Comment on "Exact three-dimensional wave function and the on-shell t matrix for the sharply cut-off Coulomb potential: Failure of the standard renormalization factor"

Konstantin A. Kouzakov,[1,*] Yuri V. Popov,[2,†] and Vladimir L. Shablov[3]

[1] *Department of Nuclear Physics and Quantum Theory of Collisions, Faculty of Physics, Moscow State University, Moscow RU-119991, Russia*

[2] *Nuclear Physics Institute, Moscow State University, Moscow RU-119991, Russia*

[3] *Obninsk State Technical University for Nuclear Power Engineering, Obninsk, Kaluga Region RU-249040, Russia*

(Received 26 May 2009; revised manuscript received 28 September 2009; published 20 January 2010)

The solutions analytically derived by W. Glöckle, J. Golak, R. Skibiński, and H. Witala [Phys. Rev. C **79**, 044003 (2009)] for the three-dimensional wave function and on-shell t matrix in the case of scattering on a sharply cut-off Coulomb potential appear to be fallacious if finite values of a cut-off radius are concerned. And the analysis carried out for an infinite cut-off radius limit is incomplete.

DOI: 10.1103/PhysRevC.81.019801 PACS number(s): 21.45.Bc, 25.10.+s, 25.40.Cm

In a recent article [1], nonrelativistic scattering of two equally charged particles with mass m interacting via potential $V(r) = \frac{e^2}{r}\Theta(R-r)$ was considered. It was stated that the exact wave function and scattering amplitude were analytically derived for arbitrary values of a cut-off radius R. On this basis a renormalization factor that relates the scattering amplitude in the limit $R \to \infty$ with the physical Coulomb scattering amplitude was obtained. The purpose of this Comment is threefold: (i) to point out that the analytical results of Ref. [1] are erroneous for finite values of R, (ii) to show that the analysis performed there for the limit $R \to \infty$ lacks completeness, and (iii) to indicate a different renormalization approach that is free from uncertainties associated with the cut-off renormalization.

In Ref. [1] the solution of the Lippmann-Schwinger equation for $r < R$ was incorrectly assumed to be of the form

$$\Psi_R^{(+)}(\vec r) = Ae^{i\vec p\cdot\vec r}{}_1F_1(-i\eta, 1, i(pr - \vec p\cdot\vec r)), \tag{1}$$

where $\eta = \frac{me^2}{2p}$ is a Sommerfeld parameter. The constant[1]

$$A = \frac{1}{{}_1F_1(-i\eta, 1, 2ipR)} \tag{2}$$

was determined in Ref. [1] by inserting Eq. (1) into the Lippmann-Schwinger equation,

$$\Psi_R^{(+)}(\vec r) = e^{i\vec p\vec r} - \frac{\mu e^2}{2\pi}\int \frac{d^3r'}{r'}\frac{e^{ip|\vec r-\vec r'|}}{|\vec r-\vec r'|}\Theta(R-r')\Psi_R^{(+)}(\vec r'), \tag{3}$$

and solving the latter at $r = 0$. The correct form of the solution in the interior region $r < R$ is

$$\Psi_R^{(+)}(\vec r) = \frac{1}{4\pi}\int d^2\hat k\mathcal{A}(\hat k)e^{ip\hat k\cdot\vec r}{}_1F_1(-i\eta, 1, i(pr - p\hat k\cdot\vec r)), \tag{4}$$

where the function $\mathcal{A}(\hat k)$ is defined on a unit sphere. To demonstrate that $\mathcal{A}(\hat k) \neq 4\pi A\delta^2(\hat k - \hat p)$, that is, that Eq. (4) does not reduce to Eq. (1), one may employ the usual partial wave formalism (see, for instance, Ref. [2]). Consider the following expansion in Legendre polynomials:

$$\mathcal{A}(\hat k) = \sum_l (2l+1)A_l P_l(\hat p\cdot\hat k). \tag{5}$$

Matching the interior Lippmann-Schwinger solution and its derivative to the exterior ones at $r = R$ yields

$$A_l = \frac{i(pR)^{-2}}{W(\psi_l, h_l^{(1)})(pR)}, \tag{6}$$

where $h_l^{(1)}$ is a spherical Hankel function of the first kind [3] and

$$\psi_l(pr) = e^{i\sigma_l}\frac{|\Gamma(l+1+i\eta)|}{\Gamma(1+i\eta)}\frac{(2pr)^l}{(2l+1)!} \times e^{-ipr}{}_1F_1(l+1-i\eta, 2l+2, 2ipr),$$

with the Coulomb phase shift $\sigma_l = \arg\Gamma(l+1+i\eta)$. It can be checked that $A_0 = A$ but $A_{l\geqslant 1} \neq A$; that is, Eq. (1) is invalid for any finite value of R. The expression for the scattering amplitude (the on-shell t matrix) obtained in Ref. [1] for the case of finite values of R is invalid as well, because it derives from the wave function (1).

In the following, we discuss the limit $R \to \infty$ considered in Ref. [1]. The wave function (1) can be presented as the product $C_R\Psi_c^{(+)}$, where $\Psi_c^{(+)}$ is a Coulomb wave and C_R is a constant ($C_{R\to\infty} \to e^{-i\eta\ln(2pR)}$). The Coulomb wave satisfies a homogeneous Lippmann-Schwinger equation [4]:

$$\Psi_c^{(+)}(\vec r) = -\frac{\mu e^2}{2\pi}\int \frac{d^3r'}{r'}\frac{e^{ip|\vec r-\vec r'|}}{|\vec r-\vec r'|}\Psi_c^{(+)}(\vec r'). \tag{7}$$

Let us introduce an auxiliary function which is a difference between the exact wave function (4) and the wave function (1) in the limit $R \to \infty$:

$$\psi_R(\vec r) = \Psi_R^{(+)}(\vec r) - e^{-i\eta\ln(2pR)}\Psi_c^{(+)}(\vec r). \tag{8}$$

According to Eqs. (3) and (7), this function satisfies the following equation ($r < R$):

$$\psi_R(\vec r) = \psi_R^{(0)}(\vec r) - \frac{\mu e^2}{2\pi}\int \frac{d^3r'}{r'}\frac{e^{ip|\vec r-\vec r'|}}{|\vec r-\vec r'|}\Theta(R-r')\psi_R(\vec r'), \tag{9}$$

*kouzakov@srd.sinp.msu.ru

†popov@srd.sinp.msu.ru

[1] The normalization factor $\frac{1}{(2\pi)^{3/2}}$ is suppressed throughout.

with the inhomogeneous term

$$\psi_R^{(0)}(\vec{r}) = e^{i\vec{p}\vec{r}} + \frac{\mu e^2}{2\pi} e^{-i\eta \ln(2pR)} \times \int \frac{d^3r'}{r'} \frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \Theta(r'-R)\Psi_c^{(+)}(\vec{r}'). \quad (10)$$

For $r \ll R$ one has approximately

$$\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{e^{ipr'}}{r'} e^{-i\vec{p}'\vec{r}} \quad (\vec{p}' = p\hat{r}'),$$

and thus it can be shown that $\psi_R^{(0)}(\vec{r}) \approx 0$. However this observation does not imply that $\psi_R^{(0)}(\vec{r}) \approx 0$ for any $r < R$. In fact, the situation is nontrivial in the region $r \lesssim R$ (within the partial wave formalism this amounts to the $l \lesssim pR$ terms [5]).

Using Eq. (8), the scattering amplitude can be presented as

$$f_R = -\frac{\mu e^2}{2\pi} e^{-i\eta \ln(2pR)} \int \frac{d^3r'}{r'} e^{-i\vec{p}'\vec{r}'} \Theta(R-r')\Psi_c^{(+)}(\vec{r}') - \frac{\mu e^2}{2\pi} \int \frac{d^3r'}{r'} e^{-i\vec{p}'\vec{r}'} \Theta(R-r')\psi_R(\vec{r}'), \quad (11)$$

where $\vec{p}' = p\hat{r}$. The asymptotic behavior of the first term was examined in Ref. [1] but the second term was not considered there at all. However, because of nontrivial properties of ψ_R in the region $r \lesssim R$, the latter term might yield a nonvanishing contribution to the scattering amplitude in the limit $R \to \infty$. Thus, the analysis of the case $R \to \infty$ carried out in Ref. [1] is incomplete and the obtained renormalization factor requires more rigorous substantiation.

Finally, it is useful to note that the renormalization treatments involving cut-off Coulomb potentials are of doubtful value from a practical viewpoint, especially in the case of many-body Coulomb scattering. In this respect, the methods based on regularization and renormalization of the Lippmann-Schwinger equations in the on-shell limit are more efficient. The two-particle case is fully explored: (i) the Green's function is derived analytically both in coordinate and in momentum representations [6], (ii) an off-shell amplitude is known [7], and (iii) the rules for taking the on-shell limit are formulated [8]. The two-particle results can be straightforwardly generalized to the many-particle case (see, for example, Ref. [9]).

We thank Sergue I. Vinitsky for drawing our attention to the article [1] and Akram Mukhamedzhanov for useful discussions.

[1] W. Glöckle, J. Golak, R. Skibiński, and H. Witała, Phys. Rev. C **79**, 044003 (2009).
[2] J. R. Taylor, *Scattering Theory: The Quantum Theory of Nonrelativistic Collisions* (Wiley, New York, 1972).
[3] *Handbook of Mathematical Functions*, edited by Milton Abramowitz and Irene A. Stegun (Dover Publishers, New York, 1972).
[4] L. D. Faddeev and S. P. Merkuriev, *Quantum Scattering Theory for Several Particle Systems* (Kluwer, Dordrecht, 1993).
[5] W. F. Ford, Phys. Rev. **133**, B1616 (1964).
[6] J. Schwinger, J. Math. Phys. (NY) **5**, 1606 (1964).
[7] C. S. Shastri, L. Kumar, and J. Callaway, Phys. Rev. A **1**, 1137 (1970).
[8] Yu. Popov, J. Phys. B: At. Mol. Phys. **14**, 2449 (1981).
[9] V. L. Shablov, V. A. Bilyk, and Yu. V. Popov, Phys. Rev. A **65**, 042719 (2002).