Vacuum pseudoscalar susceptibility

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We derive a novel model-independent result for the pion susceptibility in QCD via the isovector-pseudoscalar vacuum polarization. In the neighbourhood of the chiral limit, the pion susceptibility can be expressed as a sum of two independent terms. The first expresses the pion-pole contribution. The second is identical to the vacuum chiral susceptibility, which describes the response of QCD's ground state to a fluctuation in the current-quark mass. In this result one finds a straightforward explanation of a mismatch between extant estimates of the pion susceptibility.

Color-singlet current-current correlators or, equivalently, the associated vacuum polarizations, play an important role in QCD because they are directly related to observables. The vector vacuum polarization, for example, couples to real and virtual photons. It is thus basic to the analysis and understanding of the process $e^+e^- \to$ hadrons [1,2]. In addition, analysis of the large Euclidean-time behavior of a carefully chosen correlator can yield a hadron's mass [3,4], and correlators are also amenable to analysis via the operator product expansion and are therefore fundamental in the application of QCD sum rules [5].

In the latter connection, the vacuum pseudoscalar susceptibility (also called the pion susceptibility) plays a role in the sum-rules estimate of numerous meson-hadron couplings, for example, the strong and parity-violating pion-nucleon couplings, $g_{\pi NN}$ and $f_{\pi NN}$, respectively [6–8]. Furthermore, as will become plain herein, the pion susceptibility is as intimate a probe of QCD vacuum structure as the scalar susceptibility [9] but its veracious analysis is more subtle, with conflicts and misconceptions being common [7,8,10–12].

We approach the vacuum pseudoscalar susceptibility via the isovector-pseudoscalar vacuum polarization, which can be written[1]

$$\omega_5^{ij}(P;\zeta) = N_c \mathrm{tr} Z_4 \int_q^\Lambda \frac{i}{2}\gamma_5\tau^i S(q_+) i\Gamma_5^j(q;P)S(q_-), \quad (1)$$

where the trace is over flavor and spinor indices; ζ is the renormalization scale; $Z_4(\zeta, \Lambda)$ is the Lagrangian mass-term renormalization constant, which depends implicitly on the gauge parameter;[2] and $\int_q^\Lambda := \int^\Lambda d^4q/(2\pi)^4$ represents a

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[1]In our Euclidean metric: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma_\mu^\dagger = \gamma_\mu, \gamma_5 = \gamma_4\gamma_1\gamma_2\gamma_3$, $ab = \sum_{i=1}^4 a_i b_i$, and P_μ timelike $\Rightarrow P^2 < 0$.
[2]Physical quantities obtained from Eq. (1) are manifestly gauge invariant.

symmetry-preserving regularization of the integral, with Λ the regularization mass-scale which is taken to infinity as the last step in a complete calculation.

Herein we will subsequently assume isospin symmetry, namely, equal u- and d-quark current masses, in considering the isovector channel. An extension to three flavors and the flavor-singlet channel can be pursued following the methods of Ref. [13].

In Eq. (1), S is the dressed-quark propagator and Γ_5 is the fully dressed pseudoscalar vertex, both of which depend on the renormalization point. The propagator is obtained from QCD's gap equation, namely,

$$S(p)^{-1} = Z_2(i\gamma \cdot p + m^{\mathrm{bm}}) + \Sigma(p),$$
$$\Sigma(p) = Z_1 \int_q^\Lambda g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S(q)\frac{\lambda^a}{2}\Gamma_\nu(q,p), \quad (2)$$

where $D_{\mu\nu}(k)$ is the dressed-gluon propagator, $\Gamma_\nu(q,p)$ is the dressed-quark-gluon vertex, and m^{bm} is the Λ-dependent u- and d-quark current-quark bare mass. The quark-gluon vertex and the quark wave function renormalization constants, $Z_{1,2}(\zeta, \Lambda)$, also depend on the gauge parameter.

The gap equation's solution has the form

$$S(p)^{-1} = i\gamma \cdot p A(p^2; \zeta^2) + B(p^2; \zeta^2) \quad (3)$$

and the mass function $M(p^2) = B(p^2, \zeta^2)/A(p^2, \zeta^2)$ is renormalization point independent. The propagator is obtained from Eq. (3) augmented by a renormalization condition.

Since QCD is asymptotically free, the chiral limit is defined by

$$Z_2(\zeta, \Lambda)m^{\mathrm{bm}}(\Lambda) \equiv 0, \quad \forall \Lambda \gg \zeta, \quad (4)$$

which is equivalent to requiring that the renormalization point invariant current-quark mass is zero; that is, $\hat{m} = 0$. A mass-independent renormalization scheme can then be implemented by fixing all renormalization constants in the chiral limit [14];

namely, one solves the chiral limit gap equation subject to the requirement

$$S_{\hat{m}=0}^{-1}(p)\big|_{p^2=\zeta^2} = i\gamma \cdot p. \qquad (5)$$

This is implicit in the subsequent analysis. We note that

$$Z_2(\zeta, \Lambda)m^{\text{bm}}(\Lambda) = Z_4(\zeta, \Lambda)m(\zeta), \qquad (6)$$

where $m(\zeta)$ is the familiar running current-quark mass.

The pseudoscalar vertex is determined from an inhomogeneous Bethe-Salpeter equation, namely,

$$\left[\Gamma_5^j(k; P)\right]_{tu} = Z_4\left[\frac{1}{2}\gamma_5\tau^j\right]_{tu} + \int_q^{\Lambda} \left[\chi_5^j(q; P)\right]_{sr} K_{tu}^{rs}(q, k; P), \qquad (7)$$

where k is the relative momentum and P is the total momentum of the quark-antiquark pair; $r, s, t,$ and u represent color, flavor, and spinor indices;

$$\chi_5^j(k; P) = S(k_+)\Gamma_5^j(k; P)S(k_-), \qquad (8)$$

$k_\pm = k \pm P/2$, without loss of generality owing to the symmetry-preserving nature of the regularization scheme; and $K(q, k; P)$ is the fully amputated two-particle-irreducible quark-antiquark scattering kernel.

Much of the preceding material recapitulates results familiar from QCD Dyson-Schwinger equations (DSEs) [15,16], which also provide the foundation for our subsequent analysis. Consider, then, that in the presence of a spacetime-independent pseudoscalar source, $\vec{s}_5 \neq 0$, associated with the term

$$\int d^4x \bar{q}(x)\frac{i}{2}\gamma_5\vec{\tau} \cdot \vec{s}_5 q(x) \qquad (9)$$

in the action, one can define a vacuum pseudoscalar condensate, whose gauge-invariant, properly renormalized form in QCD is

$$\sigma_5^j(\vec{s}_5, m; \zeta, \Lambda) = Z_4 N_c \text{tr} \int_q^{\Lambda} \frac{i}{2}\gamma_5\tau^j S(q; \vec{s}_5, m; \zeta). \qquad (10)$$

This is analogous to the vacuum quark condensate [17]

$$\sigma(m; \zeta, \Lambda) = Z_4 N_c \text{tr} \int_q^{\Lambda} \frac{1}{2}\tau^0 S(q; m; \zeta), \qquad (11)$$

$\tau^0 = \text{diag}[1, 1]$, whose source-term in the QCD action is that associated with the current-quark mass.

It should be emphasized that when $\hat{m} = 0$, it is only foreknowledge of nonzero current-quark masses via the Higgs mechanism that leads one to express dynamical chiral symmetry breaking (DCSB) as

$$-\langle\bar{q}q\rangle_\zeta^0 = \lim_{m\to 0} \sigma(m; \zeta, \Lambda) \neq 0, \qquad (12)$$

instead of choosing a different vacuum vector, for example, $(\sigma; \vec{s}_5) \propto (0; 1, -i, 0)$. Moreover, Eq. (12) defines what we mean by an isoscalar-scalar configuration: isovector-pseudoscalar correlations are by convention measured with respect to this configuration.

These observations highlight the importance of the pseudoscalar susceptibility

$$\mathcal{X}_5^{ij}(\zeta) := \frac{\partial}{\partial s_5^i}\sigma_5^j(\vec{s}_5, m; \zeta, \Lambda)\bigg|_{\vec{s}_5=0}. \qquad (13)$$

Following the method of Ref. [9], it is straightforward to show that

$$\mathcal{X}_5^{ij}(\zeta) = -2\omega_5^{ij}(P = 0; \vec{s}_5 = 0, \hat{m}; \zeta). \qquad (14)$$

NB. While hitherto we have not specified a regularization procedure for the susceptibility, it can rigorously be defined via a Pauli-Villars procedure [9].

We are interested in the value of $\mathcal{X}_5^{ij}(\zeta)$ in the neighborhood of the chiral limit. Therein one may write [18]

$$i\Gamma_5^j(k; 0) = \frac{1}{2}i\gamma_5\tau^j E_5^R(k; 0) + \frac{r_\pi}{m_\pi^2}\Gamma_\pi^j(k; 0), \qquad (15)$$

where $\Gamma_\pi^j(k; P)$ is the pion bound state's canonically normalized Bethe-Salpeter amplitude; $r_\pi(\zeta)$, determined by

$$i\delta^{jk}r_\pi(\zeta) = \langle 0|\bar{q}\frac{1}{2}\gamma_5\tau^k q|\pi^j\rangle \qquad (16)$$

$$= N_c \text{tr} Z_4 \int_q^{\Lambda} \frac{1}{2}\gamma_5\tau^k \chi_\pi^j(q; P), \qquad (17)$$

is the residue of this bound state in the inhomogeneous pseudoscalar vertex; and $E_5^R(k; P)$ is a part of the inhomogeneous pseudoscalar vertex that is regular as $P^2 + m_\pi^2 \to 0$.

These statements will not be surprising once one recalls that the solution of a linear, inhomogeneous integral equation is a sum, namely, the regular solution of the inhomogeneous equation plus a solution of the homogeneous equation, here, naturally, the canonically normalized solution. In terms of this solution, the pion's leptonic decay constant is expressed through

$$\delta^{jk}f_\pi P_\mu = \langle 0|\bar{q}\frac{1}{2}\gamma_5\gamma_\mu\tau^k q|\pi^j\rangle \qquad (18)$$

$$= N_c \text{tr} Z_2 \int_q^{\Lambda} \frac{1}{2}\gamma_5\gamma_\mu\tau^k \chi_\pi^j(q; P). \qquad (19)$$

One can turn to the axial-vector Ward-Takahashi identity to determine $E_5^R(k; P)$; that is,

$$P_\mu\Gamma_{5\mu}^j(k; P) + 2m(\zeta)i\Gamma_5^j(k; P)$$
$$= S^{-1}(k_+)\frac{1}{2}i\gamma_5\tau^j + \frac{1}{2}i\gamma_5\tau^j S^{-1}(k_-), \qquad (20)$$

where $\Gamma_{5\mu}^j(k; P)$ is the inhomogeneous axial-vector vertex. At $P = 0$ with $\hat{m} \neq 0$ there is no pole contribution on the left-hand side and hence Eq. (20) states

$$m(\zeta)E_5^R(k; P = 0) = B(k^2; m; \zeta^2), \qquad (21)$$

namely, this regular piece of the pseudoscalar vertex is completely determined by the scalar part of the $\hat{m} \neq 0$ quark self-energy. NB. It is straightforward to verify Eq. (21), order-by-order, via the gap and Bethe-Salpeter equations using the systematic, nonperturbative, symmetry-preserving DSE truncation scheme introduced in Refs. [19,20].

We insert Eq. (15) into Eq. (14) to obtain

$$\mathcal{X}_5^{ij}(\zeta) \overset{\hat{m}\simeq 0}{=} \delta^{ij}\mathcal{X}_5(\zeta), \tag{22}$$

$$\mathcal{X}_5(\zeta) = \mathcal{X}_5^\pi(\zeta) + \mathcal{X}_5^R(\zeta) + \mathrm{O}(\hat{m}); \tag{23}$$

that is, in the neighbourhood of $\hat{m} = 0$ the susceptibility splits into a sum of two terms. The first of these expresses the contribution of the pion pole, $\mathrm{O}(\hat{m}^{-1})$, and can readily be expressed in the closed form

$$\mathcal{X}_5^\pi(\zeta) = \frac{2r_\pi(\zeta)^2}{m_\pi^2} \overset{\hat{m}=0}{=} -\frac{\langle\bar{q}q\rangle_\zeta^0}{m(\zeta)}, \tag{24}$$

where the last equality is proved in Ref. [18]. The second term in Eq. (23), $\mathrm{O}(\hat{m}^0)$, is implicitly determined via

$$m(\zeta)\mathcal{X}_5^R(\zeta)\delta^{jk}$$
$$\overset{\hat{m}\simeq 0}{=} -N_c\mathrm{tr}Z_4\int_q^\Lambda i\gamma_5\tau^k S(q)\frac{i}{2}\gamma_5\tau^j B(q^2;m)S(q)$$
$$= \delta^{jk}\sigma(m;\zeta,\Lambda), \tag{25}$$

where the last line is obtained using $\{\gamma_5, \gamma_\mu\} = 0$.

We can now proceed to our desired result. Equation (25) entails

$$\mathcal{X}_5^R(\zeta;m) = \mathcal{X}(\zeta) + \mathrm{O}(\hat{m}), \tag{26}$$

where the vacuum chiral susceptibility is [9]

$$\mathcal{X}(\zeta) = \left.\frac{\partial}{\partial m(\zeta)}\sigma(m;\zeta,\Lambda)\right|_{\hat{m}=0}. \tag{27}$$

Hence we arrive at a model-independent consequence of chiral symmetry and the pattern by which it is broken in QCD, namely,

$$\mathcal{X}_5(\zeta) \overset{\hat{m}\simeq 0}{=} -\frac{\langle\bar{q}q\rangle_\zeta^0}{m(\zeta)} + \mathcal{X}(\zeta) + \mathrm{O}(\hat{m}). \tag{28}$$

For illustration, in Table I we report numerical values computed from two models for the gap equation's kernel. Namely, we simplify the renormalization-group-improved effective interaction in Ref. [22],

$$Z_1g^2D_{\rho\sigma}(p-q)\Gamma_\sigma^a(q,p)$$
$$= \mathcal{G}((p-q)^2)D_{\rho\sigma}^{\mathrm{free}}(p-q)\frac{\lambda^a}{2}\Gamma_\sigma(q,p), \tag{29}$$

TABLE I. Vacuum pseudoscalar susceptibility and related quantities computed using the two kernels of the Bethe-Salpeter equation described in connection with Eqs. (30), (31), and (32). Dimensioned quantities are listed in GeV, $\kappa := -(\langle\bar{q}q\rangle_\zeta^0)^{1/3}$, and f_π^0 is the pion's chiral-limit leptonic decay constant. The entries were compiled from Refs. [9,21]. NB. For quantitative comparison with some other studies [7,8,10,12], our results for $\mathcal{X}$ should be multiplied by $(2\pi)^2$.

Vertex	$\sqrt{D}$	ω	κ	f_π^0	m	$\sqrt{\mathcal{X}_5^\pi}$	$\sqrt{\mathcal{X}_5^R}$
RL [Eq. (31)]	1	$\frac{1}{2}$	0.25	0.091	0.0050	1.77	0.39
BC [Eq. (32)]	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0.26	0.11	0.0064	1.66	0.28

wherein $D_{\rho\sigma}^{\mathrm{free}}(p-q)$ is the Landau-gauge free gauge-boson propagator, through the choice

$$\frac{\mathcal{G}(s)}{s} = \frac{4\pi^2}{\omega^6}Ds\mathrm{e}^{-s/\omega^2}, \tag{30}$$

which is a finite width representation of the form introduced in Ref. [23]. This interaction has been rendered as an integrable regularization of $1/k^4$ [24]. Equation (30) delivers an ultraviolet finite model gap equation. Hence, the regularization mass-scale can be removed to infinity and the renormalization constants set equal to one.

The kernel is completed by specifying the dressed-quark-gluon vertex. At leading-order in the systematic DSE truncation scheme [19,20] the vertex is

$$\Gamma_\sigma(q,p) = \gamma_\sigma. \tag{31}$$

This defines the rainbow-ladder (RL) truncation. One can alternatively employ *ansätze* for the vertex whose diagrammatic content is unknown. A class of such models, which has seen much use in diverse applications (e.g., Refs. [9,21,25–28]), can be characterized by [29]

$$i\Gamma_\sigma(k,\ell) = i\Sigma_A(k^2,\ell^2)\gamma_\sigma + (k+\ell)_\sigma$$
$$\times\left[\frac{i}{2}\gamma\cdot(k+\ell)\Delta_A(k^2,\ell^2) + \Delta_B(k^2,\ell^2)\right], \tag{32}$$

where

$$\Sigma_F(k^2,\ell^2) = \frac{1}{2}[F(k^2) + F(\ell^2)], \tag{33}$$

$$\Delta_F(k^2,\ell^2) = \frac{F(k^2) - F(\ell^2)}{k^2 - \ell^2}, \tag{34}$$

with $F = A, B$, namely, the scalar functions in Eq. (3). This *ansatz* satisfies the vector Ward-Takahashi identity and is often referred to as the BC vertex.

Equation (28) is a remarkable result, which is nonetheless readily understood. Recall that, in the absence of a current-quark mass, the two-flavor action has a $\mathrm{SU}_L(2)\otimes\mathrm{SU}_R(2)$ symmetry and, moreover, that ascribing scalar-isoscalar quantum numbers to the QCD vacuum is a convention contingent upon the form of the current-quark mass term.

It follows that the massless action cannot distinguish between the continuum of sources specified by

$$\mathrm{constant}\times\int d^4x\bar{q}(x)\mathrm{e}^{i\gamma_5\vec{\tau}\cdot\vec{\theta}}q(x), \quad |\theta|\in[0,2\pi). \tag{35}$$

Hence, the regular part of the vacuum susceptibility must be identical when measured as the response to any one of these sources, so that $\mathcal{X}_R = \mathcal{X}$ for all choices of $\vec{\theta}$. This is the content of the so-called "Mexican hat" potential, which is used in building models for QCD. The magnitude of $\mathcal{X}$ depends on whether the chiral symmetry is dynamically broken and the strength of the interaction as measured with respect to the critical value required for DCSB [9]. When the symmetry is dynamically broken, then the Goldstone modes appear, by convention, in the pseudoscalar-isovector channel, and thus

the pole contributions appear in $\mathcal{X}_5$ but not in the chiral susceptibility. It is valid to draw an analogy with the Weinberg sum rule [25,30].

With Eq. (28) we have, in addition, provided a novel, model-independent perspective on a mismatch between the evaluation of the pion susceptibility using either a two-point or three-point sum rule. Namely, the two-point study [10] produces the pion pole contribution, $\mathcal{X}_5^\pi$, which is also the piece emphasized in Ref. [11], whereas a three-point method [8] isolates the regular piece, $\mathcal{X}_5^R$, because a vacuum saturation *ansatz* is implemented in the derivation. Thus, the analyses are not essentially in conflict. Instead, they emphasize different, independent pieces of the susceptibility, which, with care, can be distinguished.

However, in a sum-rules estimate of pion-nucleon coupling constants, only the regular piece should be retained [6].

We note in closing that the vacuum tensor susceptibilities, which can be related to the nucleon's tensor charges [31], can similarly be analyzed. Such a study is under way.

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