldsymbol{k}, \boldsymbol{0}) = -\frac{1}{\sqrt{4\pi}}\frac{m^2}{M}\int_0^\infty \frac{\left[M^2 + 6(\gamma + \boldsymbol{k}^2 + m^2 - \frac{1}{4}M^2)\right]}{\left[M^2 + 4(\gamma + \boldsymbol{k}^2 + m^2 - \frac{1}{4}M^2)\right]^{3/2}} \times \frac{g(\gamma, 0)d\gamma}{\left(\gamma + \boldsymbol{k}^2 + m^2 - \frac{1}{4}M^2\right)^2}. \tag{45}$$

The ratio of bracketed terms in Eq. (45) reduces to $1/(2m)$ in the NR limit. In that case, the results of Eqs. (45) and (44) become identical. Thus, in general, the correspondence between the instant-form and front-form wave functions is obtained when the nonrelativistic limit is valid. This is expected because in the nonrelativistic limit the wave functions are frame-independent.

To demonstrate the equivalence of the light-front wave function and the equal-time wave function in the infinite momentum frame, we return to Eq. (41) to derive the equal-time wave function in an arbitrary frame. We find

$$\psi_{\rm IF}(\boldsymbol{k},\boldsymbol{P}) = -\frac{1}{\sqrt{4\pi}}\frac{3E_{\boldsymbol{k}}E_{\boldsymbol{P}-\boldsymbol{k}}}{8P^0}\int_{-1}^{1}dz\int_0^\infty d\gamma\; g(\gamma,z) \times\left[\gamma+\boldsymbol{k}^2-(1-z)\boldsymbol{k}\cdot\boldsymbol{P}+\frac{1}{4}(1-z)^2\boldsymbol{P}^2 + m^2-\frac{1}{4}(1-z)^2M^2\right]^{-5/2}. \tag{46}$$

Using the limits in Eqs. (37)–(39), the wave function vanishes as $1/P^4$ when $P\to\infty$. This is true for all values of z, except in the region around $z=1-2x$. To obtain the nonvanishing contribution in the infinite momentum frame, we must thus replace $g(\gamma,z)=g(\gamma,1-2x)$. This replacement enables us to perform the z integration explicitly and subsequently take the $P\to\infty$ limit. This procedure yields the equivalence

$$\lim_{P\to\infty}\psi_{\rm IF}(\boldsymbol{k},\boldsymbol{P}) = \psi_{\rm LF}(x,\boldsymbol{k}_\perp) \tag{47}$$

for any wave function for which the Nakanishi integral representation Eq. (41) is valid. To compare the ETRF wave function to the light-front wave function using the recipe in Eq. (1), however, we need to know about the functional form of $g(\gamma,z)$. This is most easily done using specific models, to which we now turn.

A. Rotationally invariant light-front model

To investigate further the relation between the wave functions in Eqs. (42) and (43), we adopt a model. We may enforce rotational invariance RI in the light-front wave function by choosing $g(\gamma,z)$ to have a particular form,

$$g^{\rm RI}(\gamma,z) = 4g_0\delta(\gamma)(1-z^2), \tag{48}$$

where g_0 is a constant. Using Eq. (48) in Eq. (43) leads to the light-front wave function

$$\psi^{\rm RI}_{\rm LF}(\boldsymbol{k}_\perp,x) = -\frac{g_0}{\sqrt{4\pi}}\frac{16}{\left[M^2-\frac{k_\perp^2+m^2}{x(1-x)}\right]^2}. \tag{49}$$

With the help of the variable κ, defined by

$$\kappa^2 = m^2-\tfrac{1}{4}M^2, \tag{50}$$

we can cast the light-front wave function into a suggestive form. Using the inverse of the recipe [Eq. (27)], we can introduce the variable k^3 to make the light-front wave function appear as a rotationally invariant instant-form wave function

$$\psi^{\rm RI}_{\rm LF}(\boldsymbol{k}_\perp,x) \to -\frac{g_0}{\sqrt{4\pi}}\frac{1}{(\boldsymbol{k}^2+\kappa^2)^2}. \tag{51}$$

This wave function has the same form as that for the lowest s state of a hydrogenic atom.

The corresponding rest-frame, instant-form wave function is obtained by using Eq. (48) in Eq. (42):

$$\psi^{\rm RI}_{\rm IF}(\boldsymbol{k}) = -\frac{g_0}{\sqrt{4\pi}}\frac{2}{M}\frac{\sqrt{\boldsymbol{k}^2+m^2}}{(\boldsymbol{k}^2+\kappa^2)^2}. \tag{52}$$

In this case, one can compare the two forms [Eqs. (52) and (51)], having already used Eq. (1). It is readily apparent that the two forms are very different. For example, for large values of $\boldsymbol{k}^2$, the former falls as $1/|\boldsymbol{k}|^3$, while the latter falls as $1/\boldsymbol{k}^4$. Once again, we see that the relation between the rest-frame wave function and the light-front wave function cannot be seen using a simple transformation.

As with the toy model, including the Jacobian factor in converting the instant-form wave function, as in Eq. (35), does not produce the light-front wave function. The ratio of the Jacobian modified instant-form wave function to the true light-front wave function is not unity,

$$\frac{\psi^{\rm RI}_{\rm JIF}(x,\boldsymbol{k}_\perp)}{\psi^{\rm RI}_{\rm LF}(x,\boldsymbol{k}_\perp)} = \frac{1}{\sqrt{M}}\left[\frac{\boldsymbol{k}_\perp^2+m^2}{x(1-x)}\right]^{1/4}. \tag{53}$$

Curiously enough, this ratio, while not unity, is the same in the RI model as in the toy model of Sec. III. This coincidence owes to the simplicity of the models considered, however, not an underlying principle, as the final example demonstrates.

B. Wick-Cutkosky (WC) model

Let us consider a field theoretic example. Exact solutions to the Bethe-Salpeter equation in the ladder approximation are known. In the WC model [45,46], two scalars are bound by scalar exchange, and the function $g(\gamma,z)$ has the form

$$g^{\rm WC}(\gamma,z) = \delta(\gamma)\lambda(1-|z|), \tag{54}$$

with the constant λ defined in terms of parameters of the model, $\lambda = 2^6\pi\sqrt{m}\kappa^{5/2}$. Given this form for $g(\gamma,z)$, we evaluate the instant and light-front wave functions by using Eq. (54) in Eqs. (42) and (43). We find the instant-form wave function to be

$$\psi^{\rm WC}_{\rm IF}(\boldsymbol{k}) = -\frac{\lambda}{\sqrt{4\pi}M^3}\frac{\sqrt{\boldsymbol{k}^2+m^2}}{(\boldsymbol{k}^2+\kappa^2)^2}\left[\boldsymbol{k}^2+\kappa^2+\frac{1}{2}M^2 -\sqrt{(\boldsymbol{k}^2+m^2)(\boldsymbol{k}^2+\kappa^2)}\right]. \tag{55}$$

In the nonrelativistic limit, this wave function becomes identical to that of the ground-state hydrogenic atom. Away from this limit, the wave function contains relativistic phase-space factors, and the effects of retardation. In the asymptotic limit, the wave function has the behavior

$$\lim_{|\boldsymbol{k}|\to\infty}\psi^{\rm WC}_{\rm IF}(\boldsymbol{k}) = \frac{3\lambda}{8\sqrt{4\pi}M}\frac{1}{|\boldsymbol{k}|^3}. \tag{56}$$

We find the light-front wave function to be given by

$$\psi^{\rm WC}_{\rm LF}(\boldsymbol{k}_\perp,x) = -\frac{\lambda}{\sqrt{4\pi}}\frac{1-|1-2x|}{x(1-x)}\frac{1}{\left[M^2-\frac{k_\perp^2+m^2}{x(1-x)}\right]^2}. \tag{57}$$

Immediate inspection indicates that the wave functions of Eq. (55) and Eq. (57) are very different. The light-front wave function falls off faster than the instant-form wave function at large transverse momentum. We can try to relate the two wave functions by using the relation in Eq. (28). The ratio of the transformed instant-form wave function to the light-front wave function is considerably different than unity:

$$\frac{\psi_{\rm IF}^{\rm WC}(x,\boldsymbol{k}_\perp)}{\psi_{\rm LF}^{\rm WC}(x,\boldsymbol{k}_\perp)} = \frac{2}{M^3}\frac{x(1-x)}{1-|1-2x|}\sqrt{\frac{\boldsymbol{k}_\perp^2+m^2}{x(1-x)}} \times\left\{\frac{\boldsymbol{k}_\perp^2+m^2}{x(1-x)}+M^2-\sqrt{\frac{\boldsymbol{k}_\perp^2+m^2}{x(1-x)}\left[\frac{\boldsymbol{k}_\perp^2+m^2}{x(1-x)}-M^2\right]}\right\}. \quad (58)$$

A simple substitution as given by Eq. (1) cannot relate the instant and light-front wave functions. Including the Jacobian factor via Eq. (35) does not simplify the ratio $\psi_{\rm JIF}^{\rm WC}(x,\boldsymbol{k}_\perp)/\psi_{\rm LF}^{\rm WC}(x,\boldsymbol{k}_\perp)$. This ratio, moreover, is considerably different than the common value [Eq. (53)] found in the two simpler toy models.

V. SUMMARY

We use simple covariant models for which the solutions of the Bethe-Salpeter equation can be obtained. This allows us to explore both the instant and the front-form wave functions. The structure of these wave functions is related to the respective kinematic subgroups of the Poincaré algebra. Moreover, a fully covariant starting point allowed us a simple way to correctly formulate three-dimensional dynamics. We find that it is not possible to use the simple transformation in Eq. (1) to relate the rest-frame instant-form wave function with the light-front wave function. Instead, one may do this by boosting the rest-frame instant-form wave function to the infinite momentum frame.

There is an interesting relation between integrals of IF and LF wave functions that can be derived; similar relations have been suggested in Refs. [35,47]. The projection onto the space-time point $x^0 = x^3 = 0$ is a unique place where the IF wave function can be related to the LF wave function. This is because at this point we also have $x^+ = x^- = 0$, so that equal time also corresponds to equal light-front time. Consider the bound state in an arbitrary frame with $P^\mu = (\sqrt{\boldsymbol{P}^2+M^2}, \boldsymbol{P})$. Integrating the IF wave function over the third component of momentum projects onto $x^3 = 0$. Carrying out this projection, we find

$$\begin{aligned}\int_{-\infty}^{\infty} dk^3 \frac{P^0}{E_{\boldsymbol{k}}E_{\boldsymbol{P}-\boldsymbol{k}}}\psi_{\rm IF}(\boldsymbol{k},\boldsymbol{P}) &= \frac{1}{\pi}\int_{-\infty}^{\infty} dk^0 dk^3 \Psi(k,P)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk^- dk^+ \Psi(k,P)\\ &= \int_0^1 \frac{dx}{x(1-x)}\psi_{\rm LF}(x,\boldsymbol{k}_\perp - x\boldsymbol{P}_\perp),\end{aligned} \quad (59)$$

which shows that integrals over the IF and LF wave functions are identical. This relation also elucidates why the IF and LF wave functions vanish with different powers of $|\boldsymbol{k}_\perp|$.

In the rest frame, $\boldsymbol{P} = \boldsymbol{0}$, one can derive a relation for the impact-parameter-dependent LF wave function, $\psi_{\rm LF}(x,\mathbf{b}_\perp)$, defined by

$$\psi_{\rm LF}(x,\mathbf{b}_\perp) = \int \frac{d\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\mathbf{b}_\perp\cdot\boldsymbol{k}_\perp}\psi_{\rm LF}(x,\boldsymbol{k}_\perp). \quad (60)$$

From Eq. (59), we find

$$\int_{-\infty}^{\infty} dk^3 \int_{-\infty}^{\infty}\frac{d\boldsymbol{k}_\perp}{(2\pi)^2}\frac{M}{\boldsymbol{k}^2+m^2}e^{i\mathbf{b}_\perp\cdot\boldsymbol{k}_\perp}\psi_{\rm IF}(\boldsymbol{k}) = \int_0^1 \frac{dx}{x(1-x)}\psi_{\rm LF}(x,\mathbf{b}_\perp), \quad (61)$$

which is similar to the transversity relation found in Ref. [48]. As a consistency check, it is trivial to verify this identity using the Nakanishi integral representation of the IF and LF wave functions. Although there is no simple recipe for cooking up a light-front wave function from an equal-time, rest-frame wave function, Eqs. (59) and (62) provide rigorous relations between their integrals. Given the phenomenological utility of light-front wave functions, we intend to explore whether further such relations exist.

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