Effects of bulk viscosity at freezeout

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We investigate particle spectra and elliptic flow coefficients in relativistic heavy-ion collisions by taking into account the distortion of phase space distributions by bulk viscosity at freezeout. We first calculate the distortion of phase space distributions in a multicomponent system with Grad's 14-moment method. We find some subtle issues when macroscopic variables are matched with microscopic momentum distributions in a multicomponent system, and we develop a consistent procedure to uniquely determine the corrections to the phase space distributions. Next, we calculate particle spectra by using the Cooper-Frye formula to see the effect of the bulk viscosity. Despite the relative smallness of the bulk viscosity, we find that it is likely to have a visible effect on particle spectra and elliptic flow coefficients. This indicates the importance of taking into account bulk viscosity together with shear viscosity to constrain the transport coefficients with better accuracy from comparison with experimental data.

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I. INTRODUCTION

One of the major discoveries at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory [1] is that the quark-gluon plasma (QGP) behaves like a "perfect liquid" [2–4]; this finding has attracted much theoretical interest in various fields. Hydrodynamic modeling of heavy-ion collisions plays an important role in deducing this fact [5,6]. Anisotropy of radial flow, namely elliptic flow [7], is found to be large in Au + Au and Cu + Cu collisions at RHIC energies [8–10]. Elliptic flow is quantified by the second harmonics of the azimuthal angle distribution, $v_2 = \langle \cos 2(\phi_p) \rangle$ [11]. The centrality, pseudorapidity, and transverse momentum dependences (up to $\sim$1 GeV/c) of v_2 data are reproduced reasonably well by using Glauber-type initial conditions and implementing hadronic dissipative effects in ideal hydrodynamic models [12]. However, when initial conditions from the Glauber model are replaced with those from a model based on the color glass condensate, elliptic flow coefficients overshoot the experimental data because of eccentricity larger than that from the conventional Glauber-model calculations [12]. Given that the initial conditions in relativistic heavy ion collisions are not known precisely, this discrepancy strongly suggests that the effects of viscosities in the QGP phase are required to reproduce the v_2 data in this particular case. Even though ideal hydrodynamic models have been successful in reproducing the v_2 data, the next important tasks are to constrain the transport coefficients by comparing theoretical predictions with the experimental data and to comprehensively understand the transport properties of the QGP.

Shear viscosity has already been taken into account in several hydrodynamic simulations to investigate its effect on elliptic flow in relativistic heavy ion collisions [13–18]. However, few studies have been done so far to investigate the effect of bulk viscosity in dynamical simulations of relativistic viscous fluids [19–23]. Many years ago, it was found that the bulk viscous coefficient ζ has a prominent peak in the vicinity of the QCD phase transition, caused by the reduction of the sound velocity [24]. Recently, the importance of bulk viscosity has been realized again, in the context of the violation of scale invariance in QCD [25]. The smallness of the sound velocity is intimately related to the violation of scale invariance, so it would be a universal feature that the bulk viscous coefficient becomes large around the critical region. From a phenomenological point of view, large bulk viscosity could trigger a violation of the applicability of the hydrodynamic framework around the (pseudo)critical temperature [26] and support the success of hybrid approaches [3,12,27] in which the hydrodynamic description of the QGP is followed by the kinetic description of the hadron gas.

In this paper, we focus on the effects of bulk viscosity in the late stage of relativistic heavy-ion collisions. To demonstrate these effects, we calculate transverse momentum spectra and differential elliptic flow coefficients. In ideal hydrodynamic calculations, the Cooper-Frye formula [28] is conventionally used to calculate the particle spectra at freezeout. In the hybrid calculations, this formula plays a role in describing the transition from macroscopic hydrodynamics to microscopic hadron cascade models. In both cases, the Cooper-Frye formula is indispensable in comparing hydrodynamic results with experimental data. In dissipative hydrodynamics, viscous effects are taken into account in the Cooper-Frye formula in two ways: One is a variation of the flow profile as a result of the viscous correction to the dynamical evolution, and the other is the distortion of the phase space distribution from its equilibrium form. So far, no full three-dimensional viscous hydrodynamic simulations are available. Given this situation, we take the flow profile from a full (3 + 1)-dimensional ideal hydrodynamic simulation [4,12], assuming that its variation due to viscosity is not significant. We estimate the correction to the distribution function from bulk viscosity as well as from shear viscosity by matching macroscopic quantities with those calculated from kinetic theory, and see how this

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affects transverse momentum (p_T) spectra and the transverse momentum dependence of the elliptic flow coefficient $v_2(p_T)$.

This paper is organized as follows. In Sec. II, we give a brief overview of relativistic kinetic theory by putting emphasis on subtle issues in its application to relativistic multicomponent gases and to gases with a zero (net baryon) number density limit. In Sec. III, we focus on a hadron resonance gas model, which is widely exploited in hydrodynamic analyses of relativistic heavy ion collisions, and introduce simple models for shear and bulk viscous coefficients. We demonstrate that the effects of bulk viscosity at freezeout are possibly large by calculating the prefactors in the correction to the distribution function. In Sec. IV, we calculate p_T spectra and $v_2(p_T)$ by taking flow profiles from three cases: (i) Björken flow, (ii) a blast-wave model, and (iii) a full $(3+1)$-dimensional ideal hydrodynamic simulation. Section V is the conclusion.

II. RELATIVISTIC KINETIC THEORY

We would like to express the viscous corrections to the phase space distribution for a relativistic multicomponent gas in terms of macroscopic variables by matching the macroscopic variables with those obtained from kinetic theory. For this purpose, we use Grad's 14-moment method, which is a common method in nonequilibrium statistical mechanics [29]. This is a generalization of the treatment developed by Israel and Stewart [30] from a single-component to a multicomponent system.

We will see that some nontrivial facts arise when a multicomponent gas is considered. First, one usually considers a scalar term, a vector term, and a traceless tensor term for a momentum expansion of the nonequilibrium part of the distribution and assumes that the trace part of the tensor term can be absorbed into the scalar term. However, this is no longer the case for a multicomponent system because the equivalence of the scalar term and the trace part holds only for a single-component gas. Second, the number of equations that determine the modification of the distribution does not change when a zero net baryon density limit is taken. This means we do not have to suffer the loss of conditions to obtain the unique expression of the distortion within the framework of Grad's 14-moment method. Therefore, we should not use the ansatz appearing in Ref. [18], which leads to a violation of thermodynamic stability, as we will show later. As in Ref. [18], we also call this ansatz simply "the quadratic ansatz" in the sense that it has only quadratic terms and has neither linear nor constant terms with respect to momentum. Consideration of the zero net baryon density limit is of practical importance because the limit is often used in hydrodynamic models of the hot QCD matter created in relativistic heavy-ion collisions.

It should be emphasized here that our aim is to develop a consistent and convenient method for calculating the viscous modification of particle spectra at freezeout within the framework of Grad's 14-moment method for hydrodynamic calculations. Therefore, it is beyond the scope of this study to consider the microscopic nature of each particle reaction or any dynamics that cannot be determined uniquely from the hydrodynamic picture.

A. Distortion of the distribution function

Tensor decompositions of the energy-momentum tensor $T^{\mu\nu}$ and the net baryon number current N_B^μ give the definitions of the macroscopic variables. The decomposed terms are divided into an equilibrium part and a nonequilibrium part:

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}, \tag{1}$$

where

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - P\Delta^{\mu\nu}, \tag{2}$$

$$\delta T^{\mu\nu} = -\Pi\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}, \tag{3}$$

and

$$N_B^\mu = N_{B0}^\mu + \delta N_B^\mu, \tag{4}$$

where

$$N_{B0}^\mu = n_{B0}u^\mu, \quad \delta N_B^\mu = V^\mu. \tag{5}$$

The timelike and spacelike projection operators are, respectively, defined as u^μ and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ with the Minkowski metric $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. We define the dot product as $a\cdot b = a_\mu b_\nu g^{\mu\nu}$. The parentheses represent symmetrization: $A^{(\mu}B^{\nu)} = \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu)$. Also, $e_0 = u_\mu T_0^{\mu\nu} u_\nu$, $P = -\frac{1}{3}\Delta_{\mu\nu}T_0^{\mu\nu}$, and $n_{B0} = u_\mu N_{B0}^\mu$ denote the energy density, the hydrostatic pressure, and the charge density, respectively. $W^\mu = \Delta^\mu{}_\alpha \delta T^{\alpha\beta} u_\beta$ is the energy current, $V^\mu = \Delta^\mu{}_\nu \delta N_B^\nu$ the charge current, $\Pi = -\frac{1}{3}\Delta_{\mu\nu}\delta T^{\mu\nu}$ the bulk pressure, and $\pi^{\mu\nu} = \delta T^{\langle\mu\nu\rangle} = [\frac{1}{2}(\Delta^\mu{}_\alpha\Delta^\nu{}_\beta + \Delta^\mu{}_\beta\Delta^\nu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}]\delta T^{\alpha\beta}$ the shear stress tensor. Generally, e_0 and n_{B0} would have dissipative counterparts δe and δn_B, but such terms are not considered here because we use the Landau matching conditions, which demand that $\delta e = u_\mu \delta T^{\mu\nu} u_\nu = 0$ and $\delta n_B = u_\mu \delta N_B^\mu = 0$. These conditions are necessary to make the system thermodynamically stable in first-order theory. We will see the details later. For reviews on relativistic viscous hydrodynamics, see Ref. [6].

For a system with multispecies particles, kinetic theory demands that

$$T^{\mu\nu} = \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\mu p_i^\nu f_i, \tag{6}$$

$$N_B^\mu = \sum_i \int \frac{b_i g_i d^3p}{(2\pi)^3 E_i} p_i^\mu f_i. \tag{7}$$

The label i denotes a particle species; g_i and b_i are the degeneracy and the baryon number, respectively. We define $\delta f_i = f_i - f_i^0$ to be the deviation of the distribution function from its equilibrium form, $f_i^0 = [\exp(\frac{p_i\cdot u - b_i\mu_B}{T}) - \epsilon]^{-1} = (\exp y_0 - \epsilon)^{-1}$. Here, $\epsilon = +1$ for bosons and $\epsilon = -1$ for fermions. T and μ_B denote the temperature and the baryon chemical potential, respectively.

Now we expand y, defined by $f_i = (\exp y - \epsilon)^{-1}$, up to the second order in momentum and estimate viscous corrections to the distribution function by Grad's 14-moment method:

$$\delta f^i = -f_0^i(1+\epsilon f_0^i)(p_i^\mu \varepsilon_\mu + p_i^\mu p_i^\nu \varepsilon_{\mu\nu}), \tag{8}$$

where ε_μ and $\varepsilon_{\mu\nu}$ are coefficients of the expansion. The numbers of unknown variables are 4 and 10, respectively; $\varepsilon_{\mu\nu}$ should be symmetric because any antisymmetric term cancels out when a contraction is taken with $p_i^\mu p_i^\nu$.

$\varepsilon_{\mu\nu}$ is often considered to be traceless, and the scalar term ε is introduced instead [30]. The apparent lack of a scalar term in Eq. (8) is due to the fact that we do not set any condition on the tensor term $\varepsilon_{\mu\nu}$; that is, it generally has a nonzero trace. The numbers of unknown variables are the same in both cases. If we consider a single-component system, such as a pion gas, the trace part of the tensor term is equivalent to a scalar term ε because the trace part can be separated as $\frac{\mathrm{Tr}(\varepsilon_{\mu\nu})}{4}g_{\mu\nu}$ and the metric produces a scalar $p^\mu p^\nu g_{\mu\nu} = m^2$.

This is not the case for a multicomponent system because all the viscous correction tensors ε, ε_μ, and $\varepsilon_{\mu\nu}$ are macroscopic quantities in the sense that they do not depend on any particular particle species. However, the trace part has mass dependence, which means that it is particle-species dependent:

$$\begin{aligned}\delta f^i_{\text{tensor}} &= -f^i_0\left(1+\epsilon f^i_0\right)p^\mu_i p^\nu_i \varepsilon_{\mu\nu}\\ &= -f^i_0\left(1+\epsilon f^i_0\right)p^\mu_i p^\nu_i \left\{\frac{\mathrm{Tr}(\varepsilon_{\mu\nu})}{4}g_{\mu\nu}\right.\\ &\quad\left.+\left[\varepsilon_{\mu\nu}-\frac{\mathrm{Tr}(\varepsilon_{\mu\nu})}{4}g_{\mu\nu}\right]\right\}\\ &= -f^i_0\left(1+\epsilon f^i_0\right)\left[\frac{\mathrm{Tr}(\varepsilon_{\mu\nu})m_i^2}{4}+p^\mu_i p^\nu_i \tilde{\varepsilon}_{\mu\nu}\right]. \end{aligned} \tag{9}$$

Here $\tilde{\varepsilon}_{\mu\nu}$ is a traceless tensor. The equivalence of the scalar term and the trace term holds for a single-component system because we can set a variable $\mathrm{Tr}(\varepsilon_{\mu\nu})$ to be $\frac{4\varepsilon}{m^2}$. Because the same prescription does not work for a multicomponent gas where we have the additional index i, it is problematic whether to consider a nonzero trace tensor term or the combination of a scalar term and a traceless tensor term in a system of multispecies particles. We will see in Sec. III that only the former is relevant in the case of a 16-component hadron resonance gas.

Our aim here is to uniquely determine the explicit forms of ε_μ and $\varepsilon_{\mu\nu}$. As we mentioned earlier, we introduce the Landau matching conditions $u_\mu \delta T^{\mu\nu} u_\nu = 0$ and $u_\mu \delta N^\mu_B = 0$, which ensure thermodynamic stability in first-order theory (see Appendix A for further details). These conditions demand that the energy density and the number density match the equilibrium densities. Together with the kinetic definitions of the macroscopic variables, we now have 14 equations in total to determine 14 unknowns in δf.

We first define $J^{\mu_1\mu_2\cdots\mu_m}$ and $\tilde{J}^{\mu_1\mu_2\cdots\mu_m}$ to express the conditions explicitly in terms of thermodynamic quantities:

$$\begin{aligned}J^{\mu_1\mu_2\cdots\mu_m} &= \sum_i \int \frac{g_i d^3p}{(2\pi)^3E_i} f^i_0\left(1+\epsilon f^i_0\right)p^{\mu_1}_i p^{\mu_2}_i\cdots p^{\mu_m}_i\\ &= \sum_n [(\Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2n-1}\mu_{2n}}u^{\mu_{2n+1}}\cdots u^{\mu_m})\\ &\quad + (\text{permutations})]J_{mn},\end{aligned} \tag{10}$$

$$\begin{aligned}\tilde{J}^{\mu_1\mu_2\cdots\mu_m} &= \sum_i \int \frac{b_i g_i d^3p}{(2\pi)^3E_i} f^i_0\left(1+\epsilon f^i_0\right)p^{\mu_1}_i p^{\mu_2}_i\cdots p^{\mu_m}_i\\ &= \sum_n [(\Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2n-1}\mu_{2n}}u^{\mu_{2n+1}}\cdots u^{\mu_m})\\ &\quad + (\text{permutations})]\tilde{J}_{mn}.\end{aligned} \tag{11}$$

J_{mn} is a coefficient of the expansion of $J^{\mu_1\mu_2\cdots\mu_m}$ by $(m-2n)$ u^μ's and n $\Delta^{\mu\nu}$'s. $\tilde{J}_{mn}$ is defined in the same way but has the baryon number b_i as a weight factor. These quantities should be distinguished in a mixed system of baryons, antibaryons, and mesons because they contribute differently to the energy-momentum tensor and the baryon number current, as seen in Eqs. (6) and (7).

The Landau matching condition for the energy-momentum tensor is simplified by using the expressions defined previously:

$$\begin{aligned}0 &= u_\mu u_\nu \sum_i \int \frac{g_i d^3p}{(2\pi)^3E_i} p^\mu_i p^\nu_i \delta f_i\\ &= -u_\mu u_\nu J^{\mu\nu\alpha}\varepsilon_\alpha - u_\mu u_\nu J^{\mu\nu\alpha\beta}\varepsilon_{\alpha\beta}\\ &= -J_{30}\varepsilon_* - (J_{40}-J_{41})\varepsilon_{**} - J_{41}\mathrm{Tr}(\varepsilon).\end{aligned} \tag{12}$$

From now on we use the notations $\varepsilon_* = \varepsilon_\mu u^\mu$, $\varepsilon_{**} = \varepsilon_{\mu\nu}u^\mu u^\nu$, $\mathrm{Tr}(\varepsilon) = \varepsilon^\mu_\mu$, and $\Delta^{\mu\nu}\varepsilon_{\nu *} = \Delta^{\mu\nu}u^\alpha\varepsilon_{\nu\alpha}$. The other conditions can be expressed similarly:

$$0 = -\tilde{J}_{20}\varepsilon_* - (\tilde{J}_{30}-\tilde{J}_{31})\varepsilon_{**} - \tilde{J}_{31}\mathrm{Tr}(\varepsilon), \tag{13}$$

$$\Pi = J_{31}\varepsilon_* + \left(J_{41}-\tfrac{5}{3}J_{42}\right)\varepsilon_{**} + \tfrac{5}{3}J_{42}\mathrm{Tr}(\varepsilon), \tag{14}$$

$$W^\mu = -J_{31}\Delta^{\mu\nu}\varepsilon_\nu - 2J_{41}\Delta^{\mu\nu}\varepsilon_{\nu *}, \tag{15}$$

$$V^\mu = -\tilde{J}_{21}\Delta^{\mu\nu}\varepsilon_\nu - 2\tilde{J}_{31}\Delta^{\mu\nu}\varepsilon_{\nu *}, \tag{16}$$

$$\pi^{\mu\nu} = -2J_{42}\varepsilon^{\langle\mu\nu\rangle}. \tag{17}$$

The conditions [Eqs. (12)–(17)] are classified into three independent sets of equations: (i) the definition of the bulk pressure and the Landau matching conditions for the scalars ε_*, ε_{**}, and $\mathrm{Tr}(\varepsilon)$; (ii) the definitions of the energy current and the charge current for the vectors $\Delta^{\mu\nu}\varepsilon_\nu$ and $\Delta^{\mu\nu}\varepsilon_{\nu *}$; and (iii) the definition of the shear stress tensor for the tensor $\varepsilon^{\langle\mu\nu\rangle}$. The equations are immediately solved for each group:

$$\varepsilon_* = \varepsilon_\mu u^\mu = D_0\Pi, \tag{18}$$

$$\varepsilon_{**} = \varepsilon_{\mu\nu}u^\mu u^\nu = \tilde{B}_0\Pi, \tag{19}$$

$$\mathrm{Tr}(\varepsilon) = \varepsilon^\mu_\mu = B_3\Pi, \tag{20}$$

$$\Delta^{\mu\nu}\varepsilon_\nu = D_1 W^\mu + \tilde{D}_1 V^\mu, \tag{21}$$

$$\Delta^{\mu\nu}\varepsilon_{\nu *} = B_1 W^\mu + \tilde{B}_1 V^\mu, \tag{22}$$

$$\varepsilon^{\langle\mu\nu\rangle} = B_2\pi^{\mu\nu}, \tag{23}$$

where the prefactors D_i and B_i in Eqs. (18)–(23) are functions of J_{mn} and $\tilde{J}_{mn}$. Thus, ε_μ and $\varepsilon_{\mu\nu}$ are determined to be

$$\begin{aligned}\varepsilon_\mu &= \varepsilon_* u_\mu + \Delta_{\mu\nu}\varepsilon^\nu\\ &= D_0\Pi u_\mu + D_1 W_\mu + \tilde{D}_1 V_\mu,\end{aligned} \tag{24}$$

$$\begin{aligned}\varepsilon_{\mu\nu} &= \varepsilon^{**}u_\mu u_\nu + \Delta_{\mu\nu}\left[\frac{\mathrm{Tr}(\varepsilon)-\varepsilon^{**}}{3}\right]\\ &\quad + 2u_{(\mu}\Delta_{\nu)\alpha}\varepsilon^{\alpha *} + \varepsilon_{\langle\mu\nu\rangle}\\ &= (B_0\Delta_{\mu\nu} + \tilde{B}_0 u_\mu u_\nu)\Pi\\ &\quad + 2B_1 u_{(\mu}W_{\nu)} + 2\tilde{B}_1 u_{(\mu}V_{\nu)} + B_2\pi_{\mu\nu},\end{aligned} \tag{25}$$

where $B_0 = (B_3 - \tilde{B}_0)/3$ is used. The prefactors are explicitly expressed as

$$D_0 = 3(J_{40}\tilde{J}_{31} - J_{41}\tilde{J}_{30})\mathcal{J}_3^{-1}, \tag{26}$$

$$B_0 = (J_{30}\tilde{J}_{30} - J_{40}\tilde{J}_{20})\mathcal{J}_3^{-1}, \tag{27}$$

$$\tilde{B}_0 = 3(J_{41}\tilde{J}_{20} - J_{30}\tilde{J}_{31})\mathcal{J}_3^{-1}, \tag{28}$$

$$D_1 = -2\tilde{J}_{31}\mathcal{J}_2^{-1}, \tag{29}$$

$$\tilde{D}_1 = 2J_{41}\mathcal{J}_2^{-1}, \tag{30}$$

$$B_1 = \tilde{J}_{21}\mathcal{J}_2^{-1}, \tag{31}$$

$$\tilde{B}_1 = -J_{31}\mathcal{J}_2^{-1}, \tag{32}$$

$$B_2 = \mathcal{J}_1^{-1}, \tag{33}$$

where

$$\mathcal{J}_3 = 5J_{30}J_{42}\tilde{J}_{30} + 3J_{31}J_{40}\tilde{J}_{31} + 3J_{41}J_{41}\tilde{J}_{20} - 3J_{31}J_{41}\tilde{J}_{30} - 3J_{30}J_{41}\tilde{J}_{31} - 5J_{40}J_{42}\tilde{J}_{20}, \tag{34}$$

$$\mathcal{J}_2 = 2J_{31}\tilde{J}_{31} - 2J_{41}\tilde{J}_{21}, \tag{35}$$

$$\mathcal{J}_1 = -2J_{42}. \tag{36}$$

The prefactor for the shear stress tensor [Eq. (33)] has the same form as shown in Ref. [30], whereas the prefactors for the bulk pressure [Eqs. (26)–(28)] are different from those in Ref. [30] because the finite trace tensor term is considered instead of the scalar term. Of course, those reduce to the same formulas for a single-component gas.

It is worth mentioning that if we define the heat current q^μ as

$$q_\mu = W_\mu - \frac{J_{31}}{\tilde{J}_{21}}V_\mu = W_\mu - \frac{e+P}{n_B}V_\mu, \tag{37}$$

then Eqs. (21) and (22) can be expressed as $D_1 q_\mu - V_\mu/n_B T$ and $B_1 q_\mu$, respectively.

B. Discussion

Some comments are in order here. First, we consider the special case where the Landau frame—that is, $W^\mu = 0$—and the zero net baryon density limit $N_B^\mu \to 0$ are used. These conditions are often imposed for hydrodynamic analyses of the QGP in relativistic heavy-ion collisions because the numbers of baryons and antibaryons are roughly the same for such events. In our framework, they correspond to estimating only the contributions of the shear viscosity and the bulk viscosity.

The Landau condition for the charge current [Eq. (13)] vanishes when the limit is taken, because $\tilde{J}_{mn}$ does. This could induce an ambiguity because the number of conditions for the determination of the prefactors decreases. But if we expand $\tilde{J}_{mn}$ around the baryon chemical potential $\mu_B = 0$, we have

$$\tilde{J}_{mn} = 0 + \mu_B \left.\frac{\partial \tilde{J}_{mn}}{\partial \mu_B}\right|_{\mu_B=0} + \mathcal{O}(\mu_B^2). \tag{38}$$

Then Eq. (13) is reduced to

$$\left[\frac{\partial \tilde{J}_{20}}{\partial \mu_B}\varepsilon_* + \frac{\partial(\tilde{J}_{30} - \tilde{J}_{31})}{\partial \mu_B}\varepsilon_{**} + \frac{\partial \tilde{J}_{31}}{\partial \mu_B}\mathrm{Tr}(\varepsilon)\right]_{\mu_B=0} = 0, \tag{39}$$

which provides a meaningful relation even in the zero baryon density limit. This enables us to determine the prefactors for the bulk pressure uniquely even after the limit is taken.

Second, we discuss the validity of the quadratic ansatz [18]—that is, $\varepsilon_{\mu\nu} = C_1\pi_{\mu\nu} + C_2\Delta_{\mu\nu}\Pi$—in the distortion of the distribution. We see that this particular form of viscous correction does not satisfy the Landau matching conditions:

$$u_\mu \delta T^{\mu\nu} u_\nu = -J_{41}\mathrm{Tr}(\varepsilon) \neq 0, \tag{40}$$

$$u_\mu \delta N_B^\mu = -\tilde{J}_{31}\mathrm{Tr}(\varepsilon) \neq 0. \tag{41}$$

Therefore, we should not use this ansatz, because the matching conditions are necessary conditions for thermodynamical stability and should not be violated. Also, if the Landau matching conditions are used in hydrodynamic equations, the consistency will be lost. Note that, in the quadratic ansatz, the ε_{**}'s in Eqs. (12) and (13) vanish because $\varepsilon_{\mu\nu}$ is perpendicular to u^μ in this ansatz. Also, the correction from bulk viscosity is naively expected to be smaller than that from shear viscosity, because $C_2 = -\frac{2}{5}C_1$ holds.

Third, it is important to explicitly consider a multicomponent system for estimations of the viscous corrections [Eqs. (26)–(33)] because the deviation of the distribution for the ith particle species δf_i generally depends on whether it is the only component or one of the components in a multicomponent gas. This results from the fact that the J_{mn}'s, and therefore the prefactors, like thermodynamic variables, include information on all particle species. Note that the difference becomes negligible if we use first-order theory and assume that the shear viscosity η is proportional to the entropy density s, because in the Boltzmann approximation we have

$$\begin{aligned}\delta f^i_{\mathrm{shear}} &= B_2 \times 2\eta \nabla_{\langle\mu}u_{\nu\rangle} \times p_i^\mu p_i^\nu \\ &\propto -\frac{1}{2J_{42}} \times 2s \times \nabla_{\langle\mu}u_{\nu\rangle} p_i^\mu p_i^\nu \\ &\approx -\frac{1}{2J_{42}} \times \frac{2J_{42}}{T^3} \times \nabla_{\langle\mu}u_{\nu\rangle} p_i^\mu p_i^\nu \\ &= C^i(T).\end{aligned} \tag{42}$$

Here we used the relation $J_{42} \approx sT^3$ in the Boltzmann approximation [30]. We see that $C^i(T)$ is a function dependent on the index i and the temperature but independent of what other components are in the gas.

Finally, we formulate the distortion of the phase space distribution in a model-independent way in the sense that one can adopt any model for the transport coefficients, not necessarily from the Boltzmann equation but from more general results of, for example, linear response theory. For an example of the model of shear and bulk viscosity to demonstrate the viscous effects at freezeout, see Sec. III. One may want to obtain species-dependent distortion that reflects the cross section or the mean free path. However, doing so is almost equivalent to solving the Boltzmann equation itself: One prepares collision terms among all constituents, calculates transport coefficients from the Boltzmann equation with the collision terms given previously, uses them in hydrodynamic calculations, and solves the linearized Boltzmann equation for all species. Obviously the results are very specific to the model of collision terms and, therefore, that discussion is not the purpose of the present paper. It should be noted that, as shown in Sec. II A, there is no room for microscopic information such as cross section and mean free path to appear in the current formulation. Given that hadron cascade models, which are rather suited for microscopic (species-dependent) description, are already available, the method described in this paper would play a role in providing initial conditions for these microscopic models.

III. THE MODEL

A. Equation of state and transport coefficients

Following the discussion in the previous section, we estimate the thermodynamic quantities and the prefactors appearing in δf. As the model of the equation of state, we consider the 16-component hadron resonance gas, which has mesons and baryons with mass up to $\Delta(1232)$. Hereafter, we take the Landau frame and set the system to be baryon-free; therefore, B_1, $\tilde{B}_1$, D_1, and $\tilde{D}_1$ do not appear. We thus consider the contributions of shear and bulk viscosity. Transport coefficients are needed to estimate the macroscopic variables Π and $\pi^{\mu\nu}$ in Eqs. (24) and (25). The models for the shear viscosity η and the bulk viscosity ζ are taken from Refs. [31] and [32]:

$$\eta = \frac{1}{4\pi} s, \tag{43}$$

$$\zeta = \alpha \left(\frac{1}{3} - c_s^2 \right)^2 \eta, \tag{44}$$

where $c_s = \sqrt{\frac{\partial P}{\partial e_0}}$ is the sound velocity. The factor α in the bulk viscosity is set to 15 unless otherwise stated [32]. Figure 1 shows the temperature dependence of η/s and ζ/s for a hadron resonance gas model. In the temperature range relevant to relativistic heavy-ion collisions $0.1 \lesssim T \lesssim 0.2$ GeV, ζ/s is several factors smaller than η/s. So one might expect that the effect of the bulk viscosity would be small. This is not necessarily true because the corrections in δf appear as a combination of transport coefficients and the prefactors discussed later.

Figure 2 shows the numerical results for the prefactors D_0, B_0, $\tilde{B}_0$, and B_2 appearing in δf as functions of temperature T in a hadron resonance gas model. Returning to the discussion of whether we should consider a nonzero trace tensor term or a trace part of a scalar term, we find that only the former is relevant for the 16-component hadron resonance gas case. This is because if we choose the combination of the scalar term and the traceless tensor term, D_0, B_0, and $\tilde{B}_0$ diverge because of a change in the sign of the denominator [Eq. (34)] at temperatures below 0.2 GeV.

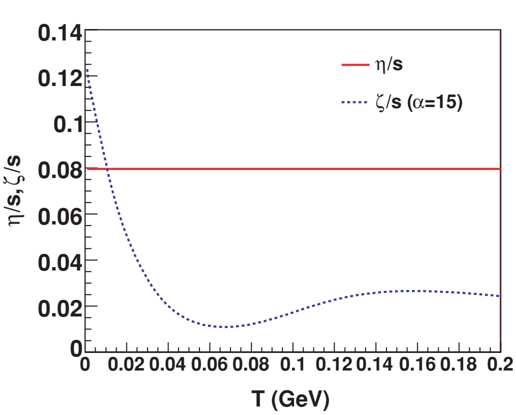

FIG. 1. (Color online) η/s (solid) and ζ/s with $\alpha = 15$ (dotted) as functions of temperature in a resonance gas model.

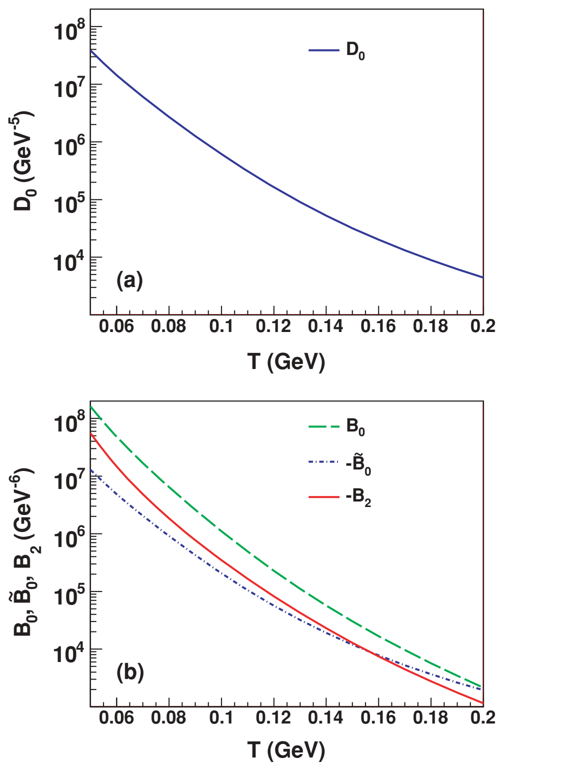

FIG. 2. (Color online) (a) The prefactor in ε_μ for the bulk pressure, D_0, as a function of temperature. (b) The prefactors in $\varepsilon_{\mu\nu}$ for the bulk pressure, B_0 (dashed line) and $\tilde{B}_0$ (dash-dotted line), and that for the shear stress tensor, B_2 (solid line), as functions of temperature.

The numerical code to generate the prefactors in ε_μ and $\varepsilon_{\mu\nu}$ will be made available [33]. We can set the components of a resonance gas and calculate the prefactors as well as any moment of $f_0(1 + \epsilon f_0)$, J_{mn}.

B. Particle spectra and flow

In hydrodynamic analyses of the QGP created in relativistic heavy-ion collisions, the Cooper-Frye formula is used at freezeout. This converts the macroscopic variables into microscopic distributions, which enables us to compare the hydrodynamic results with experimental data. From a phenomenological point of view, the formula works as an interface from a hydrodynamic model to a cascade model. The p_T spectrum of the ith particle is given, with the Cooper-Frye formula [28], as

$$\frac{d^2 N_i}{d^2 p_T dy} = \frac{g_i}{(2\pi)^3} \int_\Sigma p_i^\mu d\sigma_\mu \left(f_0^i + \delta f^i \right), \tag{45}$$

where p_T and y denote the transverse momentum and the rapidity. The equilibrium distribution f_0 is given as

$$f_0^i = \frac{1}{e^{(p^i \cdot u - b_i \mu_B)/T} - \epsilon}. \tag{46}$$

The elliptic flow parameter is the coefficient of the second harmonics $\cos(2\phi_p)$ in a Fourier expansion of the azimuthal momentum distribution, where ϕ_p is the azimuthal angle in momentum space. It is defined as

$$v_2(p_T) = \frac{\int dy \int_0^{2\pi} d\phi_p \cos(2\phi_p) \frac{dN_i}{d\phi_p p_T dp_T dy}}{\int dy \int_0^{2\pi} d\phi_p \frac{dN_i}{d\phi_p p_T dp_T dy}}. \quad (47)$$

Viscous corrections are taken into account via (i) variation of the flow and (ii) modification of the distribution function. Because we do not have a full (3 + 1)-dimensional viscous hydrodynamic flow, we focus on the latter in this study. Profiles of the flow u^μ and the freezeout hypersurface $d\sigma_\mu$, which are necessary for calculations of the Cooper-Frye formula [Eq. (45)], are taken from (i) the Björken model [34], (ii) a blast-wave model [35], and (iii) a (3 + 1)-dimensional ideal hydrodynamic simulation [4].

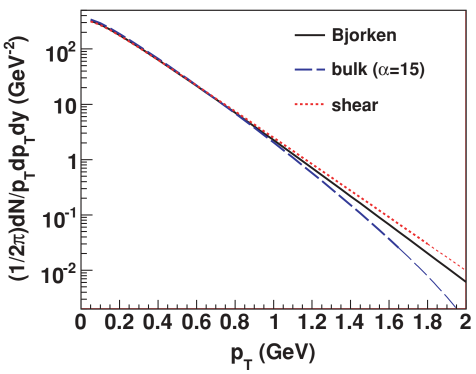

FIG. 3. (Color online) Viscous corrections to p_T spectra with Björken flow. Solid, thick dashed, and thick dotted lines are, respectively, the results without any viscous corrections, with the effect of bulk viscosity ($\alpha = 15$), and with the effect of the shear viscosity. Thin dashed and dotted lines show that the absolute value of the ratio of the correction to the ideal spectrum becomes greater than 0.5.

IV. RESULTS

We estimate the particle spectra and the elliptic flow coefficients of the negative pion with mass $m = 0.139$ GeV. The freezeout temperature T_f is set to 0.160 GeV. This temperature is sufficiently near the QCD (pseudo)critical temperature, at which the bulk viscous coefficient is expected to be large [24,25]. At $T_f = 0.160$ GeV, Eqs. (43) and (44) yield, respectively, $\eta = 1.31 \times 10^{-3}$ GeV^3 and $\zeta = 4.37 \times 10^{-4}$ GeV^3 when $\alpha = 15$. In this study the Navier-Stokes limit is taken for the shear stress tensor and the bulk pressure, which means that $\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle}$ and $\Pi = -\zeta\partial_\mu u^\mu$, where $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$.

A. Björken flow

We first consider the Bjorken model. We use the expanding coordinates (τ, r, ϕ, η_s), where proper time τ, radius r, azimuthal angle ϕ, and spacetime rapidity η_s are defined in the relations $t = \tau\cosh\eta_s$, $x = r\cos\phi$, $y = r\sin\phi$, and $z = \tau\sinh\eta_s$. In this frame, the Björken flow is written as

$$u^\tau = 1, \quad u^r = u^\phi = u^{\eta_s} = 0. \quad (48)$$

The radius of the nuclei and the freezeout time are set to $R_0 = 10.0$ fm and $\tau = 7.5$ fm, respectively. Elements of the freezeout hypersurface have only the radial component:

$$d\sigma_\tau = \tau d\eta_s r dr d\phi, \quad d\sigma_r = d\sigma_\phi = d\sigma_{\eta_s} = 0. \quad (49)$$

It is noteworthy that the Cooper-Frye formula for this model can be analytically expressed in the case of the Boltzmann approximation. See Appendix B for the details.

Figure 3 shows the particle spectra with corrections from the bulk viscosity or the shear viscosity. The mean transverse momentum $\langle p_T\rangle$ is lowered by the bulk viscosity in the case of the Björken flow. This interpretation can be made from the sign of the shear stress tensor and the bulk pressure as follows. The bulk pressure works as a negative pressure; the pressure in the energy-momentum tensor is effectively reduced to $P - |\Pi|$ because the system is expanding in the longitudinal direction and the expansion scalar $\partial_\mu u^\mu = 1/\tau$ is positive. As a result, the number of particles with lower momenta increases. Similarly, the shear viscosity is naively expected to enhance $\langle p_T\rangle$ in the midrapidity region because the pressure in the radial direction is increased by the decrease of the pressure in

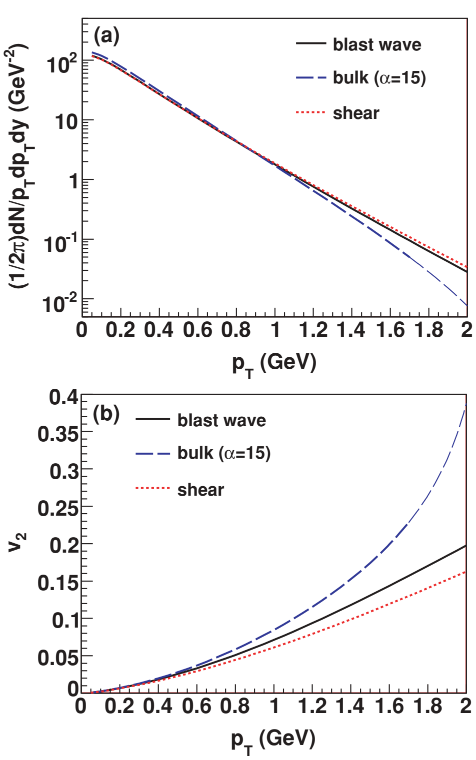

FIG. 4. (Color online) Viscous corrections to (a) p_T spectra and (b) $v_2(p_T)$ in the blast-wave model. Solid, thick dashed, and thick dotted lines are, respectively, the results without any viscous corrections, with the effect of bulk viscosity ($\alpha = 15$), and with the effect of shear viscosity. Thin dashed line shows that the absolute value of the ratio of the correction to the ideal spectrum becomes larger than 0.5.

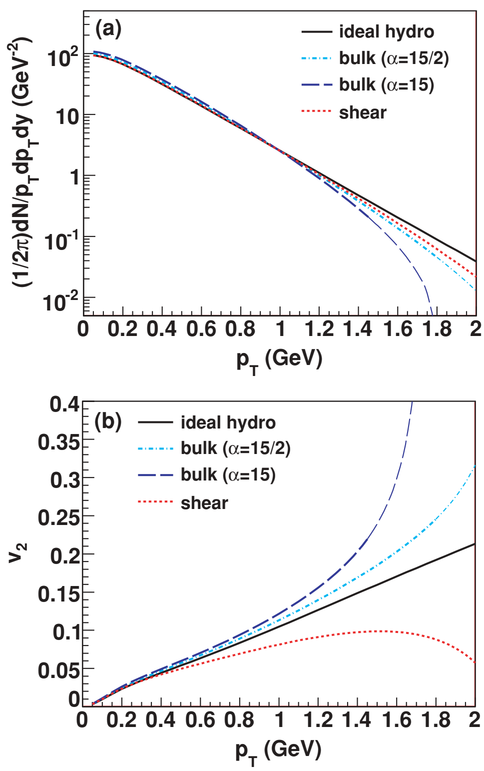

FIG. 5. (Color online) Viscous corrections to (a) p_T spectra and (b) $v_2(p_T)$ with ideal hydrodynamic flow. Solid, thick dash-dotted, thick dashed, and thick dotted lines are, respectively, the results without any viscous corrections, with the effect of bulk viscosity with $\alpha = 15/2$ and with $\alpha = 15$, and with the effect of shear viscosity. Thin dash-dotted and dashed lines show that the absolute value of the ratio of the correction to the ideal spectrum becomes larger than 0.5.

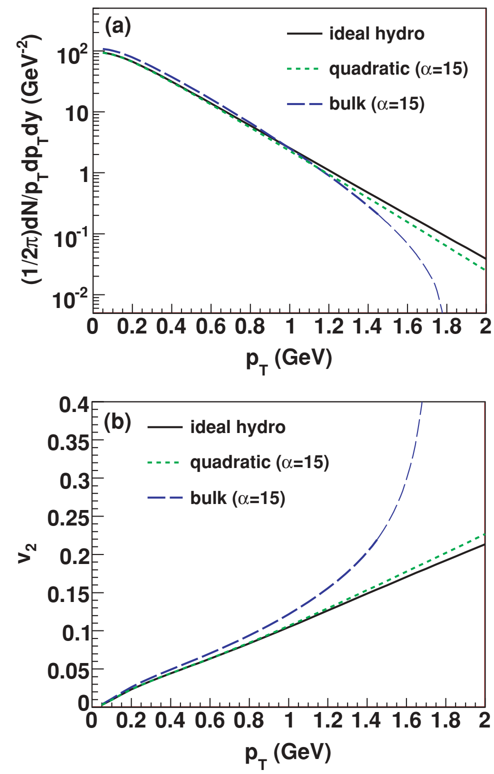

FIG. 6. (Color online) Viscous corrections to (a) p_T spectra and (b) $v_2(p_T)$ with (thick dotted) or without (thick dashed) the quadratic ansatz. Solid line is the result from an ideal hydrodynamic model. Thin dashed line shows that the absolute value of the ratio of the correction to the ideal spectrum becomes larger than 0.5.

the longitudinal direction as a result of the fact that the shear stress tensor is traceless.

B. Flow from blast-wave model

As a blast-wave model, we follow Ref. [35]:

$$u^r = u_0 \frac{r}{R_0}[1 + u_2 \cos(2\phi)]\Theta(R_0 - r), \tag{50}$$

$$u^\tau = \sqrt{1 + (u^r)^2}, \tag{51}$$

$$u^\phi = u^{\eta_s} = 0, \tag{52}$$

where $u_0 = 0.55$ and $u_2 = 0.2$. The radius of the nuclei and the freezeout time are set to $R_0 = 7.5$ fm and $\tau = 5.25$ fm, respectively. The profile of the freezeout hypersurface is the same as Eq. (49) in the Björken model.

Again the bulk viscosity lowers the spectrum, as shown in Fig. 4. The elliptic flow parameter $v_2(p_T)$, conversely, is *enhanced* by the bulk viscosity. This counterintuitive result can be interpreted as follows. The slope of the differential elliptic flow coefficient $v_2(p_T)$ can be subject to the relation $dv_2(p_T)/dp_T \simeq v_2/\langle p_T \rangle$ [36], where v_2 on the right is the p_T-integrated elliptic flow coefficient. v_2 is not much affected by the viscous correction in the case of the blast-wave model, so the slope of $v_2(p_T)$ changes solely because $\langle p_T \rangle$ changes. The bulk viscous correction softens the spectrum, as shown in Fig. 4(a), and thus $\langle p_T \rangle$ is reduced. The reduction in the denominator $\langle p_T \rangle$ leads to apparent enhancement of the slope of $v_2(p_T)$. Effects of the shear viscosity can be explained similarly.

C. (3 + 1)-dimensional ideal hydrodynamic flow

In the last example, we demonstrate the effect of the bulk viscosity by using flow profiles from a full (3 + 1)-dimensional ideal hydrodynamic simulation in Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV. The impact parameter is taken to be 7.2 fm. For details of this particular hydrodynamic model, see Refs. [4,12]. The hadron equation of state in this hydrodynamic model is exactly the same as the one considered here: a hadron resonance gas model up to the mass of $\Delta(1232)$. These flow profiles together with temperature distributions are publicly available [37]. Note that we have checked that Π/P and $\pi^{\mu\nu}/T_0^{\mu\nu}$ at each freezeout position have moderate values, at most of the order of unity.

We see in Fig. 5 that *both* shear and bulk effects lower the spectrum in the high-p_T region. The enhancement of $v_2(p_T)$ by the bulk viscosity is due to the decrease in the mean p_T of the spectrum, as discussed in Sec. IV B. The nontriviality lies in the correction from the shear viscosity: It lowers the p_T spectrum

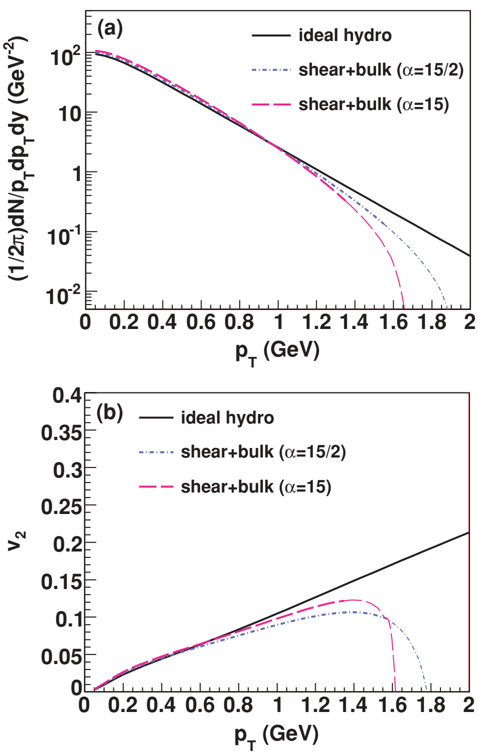

FIG. 7. (Color online) Viscous corrections from *both* shear viscosity and bulk viscosity to (a) p_T spectra and (b) $v_2(p_T)$. Solid, thick dash-dotted, and thick dashed lines are, respectively, the results without any viscous corrections, with the effect of both shear and bulk viscosities with $\alpha = 15/2$, and with both viscosities with $\alpha = 15$. Thin dash-dotted and dashed lines show that the absolute value of the ratio of the correction to the ideal spectrum becomes larger than 0.5.

and still suppresses $v_2(p_T)$. This behavior was also observed for a $(2+1)$-dimensional viscous hydrodynamic flow [14].

The viscous corrections in these calculations may have been overestimated for two reasons. First, we neglected $\delta u^\mu = u^\mu - u_0^\mu$ by considering the ideal hydrodynamic flow. Viscosity tends to reduce the difference in flow u^μ of the neighboring fluid elements; consequently, thermodynamic forces—in particular, derivatives of the flow—are expected to become smaller in the case of viscous hydrodynamic flow. This would reduce the amount of the corrections considered here. Second, the bulk pressure and the shear stress tensor are estimated in first-order theory, which naively means that they correspond to asymptotic values at times much greater than relaxation times in second-order theory. Note that because we expanded δf in momentum, the expansion is valid only where p_T is not too large. The eventual breakdown occurs when $\delta f / f_0$ exceeds unity. Thin lines in the figures indicate where the viscous correction exceeds 50% of the equilibrium particle spectra and where one should be careful about the validity of the expansion.

In Fig. 6, p_T spectra and $v_2(p_T)$ with the quadratic ansatz are compared with those without the ansatz. The results are not similar to each other in either the size of the correction or the p_T dependence. The quadratic ansatz greatly underestimates the effects of the bulk viscosity. Therefore, the proper procedure to obtain the prefactors in δf should be carried out, as discussed in Sec. II, to correctly calculate the distortion of particle spectra due to the bulk viscosity.

Finally, when *both* shear viscosity and bulk viscosity are considered, the slope of the particle spectra becomes steeper but that of $v_2(p_T)$ becomes moderate compared with that of the ideal distribution, as seen in Fig. 7. The reason for the latter would be that the effect of the bulk viscosity and that of the shear viscosity accidentally cancel each other in the low-p_T region.

V. CONCLUSIONS

We estimated the viscous corrections to the phase space distribution of a relativistic gas in a multicomponent system. We found that generalization to a multicomponent gas involves some subtleties. First, the trace part of the tensor term and the scalar term in the distortion of the distribution function are equivalent for a single-component gas but not for a multicomponent one. Numerical calculations suggest that one should have the trace part in the case of the 16-component hadron resonance gas. Second, if one takes a zero net baryon density limit, one also has to take the limit for the Landau matching condition for the baryon number current because it does yield a meaningful relation. Because the number of equations does not change, we can uniquely determine the prefactors for the bulk pressure appearing in the distortion of the distribution. It is not desirable to introduce an additional ansatz that violates the matching conditions, because such an ansatz makes the system thermodynamically unstable in first-order theory. Third, the deviation of the distribution for the ith gas component δf_i is generally different depending on whether it is the only component or one of the components. This results from the fact that the prefactors in ε_μ and $\varepsilon_{\mu\nu}$, like thermodynamic variables, include information of all the components in the gas.

For hydrodynamic models of the QGP created in relativistic heavy ion collisions, the Cooper-Frye formula is necessary to convert macroscopic variables into microscopic distributions at freezeout. This enables us to compare the hydrodynamic results with experimental data, or to see further development of the hadronic matter in a cascade model. Nonequilibrium effects are taken into account in the formula via the variation of the flow and the modification of the distribution. We focused only on the latter and estimated the viscous corrections to the p_T spectra and the elliptic flow $v_2(p_T)$ by following the previous discussion. Profiles of the flow and the freezeout hypersurface are taken from the Björken model with cylindrical geometry, the blast-wave model with azimuthally anisotropic flow, and a $(3+1)$-dimensional ideal hydrodynamic model for the numerical calculations. We found that corrections of the bulk viscosity from the distortion of the distribution have a visible effect on particle spectra and elliptic flow coefficient $v_2(p_T)$. This implies the importance of the bulk viscosity to constrain the transport coefficients with better accuracy from comparison with experimental data.

Quantitatively speaking, the viscous effects might have been overestimated because we approximated the shear stress tensor and the bulk pressure as those in first-order theory and, unlike in second-order theory, no relaxation effects are considered. As for the case of the estimation with an ideal hydrodynamic flow, the viscous corrections might be further exaggerated because the derivatives of the ideal hydrodynamic flow are generally larger than those of real viscous hydrodynamic flow. Considering the fact that the shear viscosity also has nontrivial effects on particle spectra, depending on the flow profile, a full $(3+1)$-dimensional viscous hydrodynamic flow is necessary to see more realistic behavior of p_T spectra and $v_2(p_T)$.

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APPENDIX A: THE LANDAU MATCHING CONDITIONS

The Landau matching conditions ensure thermodynamic stability in first-order theory. If the entropy current s^μ had a term proportional to Πu^μ in the nonequilibrium case, the derivative $\partial(u_\mu s^\mu)/\partial\Pi|_{\Pi=0}$ would be finite. This means that the system is not in a maximum-entropy state; that is, it is not thermodynamically stable. The matching conditions remove such unwanted terms. We can explicitly show this by inserting the phase space distribution into the definition of the entropy current in relativistic kinetic theory:

$$\begin{aligned} s^\mu &= -\sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\mu [(1+\epsilon f^i)\log(1+\epsilon f^i) + f^i \log f^i] \\ &= s u^\mu + \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\mu \delta f_i \frac{p^\beta u_\beta - b_i \mu_B}{T} + \mathcal{O}[(\delta f_i)^2] \\ &\approx s u^\mu + \frac{u^\mu u_\alpha + \Delta^\mu_\alpha}{T} u_\beta \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\alpha p_i^\beta \delta f_i \\ &\quad - \mu_B \frac{u^\mu u_\alpha + \Delta^\mu_\alpha}{T} \sum_i \int \frac{b_i g_i d^3p}{(2\pi)^3 E_i} p_i^\alpha \delta f_i. \end{aligned} \tag{A1}$$

Here we neglect higher-order terms in the third line for simplicity. By using the Landau matching conditions, we can eliminate correction terms proportional to u^μ:

$$\begin{aligned} s^\mu &= s u^\mu + \frac{u^\mu}{T} u_\alpha \delta T^{\alpha\beta} u_\beta + \frac{1}{T} \Delta^\mu_\alpha u_\beta \delta T^{\alpha\beta} \\ &\quad - \mu_B \frac{u^\mu}{T} u_\alpha \delta N_B^\alpha - \mu_B \frac{1}{T} \Delta^\mu_\alpha \delta N_B^\alpha \\ &= s u^\mu + \frac{W^\mu - \mu_B V^\mu}{T}. \end{aligned} \tag{A2}$$

This expression includes only nonequilibrium corrections of the first order with respect to dissipative currents, which are perpendicular to u^μ. This is exactly the generalization from the entropy current in ideal hydrodynamics to the one in viscous hydrodynamics, as discussed in Ref. [30]:

$$\begin{aligned} s^\mu &= \frac{s}{n_B} N_B^\mu + \frac{q^\mu}{T} \\ &= \frac{P u^\mu + T^{\mu\nu} u_\nu - \mu_B N_B^\mu}{T}. \end{aligned} \tag{A3}$$

APPENDIX B: ANALYTIC EXPRESSIONS OF SPECTRA

We can analytically express the corrections from both shear viscosity and bulk viscosity to the Cooper-Frye formula for the Björken model in the Boltzmann approximation, following Ref. [35]. The flow and the freezeout hypersurface of the Björken scaling solution with cylindrical geometry are

$$u^\tau = 1, \quad u^r = u^\phi = u^{\eta_s} = 0, \tag{B1}$$

and

$$d\sigma_\tau = \tau d\eta_s r dr d\phi, \quad d\sigma_r = d\sigma_\phi = d\sigma_{\eta_s} = 0. \tag{B2}$$

Because the momentum p^τ is given by $m_T \cosh(y-\eta_s)$, where m_T is the transverse mass, the particle spectrum is written as

$$\begin{aligned} \frac{d^2N}{d^2p_T dy} &= \frac{g}{(2\pi)^3} \int_0^{R_0} r dr \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \tau d\eta_s \\ &\quad \times m_T \cosh(y-\eta_s) f. \end{aligned} \tag{B3}$$

Here we dropped the index i of particle species for simplicity. The viscous corrections are now analytically expressed as

$$\begin{aligned} \frac{d^2\delta N_{\rm bulk}}{d^2p_T dy} &= \frac{g}{(2\pi)^3} \pi R_0^2 \cdot \tau m_T 2K_1(x) \\ &\quad \times \frac{\zeta}{\tau} \left\{ B_0 m^2 + \frac{D_0}{2} m_T \frac{K_2(x)+1}{K_1(x)} \right. \\ &\quad \left. + \frac{\tilde{B}_0 - B_0}{4} m_T^2 \left[\frac{K_3(x)}{K_1(x)} + 3 \right] \right\} \end{aligned} \tag{B4}$$

and

$$\begin{aligned} \frac{d^2\delta N_{\rm shear}}{d^2p_T dy} &= \frac{g}{(2\pi)^3} \pi R_0^2 \cdot \tau m_T 2K_1(x) \\ &\quad \times \frac{2\eta}{3\tau} B_2 \left\{ \frac{m_T^2}{2} \left[\frac{K_3(x)}{K_1(x)} - 1 \right] - p_T^2 \right\}, \end{aligned} \tag{B5}$$

where $x = \frac{m_T}{T}$ and the modified Bessel function is defined as

$$K_n(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(nt)\, dt. \tag{B6}$$

The bulk pressure and the shear stress tensor are estimated in first-order theory.

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