Renormalization of the leading-order chiral nucleon-nucleon interaction and bulk properties of nuclear matter

R. Machleidt,[1,*] P. Liu,[1] D. R. Entem,[2,†] and E. Ruiz Arriola[3,‡]
[1]*Department of Physics, University of Idaho, Moscow, Idaho 83844, USA*
[2]*Grupo de Física Nuclear, IUFFyM, Universidad de Salamanca, E-37008 Salamanca, Spain*
[3]*Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, E-18071 Granada, Spain*
(Received 22 October 2009; published 3 February 2010)

We renormalize the two-nucleon interaction at leading order (LO) in chiral perturbation theory using the scheme proposed by Nogga, Timmermans, and van Kolck—also known as modified Weinberg counting. With this interaction, we calculate the energy per nucleon of symmetric nuclear matter in the Brueckner pair approximation and obtain a converged, cutoff-independent result that shows saturation, but also substantial underbinding. We find that the renormalized LO interaction is characterized by an extraordinarily strong tensor force (from one-pion exchange), which is the major cause for the lack of binding. The huge tensor force also leads to the unusually large wound integral of 40% in nuclear matter, which implies a very slow convergence of the hole-line or coupled-cluster expansion, rendering this interaction impractical for many-body calculations. In view of the unusual properties of the renormalized LO interaction and in view of the poor convergence of the nuclear many-body problem with this interaction, there is doubt that this interaction and its predictions can serve as a reasonable and efficient starting point that is improved by perturbative corrections.

DOI: 10.1103/PhysRevC.81.024001 PACS number(s): 21.65.−f, 13.75.Cs, 21.30.−x, 12.39.Fe

I. INTRODUCTION

The problem of a proper derivation of nuclear forces is as old as nuclear physics itself, namely, almost 80 years. The modern view is that, because the nuclear force is a manifestation of strong interactions, any serious derivation has to start from quantum chromodynamics (QCD). However, the well-known problem with QCD is that it is nonperturbative in the low-energy regime characteristic for nuclear physics. For many years this fact was perceived to be the great obstacle for a derivation of nuclear forces from QCD—impossible to overcome except by lattice QCD. The effective field theory (EFT) concept has shown the way out of this dilemma. One has to realize that the scenario of low-energy QCD is characterized by pions and nucleons interacting via a force governed by spontaneously broken approximate chiral symmetry. This chiral EFT allows for a systematic low-momentum expansion known as chiral perturbation theory (ChPT) [1]. Contributions are analyzed in terms of powers of small momenta over the large scale: $(Q/\Lambda_\chi)^\nu$, where Q is generic for a momentum (nucleon three-momentum or pion four-momentum) or pion mass and $\Lambda_\chi \approx 1$ GeV is the chiral symmetry breaking scale. The early applications of ChPT focused on systems like $\pi\pi$ [2] and πN [3], where the Goldstone-boson character of the pion guarantees that the expansion converges. The past 15 years has also seen great progress in applying ChPT to nuclear forces [4–20].

However, there is a difference between the purely pionic and the one-nucleon sector, on the one hand, and two- and multinucleon systems, on the other hand. Nuclear physics is characterized by bound states that are nonperturbative in nature. Weinberg showed [4] that the strong enhancement of the amplitude arises from purely nucleonic intermediate states ("infrared enhancement"). He therefore suggested a two-step procedure: In step 1, ChPT and naive dimensional analysis are used to calculate a "potential" that consists of only irreducible diagrams, and in step 2, this potential is iterated to all orders by inserting it into a Schrödinger or Lippmann-Schwinger (LS) equation to generate the amplitude.

At leading order (LO), the potential consists of static one-pion exchange (1PE) and two nonderivative contact terms. At next-to-leading order (NLO), multipion exchange starts, which involves divergent loop integrals that need to be regularized. An elegant way of doing this is dimensional regularization, which (besides the main nonpolynomial result) typically generates polynomial terms with coefficients that are, in part, infinite or scale dependent [9]. One reason that so-called contact terms are introduced in the EFT is to absorb all infinities and scale dependencies and make sure that the final result is finite and scale independent. This is the renormalization of the perturbatively calculated NN amplitude (which, by definition, is the "NN potential"). It is very similar to what is done in the ChPT calculations of $\pi\pi$ and πN scattering, namely, a renormalization order by order, which is the method of choice for any EFT. Thus, up to this point, the NN calculation fully meets the standards of an EFT and there are no problems. The perturbative NN amplitude can be used to make model-independent predictions for peripheral partial waves [9,10,15].

For calculations of the structure of nuclear few and many-body systems, the lower partial waves are the most important ones. The fact that in S waves we have large scattering lengths and shallow (quasi-) bound states indicates that these waves need to be treated nonperturbatively. Following Weinberg's

*machleid@uidaho.edu
†entem@usal.es
‡earriola@ugr.es

prescription [4], this is accomplished by inserting the potential V into the LS equation:

$$T(\vec{p}\,', \vec{p}) = V(\vec{p}\,', \vec{p}) + \int \frac{d^3p''}{(2\pi)^3} V(\vec{p}\,', \vec{p}\,'') \times \frac{M_N}{p^2 - p''^2 + i\epsilon} T(\vec{p}\,'', \vec{p}), \quad (1)$$

where M_N denotes the nucleon mass.

In general, the integral in the LS equation is divergent and needs to be regularized. One way to achieve this is by multiplying V with a regulator function, for example,

$$V(\vec{p}\,', \vec{p}) \longmapsto V(\vec{p}\,', \vec{p})\, e^{-(p'/\Lambda)^{2n}}\, e^{-(p/\Lambda)^{2n}}. \quad (2)$$

Typical choices for the cutoff parameter Λ that appears in the regulator are $\Lambda \approx 0.5$ GeV $\ll \Lambda_\chi \approx 1$ GeV [16,17].

It is pretty obvious that results for the T matrix may depend sensitively on the regulator and its cutoff parameter. This is acceptable if one wishes to build models. For example, the meson models of the past [21,22] always depended sensitively on the choices for the cutoff parameters, which were, in fact, welcome fit parameters for achieving a good reproduction of the NN data. However, we wish for the EFT approach to be fundamental in nature and not just another model.

In field theories, divergent integrals are not uncommon and methods for dealing with them have been developed. One regulates the integrals and then removes the dependence on the regularization parameters (scales, cutoffs) by renormalization. In the end, the theory and its predictions do not depend on cutoffs or renormalization scales.

So-called renormalizable quantum field theories, like QED, have essentially one set of prescriptions that takes care of renormalization through all orders. In contrast, EFTs are renormalized order by order, in which case the number of adjustable parameters increases.

As discussed, the renormalization of *perturbative* EFT calculations is not a problem. The problem is *nonperturbative renormalization*. This problem typically occurs in *nuclear* EFT because nuclear physics is characterized by bound states that are nonperturbative in nature.

Weinberg's implicit assumption was that the counterterms introduced to renormalize the perturbatively calculated potential, based on naive dimensional analysis ("Weinberg counting"), are also sufficient to renormalize the nonperturbative resummation of the potential in the LS equation. Unfortunately, it has turned out that this assumption is not quite correct, as pointed out by Kaplan, Savage, and Wise (KSW) [23] and others. The criticism of the Weinberg counting scheme resulted in a flurry of publications on the renormalization of the nonperturbative NN problem [24–42]. The literature is too comprehensive to discuss all contributions in detail. Let us just mention some of the work that has particular relevance to our present paper.

If the potential V consists of contact terms only (a.k.a. pionless theory), then the nonperturbative summation, Eq. (1), can be performed analytically, which makes it easier to deal with the renormalization issue. However, when pion exchange is included, then Eq. (1) can be solved only numerically and the renormalization problem is less transparent. Perturbative ladder diagrams of arbitrarily high order, where the rungs of the ladder represent a potential made up from irreducible pion exchange, suggest that an infinite number of counterterms is needed to achieve cutoff independence for all the terms of increasing order generated by the iterations. For that reason, KSW [23] proposed to sum the contact interaction to all orders (analytically) and to add pion exchange perturbatively up to the given order. Unfortunately, it turned out that the order-by-order convergence of 1PE is poor in the $^3S_1$-$^3D_1$ state [24]. The failure was triggered by the $1/r^3$ singularity of the 1PE tensor force when iterated to second order. Therefore, KSW counting is no longer taken into consideration (see, however, Ref. [39]). A balanced discussion of possible solutions is given in Ref. [29].

Some researchers decided to take a second look at Weinberg's original proposal. A systematic investigation and reanalysis of Weinberg counting in LO has been conducted by Nogga, Timmermans, and van Kolck [31] in momentum space and by Pavón Valderrama and one of the present authors (E.R.A.) at LO and higher orders in configuration space [30,32,33]. A comprehensive discussion of both approaches and their equivalence is given in Refs. [36] and [40]. The LO NN potential consists of 1PE plus two nonderivative contact terms that contribute only in S waves. By numerical calculations,[1] Nogga *et al.* find that the given counterterms renormalize the S waves, that is, the naively expected infinite number of counterterms is not needed. This means that Weinberg power counting does actually work in S waves at LO (ignoring the m_π dependence of the contact interaction discussed in Refs. [23] and [29]). However, there are problems with a particular class of higher partial waves, namely, those in which the tensor force from 1PE is attractive. The first few cases of this kind of low angular momentum are $^3P_0$, $^3P_2$, and $^3D_2$. The LO (nonderivative) counterterms do not contribute in P and higher waves, which is the reason for the problem. But the second-order contact potential provides counterterms for P waves. Therefore, the promotion of, particularly, the $^3P_0$ and $^3P_2$ contacts from NLO to LO would fix the problem in P waves. To take care of the $^3D_2$ problem, a fourth-order contact needs to be promoted to LO. In this way, one arrives at a scheme of "modified Weinberg counting" [31] for the LO two-nucleon interaction.

Once cutoff independence of the on-shell NN T matrix (and NN phase shifts and observables) has been achieved, it is of interest to know whether cutoff-independent results are also obtained when this interaction is applied in nuclear few- and many-body systems. Nogga *et al.* [31] investigated the lightest such system, namely, the three-nucleon bound state, and found cutoff independence of the triton binding energy. It is the purpose of this article to conduct a similar investigation in heavier nuclear systems. Because finite nuclei are difficult to calculate, we choose nuclear matter (infinitely many nucleons).

[1]For the purposes of the present paper, we conduct the discussion in momentum space. We note, however, that all information on the necessary number of counterterms can be determined *a priori* and analytically by inspecting the potential in configuration space at short distances [30,32,33,36].

We show that the renormalized LO two-nucleon interaction leads to converged results for the energy per nucleon in nuclear matter.

Section II briefly describes and repeats the LO renormalization procedure with modified Weinberg counting introduced in Ref. [31]. In Sec. III, we present the novel point of this paper, namely, the calculation of the energy per nucleon in symmetric nuclear matter as a function of density, and in Sec. IV, we compare our results with the work of other authors. Conclusions are drawn in Sec. V.

II. RENORMALIZING THE *NN* POTENTIAL IN LEADING ORDER

In naive dimensional analysis ("Weinberg counting"), the order-by-order expansion of the chiral NN potential is given as

$$V_{\rm LO} = V_{\rm ct}^{(0)} + V_{1\pi}^{(0)}, \tag{3}$$

$$V_{\rm NLO} = V_{\rm LO} + V_{\rm ct}^{(2)} + V_{1\pi}^{(2)} + V_{2\pi}^{(2)}, \tag{4}$$

$$V_{\rm NNLO} = V_{\rm NLO} + V_{1\pi}^{(3)} + V_{2\pi}^{(3)}, \tag{5}$$

$$V_{\rm N^3LO} = V_{\rm NNLO} + V_{\rm ct}^{(4)} + V_{1\pi}^{(4)} + V_{2\pi}^{(4)} + V_{3\pi}^{(4)}, \tag{6}$$

where the superscript denotes the order ν of the low-momentum expansion. NNLO stands for next-to-next-to-leading order. Contact potentials carry the subscript "ct" and pion-exchange potentials can be identified by obvious subscripts.

The charge-independent 1PE potential reads

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4f_\pi^2}\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{q^2+m_\pi^2}, \tag{7}$$

where $\vec{p}'$ and $\vec{p}$ designate the final and initial nucleon momenta in the center-of-mass system (CMS) and $\vec{q}\equiv\vec{p}'-\vec{p}$ is the momentum transfer; $\vec{\sigma}_{1,2}$ and $\boldsymbol{\tau}_{1,2}$ are the spin and isospin operators of nucleons 1 and 2; g_A, f_π, and m_π denote the axial-vector coupling constant, pion decay constant, and pion mass, respectively. We use $f_\pi = 92.4$ MeV and $g_A = 1.29$ to correct for the Goldberger-Treiman discrepancy. Because higher-order corrections contribute only to mass and coupling constant renormalizations and because, on shell, there are no relativistic corrections, the on-shell 1PE has the form of Eq. (7) in all orders.

Here, we specifically calculate LO np scattering and take charge dependence (isospin violation) into account. Thus, the 1PE potential reads

$$V_{1\pi}^{(np)}(\vec{p}', \vec{p}) = -V_{1\pi}(m_{\pi^0}) + (-1)^{I+1}2V_{1\pi}(m_{\pi^\pm}), \tag{8}$$

where I denotes the isospin of the two-nucleon system, and

$$V_{1\pi}(m_\pi) \equiv -\frac{g_A^2}{4f_\pi^2}\,\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{q^2+m_\pi^2}. \tag{9}$$

We use $m_{\pi^0} = 134.9766$ MeV and $m_{\pi^\pm} = 139.5702$ MeV [43]. In the LS equation, Eq. (1), we apply

$$M_N = \frac{2M_pM_n}{M_p+M_n} = 938.9182\ \text{MeV}, \tag{10}$$

$$p^2 = \frac{M_p^2T_{\rm lab}(T_{\rm lab}+2M_n)}{(M_p+M_n)^2+2T_{\rm lab}M_p}, \tag{11}$$

where $M_p = 938.2720$ MeV and $M_n = 939.5653$ MeV are the proton and neutron masses [43], respectively, and $T_{\rm lab}$ is the kinetic energy of the incident neutron in the laboratory system ("laboratory energy"). The relationship between p^2 and $T_{\rm lab}$ is based on relativistic kinematics.

Besides the 1PE potential, Eq. (8), the EFT includes contact terms that represent short-range interactions that cannot be resolved at the low-energy scale. Furthermore, the contacts are needed for renormalization. Stating the contact potentials in partial-wave decomposition, we have one zero-order ($\nu = 0$) contact in each S wave:

$$V_{\rm ct}^{(0)}\left(^1S_0\right) = \widetilde{C}_{^1S_0}, \tag{12}$$

$$V_{\rm ct}^{(0)}\left(^3S_1\right) = \widetilde{C}_{^3S_1}. \tag{13}$$

Up to this point, we are still applying Weinberg counting. However, as discussed in Sec. I, higher partial waves in which the pion's tensor force is attractive need counterterms to achieve cutoff independence—which leads us to modified Weinberg counting. To be specific, two P waves receive counterterms of second order,

$$V_{\rm ct}^{(2)}\left(^3P_0\right) = C_{^3P_0}p'p, \tag{14}$$

$$V_{\rm ct}^{(2)}\left(^3P_2\right) = C_{^3P_2}p'p, \tag{15}$$

and one D wave needs a fourth-order counterterm,

$$V_{\rm ct}^{(4)}\left(^3D_2\right) = D_{^3D_2}p'^2p^2. \tag{16}$$

For solution of the LS equation, Eq. (1), a regulator function is necessary, for which we choose the one given in Eq. (2), with $n = 2$. The regulator depends on the cutoff mass Λ, which we vary over a wide range, from 0.5 to 10 GeV (Table I). In S waves, we readjust the contact parameter for each choice of Λ such that the empirical scattering lengths ($a_s = -23.748$ fm for $^1S_0$ and $a_t = 5.4170$ fm for $^3S_1$) are reproduced. In those P and D waves that carry a contact in modified Weinberg counting, the contact parameter is used to fit—for the various choices of Λ—the empirical phase shift at 50 MeV as given in Ref. [44]. In all cases, we then calculate the phase shifts for all energies below 350 MeV.

The resulting phase shifts and mixing parameters for total angular momentum $J \leqslant 2$ are shown in Fig. 1. The curves refer to $\Lambda = 0.5$ GeV (dotted line), 1 GeV (dash-dotted line), 5 GeV (dashed line), and 10 GeV (solid line). The curves for $\Lambda = 5$ GeV and $\Lambda = 10$ GeV are, in general, indistinguishable on the scale of the figure, which demonstrates that cutoff independence (nonperturbative renormalization) has been achieved. It is known from the work of Nogga *et al.* [31] in momentum space[2] that this is possible. Our results represent an independent confirmation.

[2] Again, the analysis is easier in configuration space [30,32,33]; see, in particular, Table II and the thorough convergence analysis of phase shifts with total angular momentum $J \leqslant 5$ in Ref. [33].

TABLE I. Partial-wave contact parameters as a function of the cutoff Λ for the leading-order NN potential in modified Weinberg counting. Parameters are defined in Eqs. (12)–(16).

Partial-wave contact parameter	Cutoff parameter Λ (GeV)			
	0.5	1.0	5.0	10.0
$\widetilde{C}_{^1S_0}$ (10^4 GeV^{-2})	−0.109966	−0.087189	−0.06739623	−0.064460345
$\widetilde{C}_{^3S_1}$ (10^4 GeV^{-2})	−0.076005	1.349900	−0.02692560	0.021786000
$C_{^3P_0}$ (10^4 GeV^{-4})	0.840321	−0.1722517	0.001856514	0.000384981
$C_{^3P_2}$ (10^4 GeV^{-4})	−0.2316105	−0.0700665	−0.00251447	0.001251038
$D_{^3D_2}$ (10^4 GeV^{-6})	−0.3347880	0.3899800	−0.00020581	−0.00001055

III. NUCLEAR MATTER

As discussed in Sec. I, once cutoff independence has been achieved for the two-nucleon system, a good question to ask is whether cutoff-independent predictions are also obtained in the nuclear many-body problem when this renormalized NN potential is applied. Nogga *et al.* [31] addressed this question for the three-nucleon system where they confirmed the cutoff independence of the triton binding energy at LO. We wish to turn to heavier nuclear systems and choose nuclear matter as the representative sample.

By definition, nuclear matter refers to an infinite uniform system of nucleons interacting via a strong force without electromagnetic interactions. This hypothetical system is believed to approximate conditions in the interior of heavy

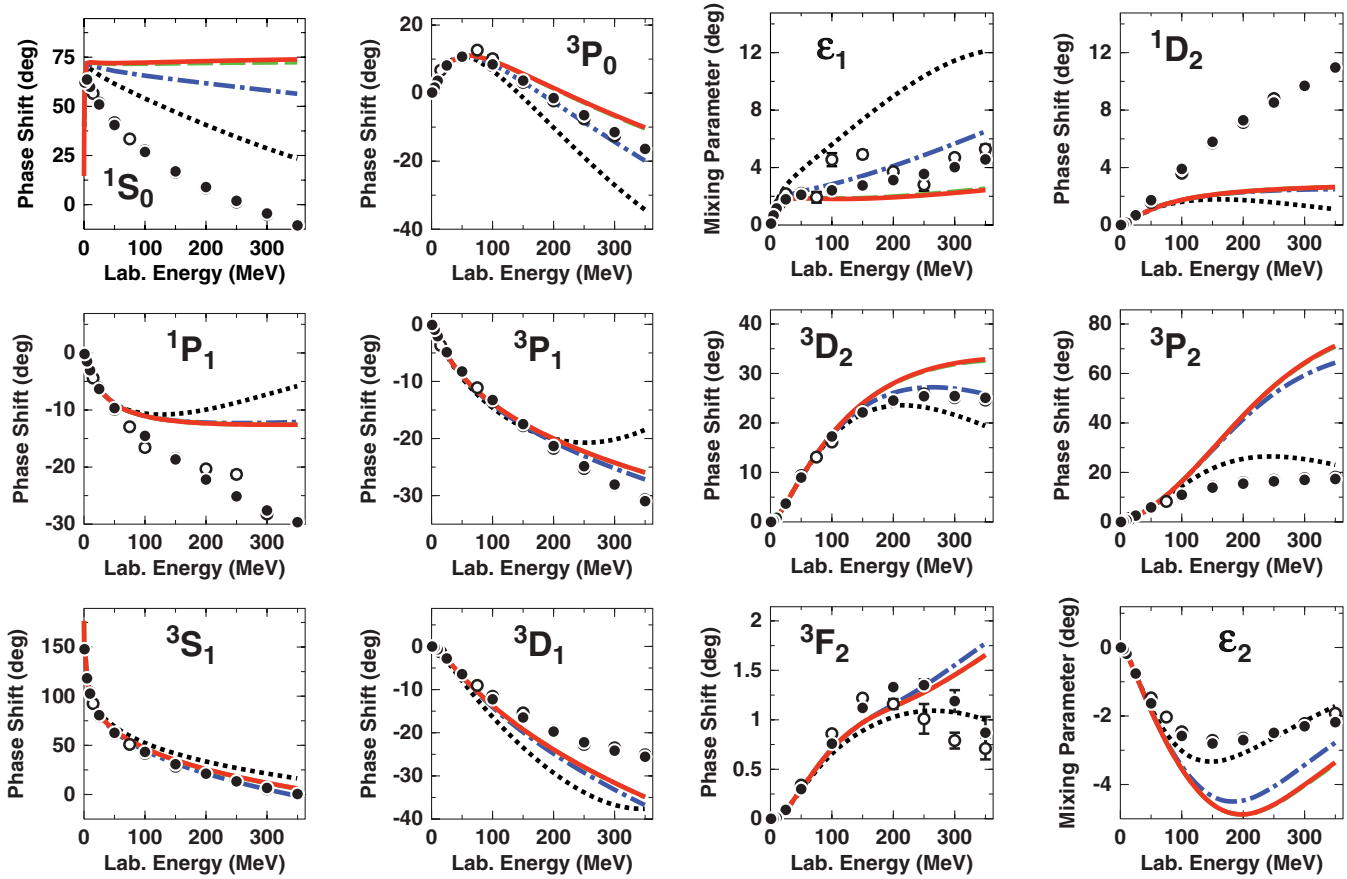

FIG. 1. (Color online) Phase shifts and mixing parameters of neutron-proton scattering for total angular momentum $J \leqslant 2$ and $T_{\rm lab} \leqslant 350$ MeV. The curves display the LO predictions for cutoff parameter $\Lambda = 0.5$ GeV [(black) dotted line], 1 GeV [(blue) dash-dotted line], 5 GeV [(green) dashed line], and 10 GeV [(red) solid line]. Note that the dashed and solid curves are, in general, indistinguishable on the scale of the figure. The filled and open circles represent the results from the Nijmegen multienergy np phase-shift analysis [44] and the VPI/GWU single-energy np analysis SM99 [45], respectively.

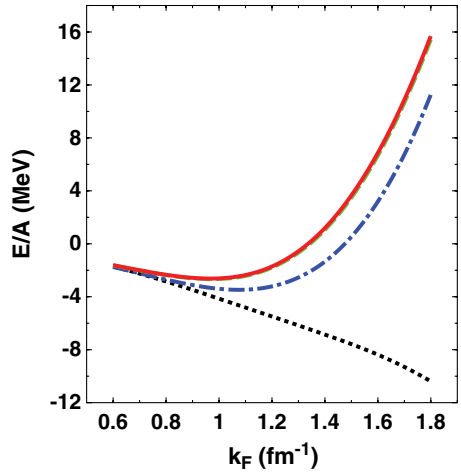

FIG. 2. (Color online) Energy per nucleon, E/A, in symmetric nuclear matter as a function of the Fermi momentum k_F applying the LO NN potentials with the various cutoffs used in the phase-shift calculations in Fig. 1. The curve patterns represent the same cutoffs as in Fig. 1.

nuclei. We assume equal neutron and proton densities, that is, we consider symmetric nuclear matter. This many-body system is characterized by its energy per nucleon as a function of the particle density.

We use the well-established Brueckner-Bethe-Goldstone method (henceforth, Brueckner theory) [22,46–48] to calculate the nuclear matter energy. In this theory, a central role is played by the Brueckner G matrix, which is a solution of the Bethe-Goldstone integral equation,

$$G(w) = V - V\frac{Q}{H_0 - w}G(w), \tag{17}$$

where w denotes the starting energy, H_0 the unperturbed Hamiltonian, and the Pauli operator Q projects onto unoccupied states. In the pair approximation, the energy per nucleon is given by

$$\frac{E}{A} = \frac{1}{A}\sum_{m\leqslant k_F}\langle m|t|m\rangle + \frac{1}{2A}\sum_{m,n\leqslant k_F}\langle mn|G(w)|mn-nm\rangle, \tag{18}$$

where A denotes the number of nucleons, t the kinetic energy operator, and k_F the Fermi momentum, which is related to the density ρ of symmetric nuclear matter by

$$\rho = \frac{2}{3\pi^2}k_F^3. \tag{19}$$

The starting energy is chosen on-shell, that is,

$$w = e(m) + e(n), \tag{20}$$

with single-particle energy

$$e(m) = t(m) + U(m) \tag{21}$$

and single-particle potential

$$U(m) = \begin{cases} \sum_{n\leqslant k_F}\langle mn|G(w)|mn-nm\rangle, & m \leqslant k_F, \\ 0, & m > k_F, \end{cases} \tag{22}$$

also known as the "gap" choice for the single-particle potential, as a gap will obviously occur at the Fermi surface. The calculations are conducted in partial-wave decomposition and the Brueckner integral equation is solved by matrix inversion; see Ref. [48] for details.

The Bethe-Goldstone method was originally devised to handle the short-distance hard core in nuclear systems. In chiral EFT, an expansion in both $1/f_\pi$ and $1/M_N$ is carried out. For short distances, $r \ll 1/m_\pi$, pion mass effects can be neglected, and on purely dimensional grounds an inverse-power short-distance singularity should be expected: to LO, $V(r) \sim 1/(f_\pi^2 r^3)$; to NLO, $V(r) \sim 1/(f_\pi^4 r^5)$; to NNLO, $V(r) \sim 1/(f_\pi^4 M_N r^6)$; etc. It is natural to ask whether the G-matrix result converges in the limit $\Lambda \to \infty$. In the appendix we show that this is indeed the case for the gap choice Eq. (22) [48]. The proof rests on the finiteness of the off-shell K matrix [49]. This gives some confidence in the stability of the numerics for increasing cutoff values.

In Fig. 2, we display our results for the energy per nucleon in symmetric nuclear matter as a function of density (measured by the Fermi momentum k_F) applying the LO NN potentials with the various cutoffs used in the phase-shift calculations in Sec. II. The same curve patterns in Figs. 1 and 2 indicate the same cutoffs. The nuclear matter curves for $\Lambda = 5$ GeV [(green) dashed line] and $\Lambda = 10$ GeV [(red) solid line] cannot be distinguished on the scale of Fig. 2, demonstrating that cutoff independence of the predictions is achieved; in other words, the (red) solid curve represents the renormalized result, which, as expected, is convergent. This curve shows saturation at a Fermi momentum $k_F \approx 1.0$ fm^{-1} and an energy per nucleon $E/A = -2.6$ MeV.

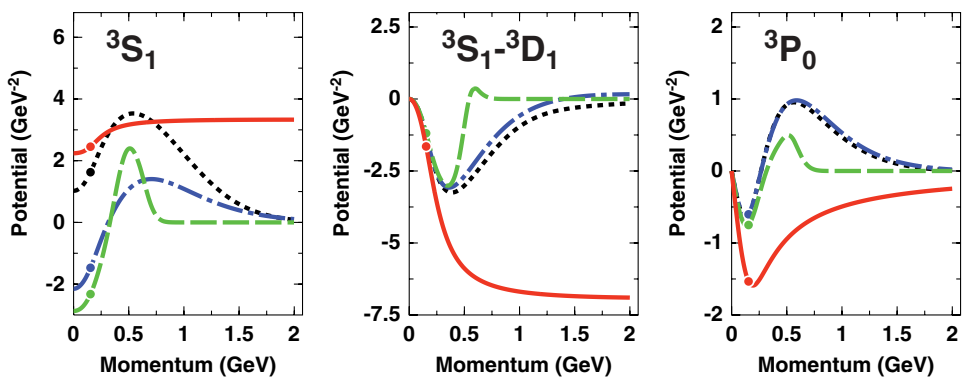

FIG. 3. (Color online) Half-off-shell NN potentials, $V/(2\pi)^3$, for partial-wave states as denoted. Potentials displayed are Argonne V18 [(black) dotted line], CD-Bonn [(blue) dash-dotted line], N3LO [(green) dashed line], and the renormalized chiral LO potential of Sec. II for $\Lambda = 10$ GeV [(red) solid line]. The on-shell point at $p = 153$ MeV/c (equivalent to $T_{\rm lab} = 50$ MeV) is marked by a filled circle.

TABLE II. Partial-wave contributions for S and P waves and total contributions to symmetric nuclear matter at a density equivalent to a Fermi momentum $k_F = 1.35\,\mathrm{fm}^{-1}$. Unless denoted otherwise, values not in parentheses or brackets are contributions to the potential energy as obtained using the Brueckner G matrix; values in parentheses are corresponding results obtained with the Born approximation, i.e., for $G = V$; and, finally, values in brackets are contributions to the wound integral.

Partial wave	LO[a]	N3LO[b]	CD-Bonn [50]	AV18 [51]
$^1S_0$	−13.42	−16.74	−16.76	−16.07
	(−5.98)	(−14.73)	(−12.64)	(−2.77)
	[0.056]	[0.008]	[0.005]	[0.017]
$^3S_1$	−13.54	−19.42	−18.96	−17.10
	(+10.65)	(−12.51)	(−8.63)	(+5.99)
	[0.090, 0.112][c]	[0.017, 0.017][c]	[0.004, 0.034][c]	[0.015, 0.053][c]
$^1P_1$	3.24	3.90	3.91	3.88
	(3.27)	(4.06)	(4.24)	(4.23)
	[0.000]	[0.001]	[0.002]	[0.001]
$^3P_0$	−1.01	−3.14	−3.08	−3.15
	(−6.30)	(−3.03)	(−2.41)	(−2.48)
	[0.109]	[0.001]	[0.002]	[0.002]
$^3P_1$	10.17	9.68	9.81	9.74
	(11.09)	(10.29)	(11.74)	(12.08)
	[0.004]	[0.003]	[0.006]	[0.007]
$^3P_2$	−5.37	−7.27	−7.05	−6.96
	(−1.49)	(−6.96)	(−6.25)	(−6.09)
	[0.015, 0.015][d]	[0.001, 0.001][d]	[0.003, 0.001][d]	[0.002, 0.002][d]
Total potential energy	−22.44	−37.02	−36.35	−33.96
	(+9.07)	(−26.72)	(−17.92)	(+6.96)
Kinetic energy	22.67	22.67	22.67	22.67
Total energy	+0.23	−14.35	−13.67	−11.29
	(+31.74)	(−4.05)	(+4.76)	(+29.63)
Total wound	[0.405]	[0.050]	[0.058]	[0.101]

[a]Renormalized LO NN potential of this work, with $\Lambda = 10$ GeV.
[b]Quantitative N^{3}LO NN potential regularized by a Gaussian with cutoff parameter $\Lambda = 0.5$ GeV [16].
[c]$^3S_1$-$^3S_1$ and $^3S_1$-$^3D_1$ contributions to the wound integral are given.
[d]$^3P_2$-$^3P_2$ and $^3P_2$-$^3F_2$ contributions to the wound integral are given.

Based on various pieces of circumstantial evidence, it is generally believed that the "empirical" saturation properties of symmetric nuclear matter are $k_F = 1.35 \pm 0.05$ fm^{-1} and $E/A = -16 \pm 1$ MeV [22]. Thus, our renormalized LO result shows considerable underbinding. In Ref. [31], a triton energy of −3.6 MeV was found for the converged LO result, which also deviates considerably from the empirical value of −8.5 MeV.

The chief reason for this lack of attraction is the fact that *the tensor force of the renormalized LO interaction is unusually strong*, as we explain now.

A simple indicator for the strength of the tensor force component contained in a given NN potential is the predicted D-state probability of the deuteron, P_D, because the transition from S to D states can only proceed via the tensor force. For the LO interaction at $\Lambda = 5$ GeV and $\Lambda = 10$ GeV, P_D comes out to be 7.2% (it is converged). Conventional potentials typically predict a lower P_D, namely, between 4% and 6%; for example, the AV18 [51], CD-Bonn [50], and N3LO [16] potentials predict 5.76%, 4.85%, and 4.51%, respectively. Historically, the highest P_D ever predicted by a "realistic" NN potential was 7.0%, by the Hamada-Johnston potential [52] of 1962.

In nuclear matter, the so-called wound integral κ is known to depend sensitively on the strength of the tensor force [22,48]. The wound integral is defined as

$$\kappa = \rho \int |\phi - \psi|^2 d\tau, \tag{23}$$

where ϕ denotes the uncorrelated two-nucleon wave function and ψ the correlated one, which are related by

$$G\phi = V\psi, \tag{24}$$

implying

$$\psi = \phi - \frac{Q}{H_0 - w} G\phi. \tag{25}$$

The physical significance of the wound integral is that it measures the probability for exiting two nucleons to states above the Fermi surface. This probability is high for "hard" and strong tensor force potentials. According to arguments conveyed by Brandow [53], a n-hole line diagram is proportional to κ^{n-1}, and hence, the convergence of the hole-line

expansion depends on the size of κ, with a large κ suggesting slow convergence.

As shown in the bottom row in Table II, the renormalized LO interaction produces a total κ of 40.5%, whereas the corresponding numbers are 10.1%, 5.8%, and 5.0% for AV18, CD-Bonn, and N3LO, respectively. The Hamada-Johnston potential generated a total κ of 21.1%. The partial-wave contributions to κ listed in Table II (numbers in brackets) confirm that the strong tensor force of the LO interaction is the main reason for the extraordinarily high κ. The $^3S_1$-$^3D_1$ transition, which depends entirely on the tensor force, contributes 11.2% to the LO κ, whereas it is 5.3%, 3.4%, and 1.7% for AV18, CD-Bonn, and N3LO, respectively. In Fig. 3, we show the $^3S_1$-$^3D_1$ transition potential of the LO interaction with $\Lambda = 10$ GeV and of conventional potentials, revealing dramatic differences, particularly for high momenta. An unusual difference also occurs in the $^3P_0$ state, where LO generates a contribution to κ of 10.9%, whereas conventional potentials have at most 0.2%. The tensor operator is known to have a large matrix element in the $^3P_0$ state. The $^3P_0$ potentials are included in Fig. 3.

The fact that a strong tensor force (and a large κ) leads to less binding energy in nuclear matter and finite nuclei can be understood as follows [22]. For the purpose of discussion, let us approximate the Brueckner G by

$$G(w) \approx V_C - V_T \frac{Q}{H_0 - w} V_T, \tag{26}$$

where V_C denotes the central force and $V_T = v_T S_{12}$ the tensor force component of a given NN potential (with S_{12} the usual tensor operator). Note now that all quantitative nuclear potentials are fit to the same NN data and, thus, produce essentially the same on-shell T matrix or, equivalently, the same on-shell K matrix, which in the preceding approximation is given by

$$K(w_f) \approx V_C - \mathcal{P} V_T \frac{1}{t - w_f} V_T, \tag{27}$$

where w_f is the free (purely kinetic) starting energy and $\mathcal{P}$ denotes the principal value.

A potential with a strong V_T (implying a large, attractive second-order tensor term) will have a less attractive central force V_C to arrive at the same on-shell K matrix compared to a potential with a weak tensor force. Now, when we enter nuclear matter and calculate the G matrix, Eq. (26), the Pauli operator Q [which is absent in the free-space Eq. (27)] and a larger energy denominator (owing to the single-particle potential in the many-body environment) reduce the magnitude of the second term in the G-matrix equation. These two medium effects are know as the Pauli and dispersion effects. The larger the attractive second-order tensor term in Eq. (26), the larger the reduction of the attraction through the medium effects. Therefore, potentials that produce large integral terms in the G-matrix equation will predict less attraction in the many-body system. When the central force is very strong ("hard" potential), this mechanism applies also to the iterations of the central term. This happens obviously in the $^1S_0$ state, where no tensor force is involved, but nevertheless, a large κ occurs for the LO interaction. This is also part of the reason why the $^3S_1$-$^3S_1$ contribution to the wound is large for LO, namely, 9.0% (cf. the very hard LO central force shown in the $^3S_1$ frame in Fig. 3).

An idea of the size of the integral term in the Brueckner equation, Eq. (17), is also obtained by comparing the Born approximation (i.e., $G = V$) with the full G. We therefore also provide in Table II the Born approximation results (numbers in parentheses) for the various partial-wave contributions.

As explained in length in Ref. [54], arguments similar to the preceding ones also apply to Faddeev calculations of three-particle energy. Thus, the substantial underbinding of the triton found in Ref. [31] is most likely also related to the huge tensor force of the renormalized LO interaction.

We note that the discussion of the severely reduced attraction in nuclear matter owing to "hard" central potentials and strong tensor force applies, of course, only to a calculation conducted in the Brueckner pair approximation. The huge wound integral suggests that there will be large three-, four-, and higher hole-line contributions, which may provide additional binding. However, evaluation of multi-hole-line contributions is extremely involved and cumbersome. Similar arguments apply when the coupled-cluster expansion is used to deal with the nuclear many-body problem [55]. Thus, an extraordinarily strong tensor force makes it very difficult to obtain converged results in the many-body system, which is one reason why a large tensor force potential is inconvenient, to say the least.

The best-studied phenomenology of nuclear forces is the one-boson-exchange model. This model includes a ρ meson, which produces a tensor force of opposite sign compared to the pion (cf. Fig. 3.7 in Ref. [22]). Careful studies have shown that the reduction of the pion's tensor force at short range by the ρ meson is crucial to arrive at a realistic strength for the nuclear tensor force [56].

In ChPT theory, contributions from heavy mesons, like the ρ, are too short-ranged to be dissolved, but instead contact terms are added to the theory. The set of contacts that appears at NLO ($\sim Q^2$) includes a tensor term that may be perceived as simulating ρ exchange. Therefore, there is a chance that, at NLO or higher order, the problem of the extraordinarily large tensor force encountered at LO will be resolved.

IV. COMPARISON WITH OTHER WORKS AND THE BROADER PERSPECTIVE

Our results clearly show that the saturation mechanism in nuclear matter is compatible with a nonperturbative renormalization of the venerable 1PE potential. However, the binding energy turns out to be rather low and the approach looks discouraging from a coupled-cluster expansion point of view. Let us therefore analyze our results in the light of other chiral approaches to nuclear matter, based in spirit on the EFT concept, where mainly perturbative schemes but also low cutoffs have been employed. We also provide some perspective on future work.

If the cutoff parameter takes sufficiently small values, perturbation theory becomes applicable, as in this case

high-momentum components are suppressed. For the smallest cutoff value presented in Fig. 2, $\Lambda = 0.5$ GeV, we see a clear attraction that is strongly dependent, in fact linearly, on the Fermi momentum. Actually, chiral-symmetry-based approaches for nuclear matter have also been pursued in Ref. [57] in a purely perturbative scheme. The main source of attraction stems from once iterated 1PE, which is proportional to the density, $\sim k_F^3$, and depends linearly also on the cutoff parameter, which needs to be fine-tuned to a value $\Lambda = 0.4$–0.5 GeV to achieve saturation. The origin of the divergence is related to vacuum amplitudes, which, to nth order with the 1PE potential, yield a contribution to the T matrix, Eq. (1), scaling by naive power counting as $\sim\Lambda^{n-1}$ (pion mass neglected). The present calculation contains iterated 1PE *with* additional counterterms to all orders for *any* value of the cutoff parameter, and as we see, it does not exhibit this very strong k_F^3-dependent attraction. Thus, in Fig. 2 only some residual cutoff effect is displayed *after* low-energy NN physics has been fixed. Clearly, the separately large perturbative contributions that scale with positive powers of Λ would not converge without inclusion of counterterms to all orders.

An NLO G-matrix calculation with a finite cutoff of $\Lambda = 500$ MeV and counterterms was also undertaken in Ref. [58], and saturation was found. As noted in Ref [32], there would be a fundamental problem of removing the cutoff at that order, as the deuteron becomes unbound owing to the strong $1/r^5$ repulsive interaction in the triplet channels. More recently, a perturbative approach has also been proposed where a power-counting scheme is introduced with extremely low cutoff values, $\Lambda \sim m_\pi$, for which saturation is achieved [59,60].

Finally, $V_{\mathrm{low}\,k}$ approaches represent a coarse graining of the interaction in the physically accessible NN elastic region for CM momenta $k \leqslant \Lambda \approx 400$ MeV, with the result that *all* high-precision potentials fitting data with $\chi^2/\mathrm{df} \approx 1$ including the 1PE tail collapse into a unique $V_{\mathrm{low}\,k}$ potential. It has been suggested [62] that low-momentum interactions in general and chiral N^{3}LO interactions (having their own cutoff $\Lambda \approx 500$–700 MeV [16,17]), in particular, can be treated perturbatively in nuclear matter calculations, as the corresponding Weinberg eigenvalues lie inside the unit circle. Given the universality of the $V_{\mathrm{low}\,k}$ approach and the fact that all interactions contain 1PE while successfully describing the phase shifts in vacuum, there seems to be no fingerprint left of chiral dynamics from the two-body sector. When three-body chiral forces determined from few-body data are included, realistic saturation properties with more controlled uncertainties are obtained [63]. A trading between two- and three-body forces is observed, and the role played by 3N forces becomes less important as the $V_{\mathrm{low}\,k}$ cutoff is increased from $\Lambda = 400$ MeV to $\Lambda = 560$ MeV, although they still produce saturation at realistic densities.

Clearly, *explicit* chiral dynamics is enhanced for larger cutoff values but also the theoretical difficulties increase. The spirit of the original proposal of Weinberg's was that the potential could be perturbatively defined according to a prescribed counting. This is clearly possible at large impact parameters, where neither the strength of the nuclear force nor our lack of knowledge of its short-distance components prevents us from using a fuzzy but sensibly large momentum cutoff value. In a sense, the G-matrix approach is close in spirit to this idea, where the strength of the nucleon force in the nuclear medium is characterized by an effective interaction. As we have shown, the LO 1PE interaction yields cutoff-independent results, is entirely parameterized by vacuum properties, and has the nice feature of saturation. The problem is to define *what* is meant by NLO, and in nuclear matter a scheme very close to the standard perturbative EFT idea is expected to face the same problems already found in the vacuum sector and described in Sec. I.

Several possible and perturbatively motivated schemes for renormalization of the chiral NN interaction already suggested in Ref. [31] include considering higher pion exchanges as perturbations around the LO calculation using distorted waves. This proposal was analyzed in Refs. [32] and [42], where it was shown to be feasible and cutoff independent, but with little gain from a practical viewpoint: many more counterterms were needed and a worse description was obtained in the $^1S_0$ channel and the deuteron. Actually, nonperturbative calculations with singular potentials behave nonanalytically in the coupling constant, that is, $1/f_\pi^\alpha$ with noninteger α [32]. An RG-based program for pion-full theories was advanced in Ref. [27], with further detailed results provided in Ref. [28]. This RG analysis yields one counterterm in each S wave of order Q^{-1}, which must be iterated, and one in each ^3P_J and ^3D_J wave of order $Q^{-1/2}$, which may be iterated. This is very similar to the power counting suggested by Nogga *et al.* [31]. Subleading counterterms occur in $^1S_0$ at Q^0 and in $^3S_1$ at $Q^{1/2}$, whereas the subleading ^3P_J and ^3D_J terms are of order $Q^{3/2}$. Subleading and higher terms ought to be treated perturbatively. Finally each S wave receives an additional term at order Q^2. The number of terms at Q^3 in the RG scheme is larger than in Weinberg counting at NNLO; however, except for the subleading D-wave interactions, it is essentially the same as Weinberg counting at N^{3}LO (Q^4).

Toy models where a $1/r^2$ singular potential is perturbed by $1/r^4$ interactions provide useful insight, but the consequences for more realistic cases have not yet been worked out [38]. A follow-up KSW scheme has been pursued in Ref. [39], where the short-distance singularity is tamed by the introduction of a Pauli-Villars-like pion mass of about the ρ-meson mass. An important prerequisite for specific calculations is the perturbative renormalizability described in Ref. [42]. However, as we discussed in Ref. [36] the huge change needed from the simple LO $^1S_0$ phase to the real observed one does not suggest any sort of small effect in the high-momentum region, as nonperturbatively renormalized calculations suggest. As already mentioned, this partial wave provides an important contribution to the energy in nuclear matter.

Turning to nonperturbative schemes, it was noted in Ref. [49] that off-shell properties of renormalized chiral potentials do not look much different from more conventional ones as the singularities are effectively removed. Furthermore, we know that the inclusion of Δ degrees of freedom in chiral potentials [10] leads to a pattern of better convergence, where, for example, the NLO-Δ and NNLO-Δ deuteron does exist [61], with an acceptable phenomenological success and

where much larger and naively more natural momentum cutoff values display a better convergence (see also the second listing in Ref. [11], where $\Lambda \sim 1$ GeV is taken). Actually, this is a case where the discussion on renormalizability becomes pointless, as the renormalized and the natural-sized cutoffs are not too far apart, because finite cutoff effects are less important the more singular the potential [36]. Indeed, at order ν in the chiral counting the potential scales as $1/(\Lambda_\chi^{\nu+2} r^{3+\nu})$ and then the finite cutoff correction, $\delta_\Lambda(k)$, to the renormalized phase shifts, $\delta_\infty(k)$, behaves as $\delta_\infty(k) - \delta_\Lambda(k) = \mathcal{O}(\Lambda^{-1/2-\nu/2})$ [36]. Moreover, unlike the Δ-less renormalized scheme, the deuteron D-state probability becomes $P_D \sim 5.8\%$, a value comparable to conventional and phenomenologically successful potentials. According to our discussion, the corresponding wound integral would be small enough to guarantee a good converging pattern of few-body correlations in the nuclear many-body problem. Thus, a G-matrix approach applied to chiral interactions including Δ-isobar degrees of freedom has some chance of furnishing the multiple theoretical requirements of renormalizability, power counting for the potential (a la Weinberg), and, presumably, convergence in the nuclear medium. At present it is unclear whether such a scheme will be phenomenologically successful, as the potentials do not contain the important spin-orbit contributions and it remains to be seen if those can be represented by appropriate counterterms.

V. CONCLUSIONS

In this work, we have renormalized the two-nucleon interaction at LO in ChPT using the scheme proposed by Nogga *et al.* [31]—also known as modified Weinberg counting. With this interaction, we have calculated the energy of symmetric nuclear matter in the Brueckner pair approximation. We find that the energy per nucleon as a function of nuclear matter density converges to a cutoff-independent (i.e., renormalized) result and shows saturation. The predicted value for the energy per nucleon at saturation shows considerable underbinding, which is in line with the converged LO prediction for the triton binding energy in Ref. [31]. We demonstrate that the LO interaction contains an unusually strong tensor force (from pion exchange), which is the main reason for the lack of binding in few- and many-body systems. In fact, the tensor force is stronger than that of any NN potential ever constructed in the 50-year history of realistic nuclear forces.

The huge tensor force of the renormalized LO interaction leads to the unusually large wound integral of 40% in nuclear matter, which implies a very slow convergence of the hole-line expansion and, similarly, the coupled-cluster expansion, rendering this interaction impractical for many-body calculations.

It is well known from the meson theory of nuclear forces that the tensor force produced by the pion needs to be damped at short range (or high momenta). In conventional models, this is achieved through πNN form factors and contributions from heavy-meson exchange (particularly, ρ exchange). ChPT, which does not include heavy mesons, provides a contact term of tensor structure at NLO. Thus, a more realistic tensor force may be expected at higher orders.

Several possible schemes for renormalization of the chiral NN interaction that have recently been given serious thought are designed to renormalize the LO interaction nonperturbatively (as done in the present work) and to add higher-order corrections in perturbation theory. However, in view of the unusual properties of the renormalized LO interaction and in view of the poor convergence of the nuclear many-body problem with this interaction, there is doubt whether this interaction and its predictions can serve as a reasonable and efficient starting point that is improved by perturbative corrections. To make the interaction more suitable for many-body calculations, one may consider applying renormalization-group methods [64] to construct a soft two-nucleon force (2NF) plus a three-nucleon force (3NF) equivalent to the original LO interaction. Then there will be no problem with the application of this "$V_{\mathrm{low}\,k}$" 2NF in the many-body system. However, it may turn out that the 3NF (needed for proper equivalence with the original LO interaction) is so strong that the issue is just shifted toward a problem with the convergence of the 3NF contribution.

ACKNOWLEDGMENTS

This research was supported in part by the US Department of Energy under Grant No. DE-FG02-03ER41270 (R.M.); the Ministerio de Ciencia y Tecnología under Contract No. FPA2007-65748; Junta de Castilla y León under Contract Nos. SA-106A07 and GR12 (D.R.E.); the European Community-Research Infrastructure Integrating Activity "Study of Strongly Interacting Matter" (HadronPhysics2 Grant No. 227431); the Spanish Ingenio-Consolider 2010 Program CPAN (CSD2007-00042); Spanish DGI and FEDER funds under Grant No. FIS2008-01143/FIS; Junta de Andalucía Grant No. FQM225-05; and EU Integrated Infrastructure Initiative Hadron Physics Project Contract No. RII3-CT-2004-506078 (E.R.A.).

APPENDIX: FINITENESS OF THE G MATRIX FOR SINGULAR INTERACTIONS

In this Appendix, we show the finiteness of the G matrix for singular interactions such as the 1PE potential for the gap choice, Eq. (22) [48], which implies that the potential energy is suppressed compared to the kinetic energy in the high-momentum region. The idea underlying the proof is that short-distance physics much below the healing distance [46] does not depend on the Fermi momentum. It is convenient to start with the extension of the G matrix to any arbitrary energy z,

$$G(z) = V + V\frac{Q}{z - H_0}G(z), \tag{A1}$$

and rewrite this equation as

$$G(z)^{-1} = V^{-1} - \frac{Q}{z - H_0}. \tag{A2}$$

The G matrix used in the main text, Eq. (17), corresponds to taking $z = w = e(m) + e(n)$. We now introduce the extended

K matrix, defined as

$$K(z) = V + V\mathcal{P}\frac{1}{z-t}K(z), \tag{A3}$$

where $\mathcal{P}$ denotes the principal value and t is the kinetic energy operator. The K matrix used in the text, Eq. (27), corresponds to using $z = w_f = t(m) + t(n)$. Below threshold, the principal value prescription can be removed, as the pole is never hit. This equation can likewise be written as

$$K(z)^{-1} = V^{-1} - \mathcal{P}\frac{1}{z-t}. \tag{A4}$$

In Ref. [49] the finiteness of the K matrix *off-shell* for short-distance singular interactions was established solely on the basis of on-shell renormalization conditions. This means that if K is finite on-shell, then the off-shell extension remains finite as well. Note that the renormalization conditions are basically equivalent to a *fixed* energy, for example, zero energy. Thus, K will remain finite *also* below threshold. Taking $z = w$ and subtracting $K(w)^{-1}$ from $G(w)^{-1}$, we have (principal value wherever necessary)

$$G(w) = K(w) + K(w) \times \left[\frac{Q-1}{w-H_0} - \frac{1}{w-t} + \frac{1}{w-H_0}\right] G(w). \tag{A5}$$

In this equation, the regulator may be effectively removed from the K matrix at the operator level even if the energy $w \neq w_f$ does not correspond to the scattering energy [49]. If the interaction is attractive, the single-particle potential $U(n) < 0$ and $w < w_f$ and the (finite-cutoff) $K(w)$ involves states below threshold. As we see, the first integral involves only states below the Fermi surface and is thus bound and cutoff independent provided that $k_F \ll \Lambda$. The only remaining piece where the cutoff enters explicitly is in the last two terms involving $(w - H_0)^{-1} - (w - t)^{-1}$, which, for the gap choice [see Eq. (22)] [48], vanishes for states above the Fermi surface, making the integral convergent as well. Thus, the finiteness of the G matrix is reduced to the finiteness of the K matrix *off-shell*.

[1] S. Weinberg, Physica A **96**, 327 (1979).
[2] J. Gasser and H. Leutwyler, Ann. Phys. (NY) **158**, 142 (1984).
[3] J. Gasser, M. E. Sainio, and A. Švarc, Nucl. Phys. **B307**, 779 (1988).
[4] S. Weinberg, Phys. Lett. **B251**, 288 (1990); Nucl. Phys. **B363**, 3 (1991).
[5] S. Weinberg, Phys. Lett. **B295**, 114 (1992).
[6] C. Ordóñez, L. Ray, and U. van Kolck, Phys. Rev. Lett. **72**, 1982 (1994); Phys. Rev. C **53**, 2086 (1996).
[7] U. van Kolck, Phys. Rev. C **49**, 2932 (1994).
[8] U. van Kolck, Prog. Part. Nucl. Phys. **43**, 337 (1999).
[9] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. **A625**, 758 (1997).
[10] N. Kaiser, S. Gerstendörfer, and W. Weise, Nucl. Phys. **A637**, 395 (1998).
[11] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A637**, 107 (1998); **A671**, 295 (2000).
[12] P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002).
[13] N. Kaiser, Phys. Rev. C **64**, 057001 (2001); **65**, 017001 (2001).
[14] D. R. Entem and R. Machleidt, Phys. Lett. **B524**, 93 (2002).
[15] D. R. Entem and R. Machleidt, Phys. Rev. C **66**, 014002 (2002).
[16] D. R. Entem and R. Machleidt, Phys. Rev. C **68**, 041001(R) (2003).
[17] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A747**, 362 (2005).
[18] R. Machleidt and D. R. Entem, J. Phys. G **31**, S1235 (2005).
[19] R. Machleidt, *Nuclear Forces from Chiral Effective Field Theory*, Lecture Series, arXiv:0704.0807 [nucl-th].
[20] E. Epelbaum, H. W. Hammer, and U.-G. Meißner, *Modern Theory of Nuclear Forces*, arXiv:0811.1338 [nucl-th].
[21] R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **149**, 1 (1987).
[22] R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989).
[23] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996); Phys. Lett. **B424**, 390 (1998); Nucl. Phys. **B534**, 329 (1998).
[24] S. Fleming, T. Mehen, and I. W. Stewart, Nucl. Phys. **A677**, 313 (2000); Phys. Rev. C **61**, 044005 (2000).
[25] D. R. Phillips, S. R. Beane, and T. D. Cohen, Ann. Phys. (NY) **263**, 255 (1998).
[26] T. Frederico, V. S. Timoteo, and L. Tomio, Nucl. Phys. **A653**, 209 (1999).
[27] M. C. Birse, Phys. Rev. C **74**, 014003 (2006).
[28] M. C. Birse, *More Effective Theory of Nuclear Forces*, arXiv:0909.4641 [nucl-th].
[29] S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. **A700**, 377 (2002).
[30] M. P. Valderrama and E. R. Arriola, Phys. Rev. C **72**, 054002 (2005).
[31] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005).
[32] M. P. Valderrama and E. R. Arriola, Phys. Rev. C **74**, 054001 (2006).
[33] M. P. Valderrama and E. R. Arriola, Phys. Rev. C **74**, 064004 (2006); **75**, 059905(E) (2007).
[34] E. Epelbaum and U.-G. Meißner, arXiv:nucl-th/0609037.
[35] M. P. Valderrama and E. R. Arriola, Ann. Phys. (NY) **323**, 1037 (2008).
[36] D. R. Entem, E. R. Arriola, M. P. Valderrama, and R. Machleidt, Phys. Rev. C **77**, 044006 (2008).
[37] C.-J. Yang, Ch. Elster, and D. R. Phillips, Phys. Rev. C **77**, 014002 (2008); arXiv:0905.4943 [nucl-th].
[38] B. Long and U. van Kolck, Ann. Phys. (NY) **323**, 1304 (2008).
[39] S. R. Beane, D. B. Kaplan, and A. Vuorinen, Phys. Rev. C **80**, 011001 (2009).
[40] M. P. Valderrama, A. Nogga, E. R. Arriola, and D. R. Phillips, Eur. Phys. J. A **36**, 315 (2008).
[41] E. Epelbaum and J. Gegelia, Eur. Phys. J. A **41**, 341 (2009).
[42] M. P. Valderrama, *Perturbative Renormalizability of Chiral Two Pion Exchange in Nucleon-Nucleon Scattering*, arXiv:0912.0699 [nucl-th].
[43] C. Amsler *et al.*, Phys. Lett. **B667**, 1 (2008).

[44] V. G. J. Stoks, R. A. M. Klomp, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. C **48**, 792 (1993).
[45] R. A. Arndt, I. I. Strakovsky, and R. L. Workman, SAID, Scattering Analysis Interactive Dial-in computer facility, George Washington University (formerly Virginia Polytechnic Institute), solution SM99 (Summer 1999); for more information see, e.g., Phys. Rev. C **50**, 2731 (1994).
[46] B. D. Day, Rev. Mod. Phys. **39**, 719 (1967).
[47] H. A. Bethe, Annu. Rev. Nucl. Sci. **21**, 93 (1971).
[48] M. I. Haftel and F. Tabakin, Nucl. Phys. **A158**, 1 (1970).
[49] D. R. Entem and E. R. Arriola, Phys. Rev. C **80**, 047001 (2009).
[50] R. Machleidt, Phys. Rev. C **63**, 024001 (2001).
[51] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995).
[52] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).
[53] B. H. Brandow, Rev. Mod. Phys. **39**, 771 (1967).
[54] R. A. Brandenburg, G. S. Chulick, R. Machleidt, A. Picklesimer, and R. M. Thaler, Phys. Rev. C **37**, 1245 (1988).
[55] G. Hagen, T. Papenbrock, D. J. Dean, and M. Hjorth-Jensen, Phys. Rev. Lett. **101**, 092502 (2008); and references therein.
[56] G. E. Brown and R. Machleidt, Phys. Rev. C **50**, 1731 (1994).
[57] N. Kaiser, S. Fritsch, and W. Weise, Nucl. Phys. **A697**, 255 (2002).
[58] P. Saviankou, S. Krewald, E. Epelbaum, and U. G. Meissner, arXiv:0802.3782 [nucl-th].
[59] J. A. Oller, A. Lacour, and U. G. Meissner, J. Phys. G **37**, 015106 (2010).
[60] J. A. Oller, A. Lacour, and U. G. Meissner, *Non-perturbative Methods for a Chiral Effective Field Theory of Finite Density Nuclear Systems*, arXiv:0906.2349 [nucl-th].
[61] M. P. Valderrama and E. R. Arriola, Phys. Rev. C **79**, 044001 (2009).
[62] S. K. Bogner, A. Schwenk, R. J. Furnstahl, and A. Nogga, Nucl. Phys. **A763**, 59 (2005).
[63] S. K. Bogner, R. J. Furnstahl, A. Nogga, and A. Schwenk, arXiv:0903.3366 [nucl-th].
[64] S. K. Bogner, R. J. Furnstahl, and R. J. Perry, Phys. Rev. C **75**, 061001(R) (2007).