# Semi-inclusive distributions in statistical models

V. V. Begun,[1,2] M. Gaździcki,[3,4] and M. I. Gorenstein[1,2]
[1]*Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine*
[2]*Frankfurt Institute for Advanced Studies, D-60438 Frankfurt, Germany*
[3]*Institut für Kernphysik, University of Frankfurt, D-60325 Frankfurt, Germany*
[4]*Jan Kochanowski University, PL-25-406 Kielce, Poland*
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The semi-inclusive properties of the system of neutral and charged particles with net charge equal to zero are considered in the grand canonical, canonical and microcanonical ensembles as well as in a microcanonical ensemble with scaling volume fluctuations. Distributions of neutral-particle multiplicity and charged-particle momentum are calculated as a function of the number of charged particles. Different statistical ensembles lead to qualitatively different dependencies. They are being compared with the corresponding experimental data on multihadron production in $p + p$ interactions at high energies.

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## I. INTRODUCTION

In relativistic high-energy collisions, many properties of produced particles follow simple rules of statistical mechanics. The single-particle momentum spectrum approximately has the Boltzmann form, $dN/d^3p \sim \exp[-(p^2+m^2)^{1/2}/T]$, in the local rest frame of produced matter [1]. The mean-particle multiplicity of heavy particles ($m \gg T$) is also governed by the Boltzmann factor, $\langle N\rangle \sim \exp(-m/T)$. Here $T$, $p$, and $m$ are the temperature parameter, the particle momentum, and its mass, respectively. The temperature parameter extracted from the data on $p + p$ interactions is in the range of 160–190 MeV [2]. Thus, almost all particles are produced at low transverse momenta, $p_T$, and with low masses ($p_T$, $m \leqslant 2$ GeV).

However, the standard statistical approach fails to reproduce the KNO scaling [3] of multiplicity distributions observed in the data on $p + p$, $p + \overline{p}$, and $e^+ + e^-$ collisions [4–6]. The other problems are a power-law behavior of the single-particle transverse momentum spectrum at large transverse momenta, $dN/d^3p \sim p_T^{-K_p}$, and a power-law dependence of a mean multiplicity of heavy particles, $\langle N\rangle \sim m^{-K_m}$, where $K_m \cong K_p - 3$ [7]. In our previous paper [8], an extension of the standard statistical approach was suggested to a region of large transverse momenta and/or large particle masses ($p_T, m \geqslant 3$ GeV) by taking into account volume fluctuations. The proposed model, the statistical microcanonical ensemble with scaling volume fluctuations (MCE/sVF), allows for solving the previously mentioned problems of the statistical approach.

Hadron production in high-energy collisions is characterized by two types of quantities: inclusive and semi-inclusive ones. Statistical models are usually used to describe inclusive quantities such as mean multiplicity or mean transverse momentum. They are calculated by the summation over all microstates of the system with corresponding statistical weights. In this article, we study selected properties of semi-inclusive quantities within statistical models. In this case, the statistical summation is restricted by the additional condition (e.g., by a requirement that charged hadron multiplicity is fixed). The grand canonical, canonical, and microcanonical ensembles as well as MCE/sVF will be used. In particular, the mean multiplicity of neutral particles and average transverse momentum of charged particles are considered at a fixed charged-particle multiplicity. The obtained model predictions are compared with the trends observed in the experimental data.

For the sake of simplicity, the system of noninteracting massless Boltzmann particles—neutral, positively charged, and negatively charged—with the total net charge equal to zero $Q = N_+ - N_- = 0$ is considered. The degeneracy factors are assumed to be $g_0 = g_+ = g_- = 1$, and the temperature parameter is set to $T = 160$ MeV for quantitative calculations.

This article is organized as follows. The joint multiplicity distributions of neutral and negatively charged particles, correlations, semi-inclusive averages, and the effects of quantum statistic are calculated in Sec. II. Semi-inclusive momentum spectra are obtained and discussed in Sec. III. A comparison with available data is presented in Sec. IV. The summary presented in Sec. V closes the article.

## II. MULTIPLICITY DISTRIBUTIONS

### A. Grand canonical ensemble

The grand canonical ensemble (GCE) is defined by the system volume $V$, temperature $T$, and charge chemical potential $\mu_Q$. The chemical potential $\mu_Q$ regulates an average value of the conserved charge $Q$. For the system with zero net charge considered here, $\mu_Q$ is equal to zero. The mean-particle multiplicities and the average energy in the GCE are as follows:

$$\langle N_0\rangle_{\rm GCE} = \langle N_+\rangle_{\rm GCE} = \langle N_-\rangle_{\rm GCE} \equiv \overline{N} = V\,T^3/\pi^2, \tag{1}$$

$$\langle E\rangle_{\rm GCE} = 3T\,\langle N_0\rangle_{\rm GCE} + 3T\,\langle N_-\rangle_{\rm GCE} + 3T\,\langle N_+\rangle_{\rm GCE} \equiv \overline{E} = 9T\,\overline{N}. \tag{2}$$

In the GCE, the neutral and charged multiplicities $N_0$ and $N_+$, $N_-$ are uncorrelated and obey the Poisson distribution. Thus, the joint distribution of neutral $N_0$ and negatively charged particles $N_-$ is given by the product of two Poisson distributions that can be approximated by the product of two

 

Gauss distributions at $\overline{N} \gg 1$:

$$P_{\rm GCE}(N_0, N_-) = \frac{\overline{N}^{N_0}}{N_0!}\exp(-\overline{N}) \times \frac{\overline{N}^{N_-}}{N_-!}\exp(-\overline{N})$$
$$\cong (2\pi\overline{N})^{-1/2}\exp\left[-\frac{(N_0-\overline{N})^2}{2\overline{N}}\right]$$
$$\times (2\pi\overline{N})^{-1/2}\exp\left[-\frac{(N_- -\overline{N})^2}{2\overline{N}}\right]. \quad (3)$$

### B. Canonical ensemble

The canonical ensemble (CE) is described by the variables $V$, $T$, and $Q$. The GCE expressions (1) and (2) for average quantities remain valid in the CE at $\overline{N} \gg 1$. From the assumption, $Q = 0$ follows that $N_+ = N_-$. Consequently, the distribution of $N_+$ and $N_-$ in the CE is narrower than in the GCE [9]. The CE distribution of neutral particles remains the same as in the GCE as it is not constrained by charge-conservation law. The joint distribution of neutral and negatively charged particles is given by [9]

$$P_{\rm CE}(N_0, N_-) = \frac{\overline{N}^{N_0}}{N_0!}\exp(-\overline{N}) \times \frac{1}{I_0(2\overline{N})}\frac{\overline{N}^{2N_-}}{(N_-!)^2}$$
$$\cong (2\pi\overline{N})^{-1/2}\exp\left[-\frac{(N_0-\overline{N})^2}{2\overline{N}}\right]$$
$$\times (\pi\overline{N})^{-1/2}\exp\left[-\frac{(N_- -\overline{N})^2}{\overline{N}}\right], \quad (4)$$

where $I_0$ is the modified Bessel function.

### C. Microcanonical ensemble

The microcanonical ensemble (MCE) is described by the variables $V$, $E$, and $Q$. The MCE partition function for $N_0$ neutral and $N_+ = N_-$ positively and negatively charged massless particles reads as follows [10]:

$$\Omega_{N_0,N_-}(E,\ V) = \frac{1}{N_0!}\frac{1}{(N_-)!^2}\left(\frac{V}{\pi^2}\right)^{N_0+2N_-}\frac{E^{3N_0+6N_- -1}}{\Gamma(3N_0+6N_-)}, \quad (5)$$

where $\Gamma$ is the Euler gamma function. The joint probability distribution of $N_0$ and $N_-$ in the MCE is

$$P_{\rm MCE}(N_0, N_-) = \frac{\Omega_{N_0,N_-}(E,V)}{\Omega(E,V)}, \quad (6)$$

where $\Omega(E, V) = \sum_{N_0,N_-}\Omega_{N_0,N_-}(E, V)$.

### D. Average multiplicities, fluctuations, and correlations

The mean quantities in different statistical ensembles can be expressed as

$$\langle X\rangle = \sum_{N_0,N_-} X(N_0, N_-)P(N_0, N_-). \quad (7)$$

For the MCE distribution (6), one obtains

$$\langle N_0\rangle_{\rm MCE} \cong \langle N_-\rangle_{\rm MCE} \cong \frac{1}{3\sqrt{3\pi}}(VE^3)^{1/4} \quad (8)$$

If the MCE energy equals the average energy of the GCE and CE, $E = \overline{E}$, then the average MCE multiplicities (8) become equal to those in the GCE and CE (1). This reflects the equivalence of the GCE, CE, and MCE in the thermodynamic limit. However, the multiplicity distributions are different in these ensembles even in the thermodynamic limit. As shown in Appendix A, the MCE distribution (6) for $\overline{N} \gg 1$ can be approximated as

$$P_{\rm MCE}(N_0, N_-) \cong \frac{\sqrt{2}}{\pi\,\overline{N}}\exp\left[-\frac{(N_0-\overline{N})^2}{\overline{N}} - \frac{2(N_0-\overline{N})(N_- -\overline{N})}{\overline{N}} - \frac{3(N_- -\overline{N})^2}{\overline{N}}\right]. \quad (9)$$

The distributions $P(N_0, N_-)$ in the GCE (3), CE (4), and MCE (6) can be written in a general form of the bivariate normal distribution as follows:

$$P(N_0, N_-) = \frac{1}{2\pi\,\overline{N}\sqrt{\omega^0\times\omega^-(1-\rho^2)}}\exp\left[-\frac{1}{2\,\overline{N}\,(1-\rho^2)}\times\left(\frac{(N_0-\overline{N})^2}{\omega^0} - 2\,\rho\,\frac{(N_0-\overline{N})(N_- -\overline{N})}{\sqrt{\omega^0\times\omega^-}} + \frac{(N_- -\overline{N})^2}{\omega^-}\right)\right], \quad (10)$$

where $\omega^0$ and $\omega^-$ are the scaled variances defined as

$$\omega^0 \equiv \frac{\langle N_0^2\rangle - \langle N_0\rangle^2}{\langle N_0\rangle}, \qquad \omega^- \equiv \frac{\langle N_-^2\rangle - \langle N_-\rangle^2}{\langle N_-\rangle}, \quad (11)$$

and $\rho$ is the correlation coefficient

$$\rho \equiv \rho^{0-} \equiv \frac{\langle N_0\,N_-\rangle - \langle N_0\rangle\,\langle N_-\rangle}{\sqrt{\left[\langle N_0^2\rangle - \langle N_0\rangle^2\right]\times[\langle N_-^2\rangle - \langle N_-\rangle^2]}} = \frac{\langle N_0\,N_-\rangle - \langle N_0\rangle\,\langle N_-\rangle}{\sqrt{\omega^0\times\langle N_0\rangle\times\omega^-\times\langle N_-\rangle}}. \quad (12)$$

The averaging in Eqs. (11) and (12) is expressed according to Eq. (7).

The scaled variances in the GCE correspond to the uncorrelated Poisson distributions (3)

$$\omega^0_{\rm GCE} = \omega^-_{\rm GCE} = \omega^+_{\rm GCE} = 1, \quad (13)$$

and the correlation coefficient (12) is obviously equal to zero. One can similarly introduce the coefficients $\rho^{0+}$ and $\rho^{+-}$. They are also equal to zero in the GCE.

In the CE,

$$\omega^0_{\rm CE} = 1, \qquad \omega^-_{\rm CE} = \omega^+_{\rm CE} = \tfrac{1}{2}, \quad (14)$$

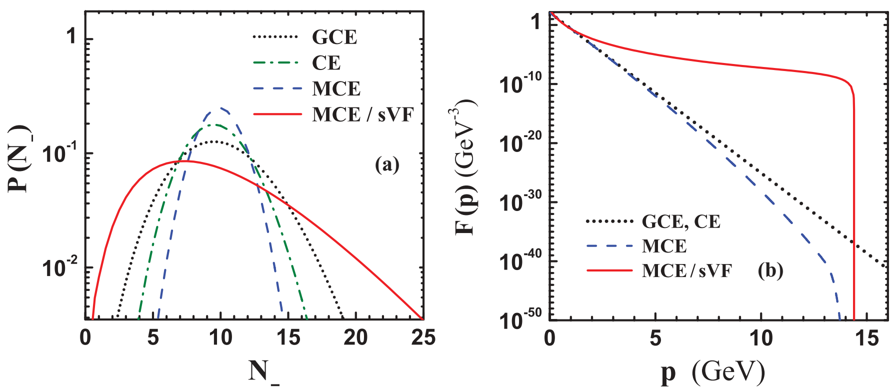


FIG. 1. (Color online) Examples of the multiplicity distributions (a) and the inclusive momentum spectra (b) of negatively charged particles obtained within the GCE, CE, MCE, and MCE/sVF. The distributions are calculated assuming $\overline{N} = 10$ and $T = 160$ MeV (see text for details).

for the distribution (4). The strong correlation, $N_+ = N_-$, in each microscopic state of the CE leads to the largest possible value of the correlation coefficient

$$\rho^{+-}_{\rm CE} = \frac{\langle N_+ N_- \rangle_{\rm CE} - \langle N_+ \rangle_{\rm CE} \langle N_- \rangle_{\rm CE}}{\sqrt{\omega^+_{\rm CE} \times \langle N_+ \rangle_{ce} \times \omega^-_{\rm CE} \times \langle N_- \rangle_{\rm CE}}} = 1. \qquad (15)$$

However, similar to the GCE, there are no correlations between neutral and charged particles, $\rho^{0\pm}_{\rm CE} = 0$.

The scaled variances in the MCE are as follows:

$$\omega^0_{\rm MCE} = \tfrac{3}{4}, \qquad \omega^-_{\rm MCE} = \omega^+_{\rm MCE} = \tfrac{1}{4}. \qquad (16)$$

They reflect the suppression of fluctuations of neutral particles in the MCE compared with the GCE and CE, and they reflect the stronger suppression of fluctuations of charged particles compared with the CE. The correlation coefficient, $\rho^{+-}_{\rm MCE} = 1$, is the same as in the CE (15). The exact energy conservation in the MCE leads to a rather strong anticorrelation between neutral and charged particles:

$$\rho^{0-}_{\rm MCE} = \rho^{0+}_{\rm MCE} = -\frac{1}{\sqrt{3}} \cong -0.577. \qquad (17)$$

In the large volume limit, the multiplicity distribution of negatively charged particles in the GCE, CE, and MCE can be approximated by normal distribution [11]:

$$P(N_-) = \sum_{N_0} P(N_-, N_0) \cong (2\pi\omega^- \overline{N})^{-1/2} \exp\left[-\frac{(N_- - \overline{N})^2}{2\omega^- \overline{N}}\right], \qquad (18)$$

with $\omega^-_{\rm GCE} = 1$, $\omega^-_{\rm CE} = 1/2$, and $\omega^-_{\rm MCE} = 1/4$ in the GCE, CE, and MCE ensembles, respectively. The $N_-$ distribution in these ensembles is presented in Fig. 1(a). As the considered system has zero charge, the distributions $P(N_+)$ are equal to $P(N_-)$ ones in all statistical ensembles.

The neutral-particle multiplicity distribution reads as follows:

$$P(N_0) = \sum_{N_-} P(N_-, N_0) \cong (2\pi\omega^0 \overline{N})^{-1/2} \exp\left[-\frac{(N_0 - \overline{N})^2}{2\omega^0 \overline{N}}\right], \qquad (19)$$

with $\omega^0_{\rm GCE} = \omega^0_{\rm CE} = 1$ and $\omega^0_{\rm MCE} = 3/4$.

### E. MCE with scaling volume fluctuations

The MCE/sVF [8] is described by the variables $E$, $Q$, and $V$ as well as by the distribution function defining the scaling volume fluctuations.[1] All quantities calculated within the MCE/sVF will be denoted by the subscript $\alpha$. For a description of the volume fluctuations it is convenient to introduce an auxiliary variable $y$ as follows:

$$y \equiv (V/\overline{V})^{1/4}, \qquad (20)$$

and describe the scaling volume fluctuations by the scaling function $\psi_\alpha(y)$ (see Ref. [8] for details). Experimental data on the multiplicity distribution of charged hadrons in $p + p$ interactions suggest a simple analytical form of the $\psi_\alpha(y)$ function [8,14,15]:

$$\psi_\alpha(y) = \frac{k^k}{\Gamma(k)}\, y^{k-1} \exp(-k\, y), \qquad (21)$$

with $k = 4$ and $\Gamma(k)$ being the Euler gamma function.

The joint $N_0$ and $N_-$ distribution in the MCE/sVF equals

$$P_\alpha(N_0, N_-) = \int_0^\infty dy\, P_{\rm MCE}(N_0, N_-)\psi_\alpha(y), \qquad (22)$$

where $P_{\rm MCE}(N_0, N_-)$ is given by Eq. (6). The analytical approximations for $P_\alpha(N_0, N_-)$ are discussed in Appendix B. The inclusive mean multiplicities in the MCE/sVF are as follows:

$$\langle N_- \rangle_\alpha = \sum_{N_-, N_0} N_-\, P_\alpha(N_0, N_-) \cong \overline{N}, \qquad \langle N_0 \rangle_\alpha = \sum_{N_-, N_0} N_0\, P_\alpha(N_0, N_-) \cong \overline{N}, \qquad (23)$$

and, thus, they coincide with those in the GCE, CE, and MCE at $\overline{N} \gg 1$. The inclusive multiplicity distributions in the MCE/sVF are as follows:

$$P_\alpha(N_-) = \sum_{N_0} P_\alpha(N_0, N_-) \cong \frac{1}{\overline{N}} \psi_\alpha\left(\frac{N_-}{\overline{N}}\right), \qquad (24)$$

$$P_\alpha(N_0) = \sum_{N_-} P_\alpha(N_0, N_-) \cong \frac{1}{\overline{N}} \psi_\alpha\left(\frac{N_0}{\overline{N}}\right). \qquad (25)$$

[1] Statistical ensembles with fluctuating extensive quantities are discussed in recent papers [12,13].

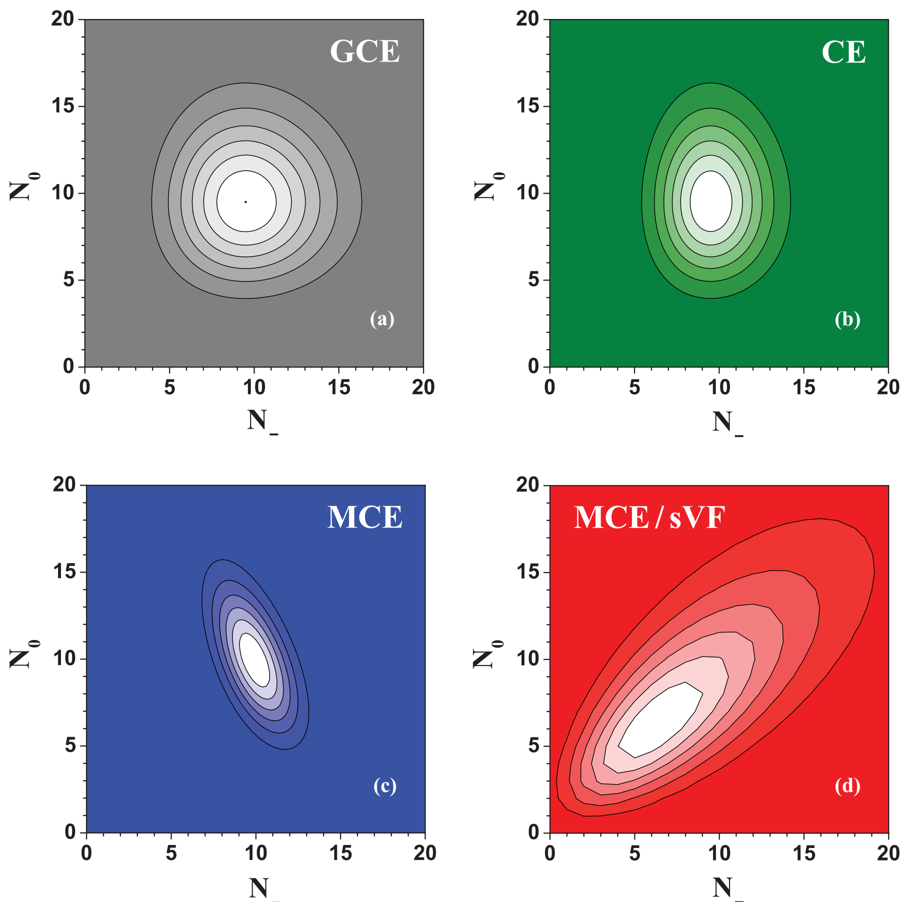


FIG. 2. (Color online) Examples of the joint $N_0$ and $N_-$ distributions calculated within the GCE (a), CE (b), MCE (c), and MCE/sVF (d). The distributions are calculated assuming $\overline{N} = 10$ (see text for details).

The $P_\alpha(N_-)$ distribution is shown in Fig. 1. It is significantly broader than the corresponding distributions for the GCE, CE, and MCE. The scaled variance for negatively charged and neutral particles is as follows:

$$\omega_\alpha^- \cong \frac{1}{k}\overline{N} + \omega_{\text{MCE}}^-, \qquad \omega_\alpha^0 \cong \frac{1}{k}\overline{N} + \omega_{\text{MCE}}^0. \tag{26}$$

Thus, in the MCE/sVF, because of the scaling volume fluctuations, the scaled variance increases in proportion to the mean multiplicity, whereas the scaled variance is approximately independent of mean multiplicity, $\omega \approx$ const, in the GCE, CE, and MCE.

For illustration of the previously discussed properties, the joint $N_0$ and $N_-$ distributions calculated within the GCE (3), CE (4), MCE (6), and MCE/sVF (22) are shown in Fig. 2. The multiplicities of neutral and negatively charged particles are uncorrelated in the GCE and CE [see Figs. 2(a) and 2(b)]. They are anticorrelated and correlated in the MCE and MCE/sVF, respectively. A positive correlation between $N_0$ and $N_-$ in the MCE/sVF is caused by the scaling volume fluctuations. Note that finite-size effects are shown in Fig. 2. For example, the Poisson distribution (3) is significantly asymmetric for large deviations from $\overline{N}$.

The distributions in Fig. 2 are rather different. Thus, it is obvious that the dependence on $N_-$ of the semi-inclusive mean multiplicity of neutral particles defined as follows:

$$\langle N_0\rangle^* \equiv \frac{\sum_{N_0} N_0 P(N_0, N_-)}{\sum_{N_0} P(N_0, N_-)} \tag{27}$$

is different in various ensembles. Namely, it is independent of $N_-$ in the GCE and CE:

$$\langle N_0\rangle^*_{\text{GCE}} = \langle N_0\rangle^*_{\text{CE}} = \overline{N}. \tag{28}$$

In the MCE, $\langle N_0\rangle^*_{\text{MCE}}$ monotonically decreases with increasing $N_-$ and equals approximately (see Appendix C)

$$\langle N_0\rangle^*_{\text{MCE}} \cong \overline{N}\left(\frac{4}{3} - \frac{N_-}{3\overline{N}}\right)^3. \tag{29}$$

Finally, the positive correlation between $N_0$ and $N_-$ in the MCE/sVF leads to an approximately linear increase[2] $\langle N_0\rangle^*_\alpha$ with increasing $N_-$:

$$\langle N_0\rangle^*_\alpha \cong N_-. \tag{30}$$

The semi-inclusive mean multiplicities of neutral particles calculated within the GCE, CE (28), MCE (29), and MCE/sVF (30) are shown as functions of $N_-$ in Fig. 3.

[2] A linear increase of $\langle N_0\rangle^*_{\text{MCE}}$ with $N_-$ is from the assumption of massless particles. For the nonzero value of mass $m$, the relation (30) is changed at large $N_-$. The maximum value of $N_-$ is $N_-^{\max} = E/2m$. Equation (30) remains approximately valid for $N_- \ll N_-^{\max}$, but $\langle N_0\rangle^*_\alpha$ approaches zero at $N_- \to N_-^{\max}$.

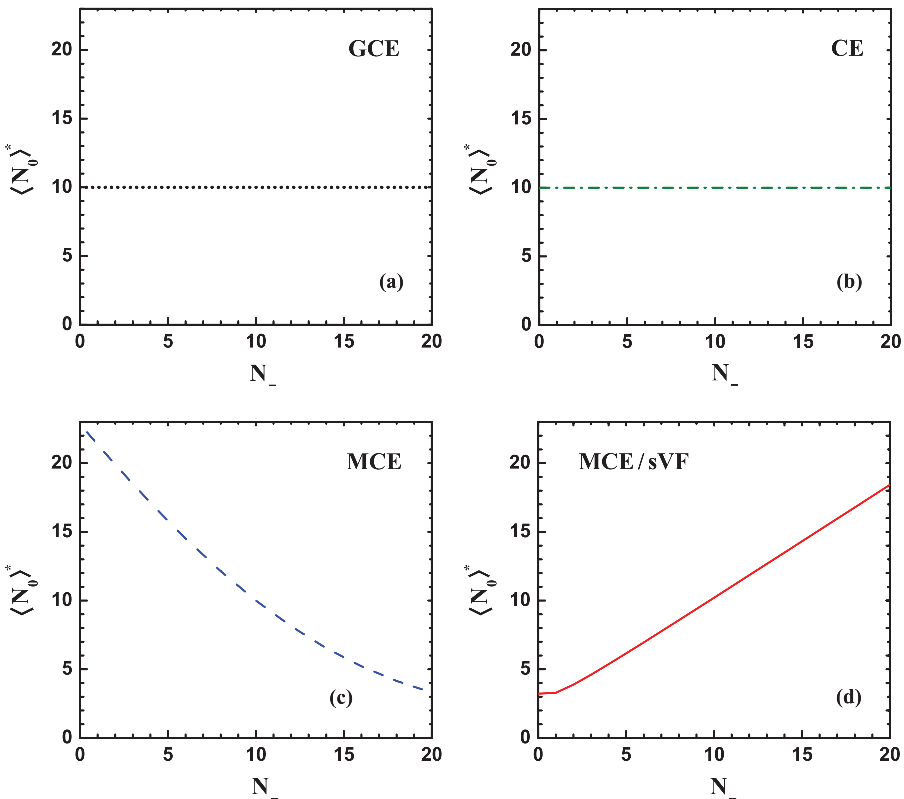


FIG. 3. (Color online) Examples of the dependence of the neutral-particle mean multiplicity on the multiplicity of negatively charged particles calculated within the GCE (a), CE (b), MCE (c), and MCE/sVF (d). The distributions are calculated assuming $\overline{N} = 10$ (see text for details).

### F. Quantum statistics

In this subsection, we illustrate the effects of quantum statistics in the GCE, CE, and MCE using the microcorrelator method of Ref. [16]. The mean multiplicities in the CE or MCE are approximately the same as in the GCE. They are given by the sum of mean occupation numbers with momentum **p** [17]:

$$\langle N^a\rangle_{\rm CE} \cong \langle N^a\rangle_{\rm MCE} \cong \langle N^a\rangle_{\rm GCE} \equiv \sum_{\mathbf p}\langle n^a_{\mathbf p}\rangle = \sum_{\mathbf p}\frac{1}{\exp(\epsilon_{\mathbf p}/T)-\gamma}, \tag{31}$$

where $a$ is $+$, $-$, or 0 and denotes positive, negative, or neutral particles, $\epsilon_{\mathbf p} = p$ is one particle energy for massless particles, $\gamma = +1$ for Bose statistics, $\gamma = -1$ for Fermi statistics, and $\gamma = 0$ corresponds to the Boltzmann approximation used throughout this article. We study a neutral system; thus, chemical potentials are zero in Eq. (31), and the average occupation numbers $\langle n^a_{\mathbf p}\rangle \equiv \langle n_{\mathbf p}\rangle$ are therefore the same for neutral and charged particles. The summation over discrete levels can be substituted by the integration in the thermodynamic limit:

$$\sum_{\mathbf p}\ldots \cong \frac{V}{2\pi^2}\int_0^\infty p^2dp\ldots. \tag{32}$$

The fluctuations and correlations in the GCE, CE, and MCE are very different; nevertheless they can be expressed in terms of the fluctuations of the occupation numbers of a single momentum level in the GCE:

$$\left\langle\left(\Delta n^a_{\mathbf p}\right)^2\right\rangle_{\rm GCE} \equiv v^{a\,2}_{\mathbf p} = \langle n_{\mathbf p}\rangle(1+\gamma\langle n_{\mathbf p}\rangle). \tag{33}$$

This is a main advantage of the microcorrelator method. It allows for calculating the fluctuations and correlations using the following microcorrelators [18]:

$$\left\langle\Delta n^a_{\mathbf p}\Delta n^b_{\mathbf k}\right\rangle_{\rm GCE} = v^{a\,2}_{\mathbf p}\delta_{\mathbf{pk}}\,\delta_{ab}, \tag{34}$$

$$\left\langle\Delta n^a_{\mathbf p}\Delta n^b_{\mathbf k}\right\rangle_{\rm CE} = v^{a\,2}_{p}\delta_{\mathbf{pk}}\,\delta_{ab} - q^aq^b\,\frac{v^{a\,2}_{\mathbf p}v^{b\,2}_{k}}{\sum_{\mathbf p,a}v^{a\,2}_{\mathbf p}q^{\alpha 2}}, \tag{35}$$

$$\left\langle\Delta n^a_{\mathbf p}\Delta n^b_{\mathbf k}\right\rangle_{\rm MCE} = v^{a\,2}_{\mathbf p}\delta_{\mathbf{pk}}\,\delta_{ab} - \frac{v^{a\,2}_{\mathbf p}v^{b\,2}_{\mathbf k}}{|A|}\times\left[q^aq^b\sum_{\mathbf p,a}v^{a\,2}_{\mathbf p}\epsilon^2_{\mathbf p} + \epsilon_{\mathbf p}\epsilon_{\mathbf k}\sum_{\mathbf p,a}v^{a\,2}_{\mathbf p}q^{\alpha 2}\right], \tag{36}$$

where $\delta_{\mathbf{pk}}$ and $\delta_{ab}$ are the Kronecker delta symbols, $q^a$, $q^b$ are particle charges, $\pm 1$ or 0, and

$$|A| \equiv \left(\sum_{p,a}v^{a\,2}_{\mathbf p}\epsilon^2_{\mathbf p}\right)\times\left(\sum_{p,a}v^{a\,2}_{\mathbf p}q^{\alpha 2}\right) \tag{37}$$

is the correlation determinant.

The variance and correlations in the GCE, CE, and MCE are calculated as the sums (integrals) over momentum of the corresponding microcorrelators (34)–(36):

$$\langle(\Delta N_a)^2\rangle = \sum_{\mathbf{p,k}}\left\langle\Delta n^a_{\mathbf p}\Delta n^a_{\mathbf k}\right\rangle, \quad \langle\Delta N_a\,\Delta N_b\rangle = \sum_{\mathbf{p,k}}\left\langle\Delta n^a_{\mathbf p}\Delta n^b_{\mathbf k}\right\rangle. \tag{38}$$

One obtains for Bosons:

$$\omega^{\pm\,\mathrm{Bose}}_{\mathrm{GCE}} = \frac{\sum_{\mathbf{p}} v_{\mathbf{p}}^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle} \cong \frac{\int_0^\infty p^2 dp e^{p/T}(e^{p/T}-1)^{-2}}{\int_0^\infty p^2 dp (e^{p/T}-1)^{-1}} = \frac{\pi^2}{6\,\zeta(3)} \cong 1.368, \tag{39}$$

$$\omega^{\pm\,\mathrm{Bose}}_{\mathrm{CE}} = \frac{\sum_{\mathbf{p}} v_{\mathbf{p}}^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle} - \frac{\left(\sum_{\mathbf{p}} v_{\mathbf{p}}^2\right)^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \sum_{p,a} v_{\mathbf{p}}^2 q^{\alpha 2}} = \frac{\sum_{\mathbf{p}} v_{\mathbf{p}}^2}{2\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle} = \frac{1}{2}\,\omega^{\pm\,\mathrm{Bose}}_{\mathrm{GCE}} \cong 0.684, \tag{40}$$

$$\omega^{\pm\,\mathrm{Bose}}_{\mathrm{MCE}} = \frac{\sum_{\mathbf{p}} v_{\mathbf{p}}^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle} - \frac{\left(\sum_{\mathbf{p}} v_{\mathbf{p}}^2\right)^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \sum_{p,a} v_{\mathbf{p}}^2 q^{\alpha 2}} - \frac{\left(\sum_{\mathbf{p}} v_{\mathbf{p}}^2 \epsilon_{\mathbf{p}}\right)^2}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \sum_{\mathbf{p},a} v_{\mathbf{p}}^2 \epsilon_{\mathbf{p}}^2} = \frac{\pi^2}{12\,\zeta(3)} - \frac{45\,\zeta(3)}{2\,\pi^4} \cong 0.407, \tag{41}$$

where $\zeta(3) \cong 1.202$ is the zeta Riemann function. In calculating Eq. (41), we use $\langle n_{\mathbf{p}}^a\rangle \equiv \langle n_{\mathbf{p}}\rangle$ and $v_{\mathbf{p}}^{+\,2} = v_{\mathbf{p}}^{-\,2} = v_{\mathbf{p}}^{0\,2} \equiv v_{\mathbf{p}}^2$ for a neutral system. This gives $\sum_{\mathbf{p},a} v_{\mathbf{p}}^{a\,2} q^{a\,2} = 2\sum_{\mathbf{p}} v_{\mathbf{p}}^2$ and $\sum_{\mathbf{p},a} v_{\mathbf{p}}^{a\,2}\epsilon_{\mathbf{p}}^2 = 3\sum_{\mathbf{p}} v_{\mathbf{p}}^2 \epsilon_{\mathbf{p}}^2$. Similarly, one can get the results for Fermions:

$$\omega^{\pm\,\mathrm{Fermi}}_{\mathrm{GCE}} = \frac{\pi^2}{9\,\zeta(3)} \cong 0.912, \tag{42}$$

$$\omega^{\pm\,\mathrm{Fermi}}_{\mathrm{CE}} = \frac{1}{2}\,\omega^{\pm\,\mathrm{Fermi}}_{\mathrm{GCE}} \cong 0.456, \tag{43}$$

$$\omega^{\pm\,\mathrm{Fermi}}_{\mathrm{MCE}} = \frac{\pi^2}{18\,\zeta(3)} - \frac{135\,\zeta(3)}{7\,\pi^4} \cong 0.218. \tag{44}$$

The scaled variances for neutral particles are as follows:

$$\omega^{0\,\mathrm{Bose}}_{\mathrm{GCE}} = \omega^{0\,\mathrm{Bose}}_{\mathrm{CE}} = \omega^{\pm\,\mathrm{Bose}}_{\mathrm{GCE}} \cong 1.368, \quad \omega^{0\,\mathrm{Fermi}}_{\mathrm{GCE}} = \omega^{0\,\mathrm{Fermi}}_{\mathrm{CE}} = \omega^{\pm\,\mathrm{Fermi}}_{\mathrm{GCE}} \cong 0.912, \tag{45}$$

$$\omega^{0\,\mathrm{Bose}}_{\mathrm{MCE}} = \frac{\pi^2}{6\,\zeta(3)} - \frac{45\,\zeta(3)}{2\,\pi^4} \cong 1.091, \quad \omega^{0\,\mathrm{Fermi}}_{\mathrm{MCE}} = \frac{\pi^2}{9\,\zeta(3)} - \frac{135\,\zeta(3)}{7\,\pi^4} \cong 0.674. \tag{46}$$

The correlation coefficients can be also calculated using the microcorrelator method:

$$\rho^{a\,b} = \frac{\langle \Delta N_a\, \Delta N_b\rangle}{\sqrt{\omega^a \times \langle N_a\rangle \times \omega^b \times \langle N_b\rangle}} = \frac{1}{\sqrt{\omega^a\,\omega^b}}\,\frac{\sum_{\mathbf{p},\mathbf{k}}\langle \Delta n_{\mathbf{p}}^a \Delta n_{\mathbf{k}}^b\rangle}{\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle}. \tag{47}$$

There are no correlations between positively and negatively charged particles in the GCE, $\rho^{+-}_{\mathrm{GCE}} = 0$, and there is the absolute correlation in the CE and MCE, $\rho^{+-}_{\mathrm{CE}} = \rho^{+-}_{\mathrm{MCE}} = 1$. These values are the same for any type of statistics. The correlation between charged and neutral particles, $\rho^{0-} = \rho^{0+}$ (12), is zero for the GCE and CE, but it has a negative value for the MCE. The correlation coefficient $\rho^{0-}_{\mathrm{MCE}}$ reads as follows:

$$\rho^{0-}_{\mathrm{MCE}} = -\frac{1}{\sqrt{\omega^0\,\omega^-}}\,\frac{\left(\sum_{\mathbf{p}} v_{\mathbf{p}}^2 \epsilon_{\mathbf{p}}\right)^2}{3\sum_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \sum_{\mathbf{p}} v_{\mathbf{p}}^2 \epsilon_{\mathbf{p}}^2}. \tag{48}$$

Equation (48) gives for Bosons and Fermions:

$$\rho^{0-\mathrm{Bose}}_{\mathrm{MCE}} = -\sqrt{2}\left[\left(\frac{\pi^6}{135\,\zeta(3)^2} - 2\right)\left(\frac{\pi^6}{135\,\zeta(3)^2} - 1\right)\right]^{-1/2} \cong -0.417, \tag{49}$$

$$\rho^{0-\mathrm{Fermi}}_{\mathrm{MCE}} = -\sqrt{2}\left[\left(\frac{7\pi^6}{1215\,\zeta(3)^2} - 2\right)\left(\frac{7\pi^6}{1215\,\zeta(3)^2} - 1\right)\right]^{-1/2} \cong -0.621. \tag{50}$$

The scaled variances and correlation coeffients for the Boltzmann approximation can be obtained from Eqs. (41) and (48) replacing $\gamma = 0$ in Eq. (31):

$$\omega^{\pm\,\mathrm{Boltz}}_{\mathrm{GCE}} = 1, \quad \omega^{\pm\,\mathrm{Boltz}}_{\mathrm{CE}} = \tfrac{1}{2}, \quad \omega^{\pm\,\mathrm{Boltz}}_{\mathrm{MCE}} = \tfrac{1}{4}, \quad \omega^{0\,\mathrm{Boltz}}_{\mathrm{GCE}} = \omega^{0\,\mathrm{Boltz}}_{\mathrm{CE}} = 1, \quad \omega^{0\,\mathrm{Boltz}}_{\mathrm{MCE}} = \tfrac{3}{4}, \tag{51}$$

$$\rho^{0-\,\mathrm{Boltz}}_{\mathrm{MCE}} = -\frac{1}{\sqrt{3}} \cong -0.577. \tag{52}$$

They, of course, coincide with our previous results in Eqs. (13), (14), (16), and (17) for the Boltzmann statistics.

One can conclude that Bose statistics always makes the fluctuations bigger and that Fermi statistics always makes them smaller: $\omega^{\mathrm{Fermi}} < \omega^{\mathrm{Boltz}} < \omega^{\mathrm{Bose}}$, in all statistical ensembles (GCE, CE, and MCE) and for all types of particles (positive, negative, and neutral). The strongest effect for the neutral system in equilibrium[3] is for the scaled variance of charged Bosons in the MCE: $\omega^{\pm\,\mathrm{Bose}}_{\mathrm{MCE}}/\omega^{\pm\,\mathrm{Boltz}}_{\mathrm{MCE}} \cong 1.6$. The only correlation coefficient that feels an influence of quantum statistics is $\rho^{0\pm}$ in the MCE: $\rho^{0\pm\mathrm{Fermi}}_{\mathrm{MCE}} < \rho^{0\pm\mathrm{Boltz}}_{\mathrm{MCE}} < \rho^{0\pm\mathrm{Bose}}_{\mathrm{MCE}} < 0$. However, the quantum statistics does not change a sign of this correlation. Thus, the main features of the GCE, CE, and MCE fluctuations and correlations found in the Boltzmann approximation—constant values of $\omega^{\pm}$ and $\omega^0$ in the thermodynamic limit, strong correlations, $\rho^{+-}_{\mathrm{CE}} = \rho^{+-}_{\mathrm{MCE}} = 1$, caused by the exact charge conservation, and anticorrelation between neutral and charged particles, $\rho^{0\pm}_{\mathrm{MCE}} < 0$, caused by the exact energy conservation—remain the same for Bose and Fermi statistics. The quantum statistics cannot simulate the MCE/sVF effects: an increase of the scaled variances in proportion to the mean multiplicities (26) and a strong *positive* correlation, $\rho^{0\pm}_{\alpha} \cong 1$, between neutral and charged particles (30). These new effects take place because of the scaling volume fluctuations in the MCE/sVF.

## III. SEMI-INCLUSIVE MOMENTUM SPECTRA

In this section, single-particle momentum spectra of negatively charged particles are considered. The inclusive spectra

[3]The effects of quantum statistics for fluctuations can be much stronger at nonzero chemical potential. The scaled variance of Bosons may rise up to infinity near the point of Bose condensation [19].

are denoted as $F(p)$, and the semi-inclusive at fixed $N_-$ are denoted as $F^*(p)$. In both cases, the spectra are normalized to unity: $\int_0^\infty p^2 dp F(p) = 1$ and $\int_0^\infty p^2 dp F^*(p) = 1$.

### A. GCE and CE

The inclusive and semi-inclusive momentum spectra in the GCE and CE are equal and read as follows[4]:

$$F_{\rm GCE}(p) = F^*_{\rm GCE}(p) = F_{\rm CE}(p) = F^*_{\rm CE}(p) = \frac{1}{2T^3}\exp\left(-\frac{p}{T}\right). \quad (53)$$

This follows from the fact that a single-particle momentum spectrum and the particle multiplicity are uncorrelated in these ensembles.

### B. Microcanonical ensemble

The inclusive single-particle momentum spectrum of negatively charged particles in the MCE reads as follows:

$$F_{\rm MCE}(p) = \frac{1}{\overline{N}}\frac{1}{2E^3}\sum_{N_0=0}^{\infty}\sum_{N_-=1}^{\infty}\frac{N_-(3N_0+6N_--1)!}{(3N_0+6N_--4)!} \times \left(1-\frac{p}{E}\right)^{3N_0+6N_--4} P_{\rm MCE}(N_0,N_-); \quad (54)$$

see also Ref. [8]. The spectrum (54) approximately has the Boltzmann form (53) at momenta $p$ significantly smaller than the total system energy $E$. However, large deviations from Eq. (53) are observed close to the threshold, $p = E$, where the MCE spectrum approaches zero. The inclusive spectra $F(p)$ in the GCE, CE (53), and MCE (54) are shown in Fig. 1(b).

The semi-inclusive momentum spectrum at a fixed number of negatively charged particles is given by

$$F^*_{\rm MCE}(p) = \frac{C}{2E^3}\sum_{N_0=0}^{\infty}\frac{(3N_0+6N_--1)!}{(3N_0+6N_--4)!} \times \left(1-\frac{p}{E}\right)^{3N_0+6N_--4} P_{\rm MCE}(N_0,N_-), \quad (55)$$

where $N_- \geqslant 1$ and $C = [\sum_{N_0} P_{\rm MCE}(N_0,N_-)]^{-1}$ is the normalization factor. Examples of the $F^*_{\rm MCE}(p)$ spectrum for three values of $N_-$ are shown in Fig. 4(c). The semi-inclusive spectra in the MCE (55) have the Boltzmann form for $p \ll E$,

$$F^*_{\rm MCE}(p) \cong \frac{1}{2T^{*3}_{\rm MCE}}\exp\left(-\frac{p}{T^*_{\rm MCE}}\right), \quad (56)$$

and the inverse slope parameter $T^*_{\rm MCE}$ depends on $N_-$. This dependence is presented in Fig. 5 for the MCE and other ensembles studied here. In the GCE and CE, the $T^*$ is independent of $N_-$ and equal to the inverse slope parameter of the inclusive spectrum, $T^*_{\rm GCE} = T^*_{\rm CE} = T = 160$ MeV. In the MCE, the inverse slope parameter decreases with increasing $N_-$ and it crosses the line $T = 160$ MeV at $N_- = \overline{N}$. Thus, the inclusive momentum spectrum $F_{\rm MCE}(p)$ (54) coincides with the semi-inclusive one, $F^*_{\rm MCE}(p)$ (55), at the crossing point.

[4]This is true for the Boltzmann statistics used here. The form of momentum spectrum in the CE becomes different from that in the GCE for quantum gases in finite volumes. For the isospin conservation, this was demonstrated in Ref. [20].

### C. MCE with scaling volume fluctuations

The inclusive single-particle momentum spectrum in the MCE/sVF equals

$$F_\alpha(p) = \frac{1}{\overline{N}}\frac{1}{2E^3}\sum_{N_0=0}^{\infty}\sum_{N_-=1}^{\infty}\frac{N_-(3N_0+6N_--1)!}{(3N_0+6N_--4)!} \times \left(1-\frac{p}{E}\right)^{3N_0+6N_--4} P_\alpha(N_0,N_-). \quad (57)$$

The structure of Eq. (57) is the same as the structure of the corresponding Eq. (54) for the MCE. The only difference is in the form of the multiplicity distribution; namely, $P_\alpha(N_0,N_-)$ is used in Eq. (57) instead of $P_{\rm MCE}(N_0,N_-)$ in Eq. (54). The inclusive spectrum $F_\alpha(p)$ is shown in Fig. 1(b). It can be well approximated by the power-law dependence:

$$F_\alpha(p) \cong \frac{k^k\Gamma(k+4)}{2\Gamma(k)}T^{k+1}(p+Tk)^{-k-4} \cong 11.27\ {\rm GeV}^5\,(p+4T)^{-8}, \quad (58)$$

where $k = 4$ is used in the last expression (see Ref. [8]).

The semi-inclusive momentum spectrum in the MCE/sVF reads as follows:

$$F^*_\alpha(p) = \frac{C}{2E^3}\sum_{N_0=0}^{\infty}\frac{(3N_0+6N_--1)!}{(3N_0+6N_--4)!} \times \left(1-\frac{p}{E}\right)^{3N_0+6N_--4} P_\alpha(N_0,N_-), \quad (59)$$

where $C = [\sum_{N_0} P_\alpha(N_0,N_-)]^{-1}$. The spectrum $F^*_\alpha(p)$ is plotted in Fig. 4 for several values of $N_-$. Similar to the MCE spectrum (56), the MCE/sVF one can be approximated as follows:

$$F^*_\alpha(p) \cong \frac{1}{2T^{*3}_\alpha}\exp\left(-\frac{p}{T^*_\alpha}\right), \quad (60)$$

with the inverse slope parameter $T^*_\alpha$. The dependence of $T^*_\alpha$ on $N_-$ is shown in Fig. 5(d). The MCE/sVF temperature $T^*_\alpha$ decreases with increasing $N_-$. For $N_- = \overline{N}$, the inverse slope parameter $T^*$ is the same in the MCE and MCE/sVF and

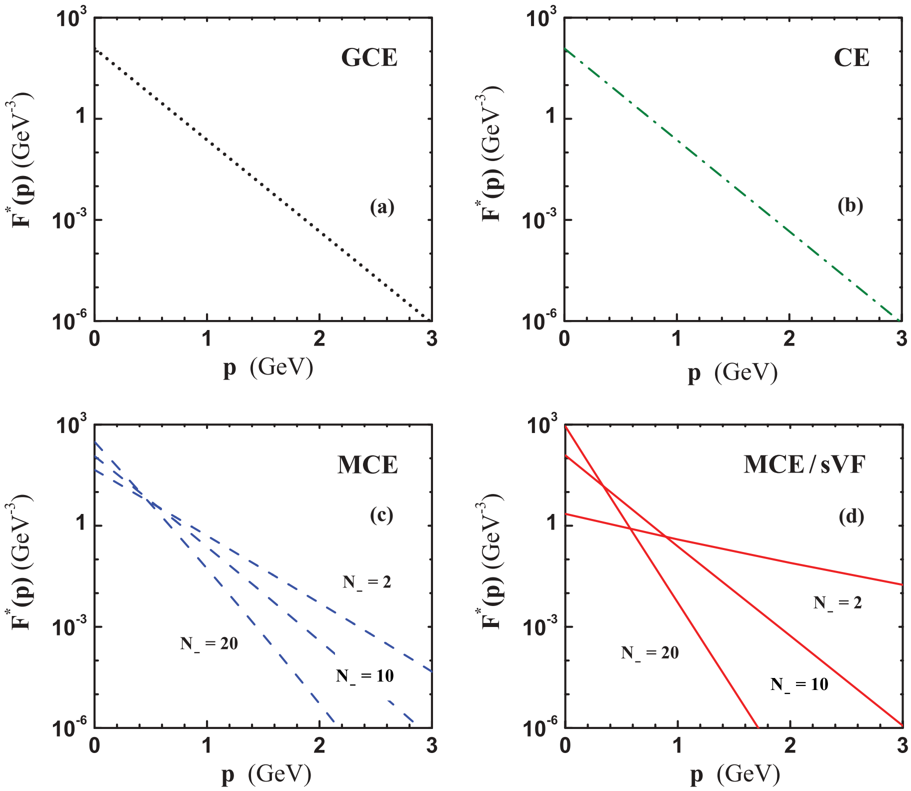


FIG. 4. (Color online) Examples of the semi-inclusive momentum spectra of negatively charged particles calculated within the GCE (a), CE (b), MCE (c), and MCE/sVF (d) for three values of $N_-$. The distributions are calculated assuming $\overline{N} = 10$ and $T = 160$ MeV (see text for details).

equals the parameter $T$ in the GCE and CE. The analytical approximations of the dependence of $T^*$ on $N_-$ in the MCE and MCE/sVF are presented in Appendix C.

## IV. COMPARISON WITH DATA

A quantitative comparison of the discussed statistical models with the experimental data requires a significant

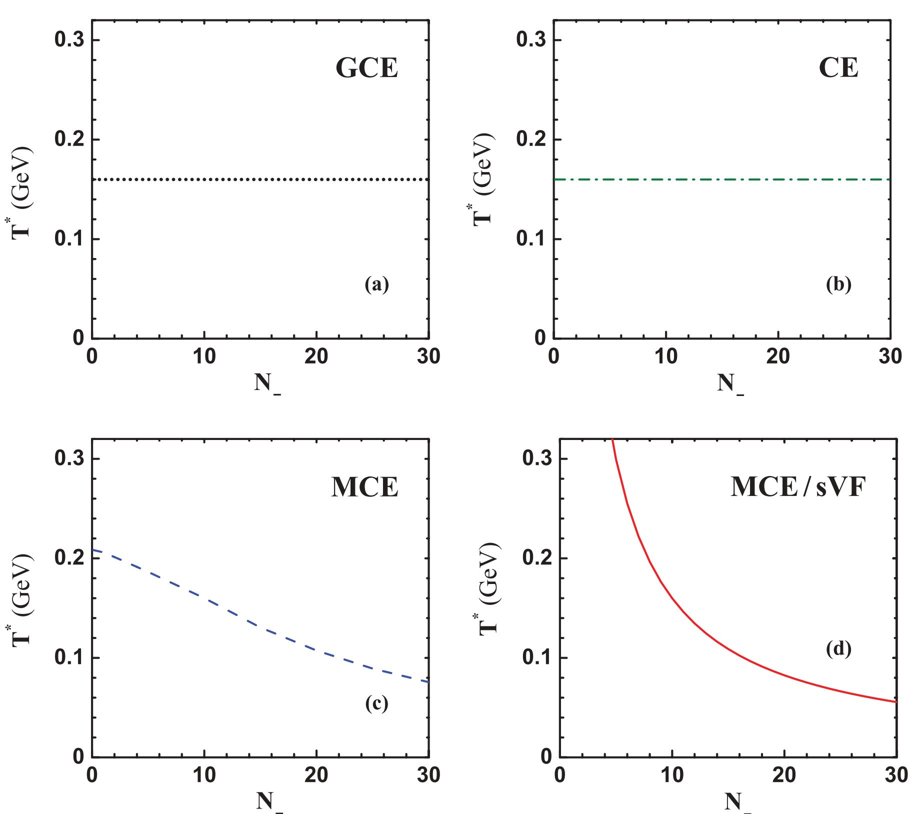


FIG. 5. (Color online) The dependence of the inverse slope parameter of the momentum spectra on the multiplicity of negatively charged particles $N_-$ calculated within the GCE (a), CE (b), MCE (c), and MCE/sVF (d). The distributions are calculated assuming $\overline{N} = 10$ and $T = 160$ MeV (see text for details).

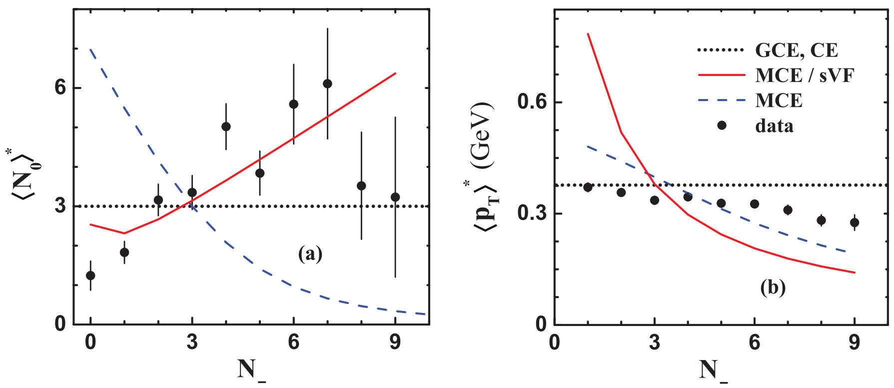


FIG. 6. (Color online) The mean multiplicity of neutral particles (a) and the transverse momentum of negatively charged particles (b) as a function of the multiplicity of negatively charged particles. The experimental data on $p + p$ interactions at 205 GeV/$c$ [23] (a) and [24] (b) are indicated by closed circles. The predictions of the GCE, CE, MCE, and MCE/sVF are shown by the lines. The calculations are performed assuming $\overline{N} = 3$ and $T = 160$ MeV (see text for details).

additional effort, which is far beyond the scope of this article. In particular, one should introduce proper degrees of freedom and all related conservation laws as well as a longitudinal collective motion of matter. Nevertheless, a qualitative comparison seems to be useful already, and consequently, it is presented in this section.

An excellent review of the experimental data on semi-inclusive properties of $p + p$ interactions at high energies can be found in Ref. [21]. The volume scaling function as well as the value of the temperature parameter used in this work in quantitative calculations were selected to approximately reproduce the results on the inclusive distributions in $p + p$ interactions. Consequently, a comparisons between these data and the model results is justified. Clearly, as the influence of global conservation laws is crucial for the considered statistical approaches, the data referring to the semi-inclusive properties measured in the full phase space are of primary importance. Several features of these data are well established [21]. Two of them are relevant for the comparison with the models discussed here, namely:

(i) the mean multiplicity of produced $\pi^0$ mesons increases with increasing multiplicity of negatively charged particles and
(ii) the average transverse momentum of negatively charged particles decreases with increasing multiplicity of these particles.

Property (ii) needs several comments. First, it is well established experimentally [21] for charged hadron multiplicity and mean transverse momentum measured in full phase space in $p + p$ interactions at 6.6–400 GeV/$c$. Clearly, the full phase-space results are relevant when effects related to the global conservation laws are of interest. Second, the mean transverse momentum increases with increasing multiplicity when midrapidity values are considered [22]. An interpretation of this dependence is, however, beyond the scope of this article.

For the purpose of the comparison between considered models and data, the mean transverse momentum was calculated as follows:

$$\langle p_T \rangle^* = 2 \int_0^\infty dp_T\, p_T^2 \int_{-\infty}^\infty dy\, p_T\, F^*(p)$$
$$\cong \frac{1}{T^{*3}} \int_0^\infty dp_T\, p_T^2 \int_{-\infty}^\infty dy\, p_T \exp\left(-\frac{p_T \cosh y}{T^*}\right)$$
$$= \frac{3\pi}{4} T^* \cong 2.36\, T^*. \qquad (61)$$

The mean multiplicity of neutral particles calculated within the models was identified with the mean $\pi^0$ multiplicity.

The model predictions are summarized in Fig. 6. In the GCE and CE, $\langle N_0 \rangle^*$ and $\langle p_T \rangle^*$ are independent of $N_-$. The MCE reproduces property (ii) but leads to a decrease of mean multiplicity $\langle N_0 \rangle^*$ with increasing $N_-$. Both features (i) and (ii) are qualitatively reproduced in the MCE/sVF.

## V. SUMMARY

Semi-inclusive distributions for the system of neutral and charged massless particles with net charge equal to zero are considered in the grand canonical, canonical, and microcanonical ensembles and in the MCE/sVF. The MCE/sVF has been included in this study as it is the only statistical ensemble that reproduces the KNO scaling of multiplicity distributions and the power-law behavior of the inclusive transverse momentum spectra measured in $p + p$ interactions. The mean multiplicity of neutral particles and momentum spectra of charged particles are calculated at fixed charged-particle multiplicity $N_-$. Different statistical ensembles lead to qualitatively different results for these semi-inclusive quantities even in the large volume limit. In other words, the semi-inclusive quantities can be different in different statistical ensembles despite the ensemble thermodynamical equivalence.

The obtained model predictions are compared with the experimental data on $p + p$ inelastic interactions at high energies. The MCE/sVF follows the trends observed in the data. This demonstrates the role of volume fluctuations in the system with exact energy and charge conservation. However, the detailed comparison with the experimental data is far beyond the scope of this article. The conclusive comparison with the experimental results would require inclusion of several neglected effects in the statistical model calculations. In particular, the hadron masses and quantum numbers, isospin symmetry, quantum statistics, and resonance decays should be taken into account.

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## APPENDIX A

The bivariate normal approximation (9) of $P_{\rm MCE}(N_0, N_-)$ can be derived as follows. Equation (5) can be rewritten as follows:

$$\Omega_{N_0,N_-}(E, V) = \frac{1}{E}\exp[f(N_0, N_-)], \qquad \text{(A1)}$$

where

$$f(N_0, N_-) = (N_0 + 2N_-)\ln[A] - \ln[N_0!] - 2\ln[N_-!] - \ln[(3N_0 + 6N_- - 1)!] \qquad \text{(A2)}$$

and $A = VE^3/\pi^2$. Using the Stirling formula, $\ln(N!) \cong (N + 1/2)\ln(N) - N + \ln(2\pi)/2$ at $N \gg 1$, the right-hand side of Eq. (A2) can be expanded with respect to $N_0$ and $N_-$ near the maximum of $f$. Then, the mean multiplicities can be calculated from the condition $\partial f/\partial N_0 = \partial f/\partial N_- = 0$. Second derivatives of $f$ with respect of $N_0$ and $N_-$ at the point of maximum are as follows:

$$\frac{\partial^2 f}{\partial N_0^2} \cong -\frac{2}{\overline{\overline{N}}}, \quad \frac{\partial^2 f}{\partial N_-^2} \cong -\frac{6}{\overline{\overline{N}}}, \quad \frac{\partial^2 f}{\partial N_0\,\partial N_-} \cong -\frac{2}{\overline{\overline{N}}}, \qquad \text{(A3)}$$

and Eq. (9) follows.

## APPENDIX B

Using the approximation (9), the integration over $y$ can be done analytically. Then the joint $N_0$ and $N_-$ distribution in the MCE/sVF reads as follows:

$$P_\alpha(N_0, N_-) \equiv \int_0^\infty dy\, P_{\rm MCE}(N_0, N_-)\psi_\alpha(y) \cong \frac{k^k}{\Gamma(k)}\frac{2\sqrt{2}}{\pi\,\overline{N}}\left(\frac{N_0^2 + 2N_0N_- + 3N_-^2}{6\overline{N}^2 + \overline{N}\,k}\right)^{(k-1)/2} \times \exp(4N_0 + 8N_-)K_{1-k}\left[2\sqrt{\left(6 + \frac{k}{\overline{\overline{N}}}\right)\left(N_0^2 + 2N_0N_- + 3N_-^2\right)}\right], \qquad \text{(B1)}$$

where $K_{1-k}$ is the Bessel function of the second kind. Equation (B1) can be simplified using the asymptotic expansion:

$$K_{1-k}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + \frac{4k^2 - 8k + 3}{8}\frac{1}{x} + O(x^{-2})\right], \quad x \gg 1. \qquad \text{(B2)}$$

Consequently,

$$P_\alpha(N_0, N_-) \simeq \frac{k^k}{\Gamma(k)}\frac{1}{\sqrt{\pi}\,\overline{N}}\left(\frac{N_0^2 + 2N_0N_- + 3N_-^2}{6\overline{N}^2 + \overline{N}\,k}\right)^{(k-1)/2} \times \frac{1}{\sqrt{N_0 + 2N_-}}\exp\left[-k\,\frac{N_0 + 2N_-}{3\overline{N}} - \frac{(N_0 - N_-)^2}{3N_-}\right]. \qquad \text{(B3)}$$

## APPENDIX C

In the MCE with a fixed multiplicity $N_-$, the system temperature $T^*_{\rm MCE}$ can be found as follows. The mean multiplicity of neutral particles equals $\langle N_0^*\rangle_{\rm MCE} = VT^{*3}_{\rm MCE}/\pi^2$, and their average energy is $\langle E_0^*\rangle_{\rm MCE} = 3T^*_{\rm MCE}\langle N_0^*\rangle_{\rm MCE}$. Thus, the total energy reads as follows:

$$\frac{3VT^{*4}_{\rm MCE}}{\pi^2} + 6\,N_-\,T^*_{\rm MCE} = E. \qquad \text{(C1)}$$

The first term in the left-hand side of Eq. (C1) corresponds to the average energy of neutral particles and the second term to that of charged particles. For the multiplicity of negatively charged particles close to the mean multiplicity, one can solve approximately Eq. (C1) with respect to temperature. Denoting $\delta N_- = N_- - \overline{\overline{N}}$ and $\delta T = T - T^*_{\rm MCE}$, the solution reads as follows:

$$\delta T \cong -T\frac{\delta N_-}{2\overline{\overline{N}} + \overline{\overline{N}}} \qquad \text{(C2)}$$

or

$$T^*_{\rm MCE} \cong T\left(\frac{4}{3} - \frac{N_-}{3\overline{\overline{N}}}\right). \qquad \text{(C3)}$$

Consequently, one gets

$$\langle N_0^*\rangle_{\rm MCE} = \frac{1}{\pi^2}VT^{*3}_{\rm MCE} \cong \overline{\overline{N}}\left(\frac{4}{3} - \frac{N_-}{3\overline{\overline{N}}}\right)^3. \qquad \text{(C4)}$$

In the MCE/sVF at fixed $N_-$, one finds

$$3T^*_\alpha\langle N_0^*\rangle_\alpha + 6\,N_-\,T^*_{\rm MCE} = E. \qquad \text{(C5)}$$

Using Eq. (30), this gives

$$T^*_\alpha \cong T\,\frac{\overline{\overline{N}}}{N_-}. \qquad \text{(C6)}$$

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