Renormalization of the off-shell chiral two-pion exchange NN interactions

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The renormalization, finiteness, and off-shellness of short distance inverse-power singular interactions are discussed. We show analytically that the renormalizability of the off-shell scattering amplitude relies completely on the corresponding on-shell amplitude without proliferation of new counterterms. We illustrate the result by complementary calculations both in coordinate as well as in momentum space in the simplest $^1S_0$ channel for chiral np interactions including two pion exchange.

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A traditional source of theoretical uncertainty in the study of nuclear physics and nuclear reactions has been the relevance and significance of off-shellness in the NN force (for a review up to the mid-1970s see, e.g., [1] and references therein). As it is well known, any off-shell ambiguity should cancel in the final results and off-shellness itself cannot be measured as a matter of principle. This does not necessarily mean, however, that off-shellness can generally be completely disposed of and most few- and many-body calculations do involve off-shell quantities as intermediate stages. This feature relies on the fundamental fact that quantum mechanics is naturally formulated in terms of wave functions while they are not directly measurable quantities except at asymptotically large distances. A parallel statement for quantum field theory applies for the fields themselves as well as the associated Green functions.

Potential approaches to the NN interaction need the half off-shell extrapolated potential and the half off-shell T-matrix is used to determine the on-shell S-matrix. Moreover, a knowledge of the off-shell T-matrix is needed, e.g., for nucleon-nucleon bremsstrahlung, the three nucleon problem as well as nuclear matter calculations and thus a phenomenological determination of the off-shell T-matrix has been the subject of intense research in the past [1]. A relevant issue in this regard is that the definition of off-shellness is largely conventional. Actually, the quantum mechanical trading between two-body off-shellness and three- and many-body forces was shown in Ref. [2], and further discussed for potential models [3] and within Lagrangian field theory [4]. Moreover, unitarity for the three-body problem rests on off-shell unitarity for the two-body problem, imposing constraints on the acceptable off-shellness [5,6].

Within the effective field theory (EFT) approach to nuclear physics based on chiral symmetry [7,8] (for comprehensive reviews see, e.g., Refs. [9–11]), the ambiguities related to off-shellness can be rephrased in the freedom to undertake field dependent transformations and using the equations of motion. Actually, in purely contact EFT's, where the interaction is represented by a polynomial in momenta and/or energy, off-shellness can be completely ignored from the start by using local field redefinitions [4,12]. This fact does not generally hold for finite range interactions stemming from particle exchange if the exchanged momentum becomes comparable to the exchanged particle mass and where nonlocal and singular field redefinitions would be needed. Similarly to phenomenological potentials, chiral potentials are not free from these ambiguities since by construction they are extracted from the (on-shell) S-matrix with a given perturbative definition and it is possible in particular to choose either energy [8] or momentum [13] dependent forms by using the on-shell condition (see also Ref. [14]). Further ambiguities and their equivalence have been discussed in Ref. [15]. Quite generally chiral potentials are based on an expansion in inverse powers of f_π (the pion weak decay constant) and M_N (the nucleon mass) and are necessarily singular at short distances by purely dimensional reasons:

$$V(r) \to \frac{1}{M_N^m f_\pi^n r^{n+m+1}}. \quad (1)$$

The problem on how these singularities should be handled from a renormalization point of view has been addressed in a series of works for the *on-shell* case where the finiteness can be established *a priori* [16,17]. In the subtractive renormalization method conducted in momentum space [18–20] numerical results seems to suggest that off-shell amplitudes are finite, a fact proven recently [21]. Still, an analytic complementary proof would be timely since there exist examples where a direct calculation of a Green functions in effective field theory does not necessarily guarantee off-shell finiteness from on shell renormalization conditions (see, e.g., Ref. [22]) and suitable field redefinitions may be requested to ensure off-shell renormalizability.

The hard core problem was the first singular potential which was treated by van Leeuwen and Reiner in the early 1960s [23]. It was shown that a finite and smooth result for the off-shell scattering amplitude could be achieved if the hard core boundary condition was also fulfilled by the off-shell wave functions. Let us note that for a nonsingular potential the standard way of going off-shell is to keep the same *regular* boundary condition as in the on-shell case. In the present paper we exploit this idea of a common boundary condition both for the on-shell and off-shell states to show

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that finiteness rests on pure on-shell properties. Moreover, we check the off-shell equivalence between the boundary condition renormalization in coordinate and the counterterm renormalization in momentum space thus extending similar findings for the on-shell case [24,25].

In the c.m. frame, where the np kinetic energy is given by $E = p^2/M$, with $M = 2\mu_{np} = 2M_n M_p/(M_p + M_n)$, the scattering process is described by using the Lippmann-Schwinger equation

$$T(E) = V + VG_0T(E), \tag{2}$$

with V the potential operator and $G_0 = (E - H_0)^{-1}$ the resolvent of the free Hamiltonian and $T(E)$ the T-matrix for energy E. The outgoing boundary condition corresponds to $E \to E + i0^+$. Using the normalization $\langle\vec{x}|\vec{k}\rangle = e^{i\vec{k}\cdot\vec{x}}/(2\pi)^{3/2}$ one has

$$\langle\vec{k}'|T(E)|\vec{k}\rangle = \langle\vec{k}'|V|\vec{k}\rangle + \int^{\Lambda} d^3q \frac{\langle\vec{k}'|V|\vec{q}\rangle\langle\vec{q}|T(E)|\vec{k}\rangle}{E - (q^2/2\mu)}. \tag{3}$$

Here Λ means a generic regulator and represents the scale below which all physical effects are taken into account *explicitly*. Here we assume that $|\vec{k}\rangle$ and $|\vec{k}'\rangle$ are plane-wave states with energies, $E_k = k^2/(2\mu)$, $E_{k'} = k'^2/(2\mu)$ different from $E_p = p^2/(2\mu)$.

In coordinate space this corresponds to solve the inhomogeneous Schrödinger equation,

$$-\frac{1}{M}\nabla^2\Psi(\vec{x}) + V(\vec{x})\Psi(\vec{x}) = E_p\Psi(\vec{x}) + (E_k - E_p)e^{i\vec{k}'\cdot\vec{x}}, \tag{4}$$

with the outgoing boundary condition

$$\Psi(\vec{x}) \to \left[e^{i\vec{k}\cdot\vec{x}} + f_p(\vec{k}',\vec{k})\frac{e^{ipr}}{r}\right]\chi^{s,m_s}_{t,m_t}, \tag{5}$$

with $f(\hat{k}',\hat{k})$ the quantum mechanical off-shell scattering matrix amplitude and χ^{s,m_s}_{t,m_t} a 4×4 spin-isospin state.

Our points are best illustrated in the simplest $^1S_0$ channel the extension to other channels being straightforward but cumbersome. In the $^1S_0$ channel the scattering process is governed by the Lippmann-Schwinger equation

$$T_p(k',k) = V(k',k) + \int_0^{\Lambda} dq q^2 M \frac{V(k',q)T_p(q,k)}{p^2 - q^2 + i0^+}, \tag{6}$$

where $T_p(k',k)$ and $V(k',k)$ are the scattering amplitude and the potential matrix elements, respectively, between off-shell momentum states k and k' in that channel. From the on-shell scattering amplitude the phase shift, $\delta_0(p)$, can be readily obtained

$$T_p(p,p) = -\frac{2}{\pi Mp}e^{i\delta_0(p)}\sin\delta_0(p). \tag{7}$$

The LS equation (6) is solved by standard matrix inversion techniques. The momentum space renormalization was addressed in Ref. [24] where we refer for further details.

In coordinate space and for a *local* potential $V(r)$ the on-shell problem can be determined without explicit reference to off-shell momentum information, although the wave function is obtained in the nonobservable interacting region. To simplify the discussion we consider the half-off-shell problem (the full off-shell case can be extended along similar lines), which in the $^1S_0$ channel follows from projecting Eqs. (4) and (5) onto partial waves and setting $k' = k$. One has to solve the Schrödinger equation (primes denote derivative with respect to the radial coordinate r) and asymptotic condition

$$-u_p''(r,k) + U(r)u_p(r,k) = p^2u_p(r,k) + (k^2 - p^2)\sin(kr), \tag{8}$$

$$u_p(r,k) \to \sin(kr) - kK(p,k)\cos(pr). \tag{9}$$

Here $U(r) = 2\mu_{np}V(r)$ is the reduced potential [in fact, the Fourier transformation of $V(q)$] and $u_p(r,k)$ the reduced wave function for an s-wave state with energy p^2/M and momentum k. In the on-shell case one has

$$u_p(r) \equiv u_p(r,p), \qquad K(p,p) = -\frac{\tan\delta_0(p)}{p}. \tag{10}$$

Anticipating the singular character of chiral potentials [see, e.g., Eq. (1)] these equations are solved for $r > r_c$ where r_c is the short distance cutoff which will eventually be removed, $r_c \to 0$, and the reduced wave function $u_p(r,k)$ is subject to a suitable boundary condition at $r = r_c$. What should this boundary condition be? For the finite energy case, $p \neq 0$, it was argued [26] that completeness of the on-shell wave functions $u_p(r)$ requires a common domain for the Hilbert space, requiring

$$\frac{u_p'(r_c)}{u_p(r_c)} = \frac{u_0'(r_c)}{u_0(r_c)}, \tag{11}$$

where the zero energy on-shell problem fulfills

$$-u_0''(r) + U(r)u_0(r) = 0, \qquad r \geqslant r_c, \tag{12}$$

$$u_0(r) \to 1 - \frac{r}{\alpha_0}. \tag{13}$$

The extended off-shell requirement of a common domain of wave functions in the physical Hilbert space

$$\frac{u_p'(r_c,k)}{u_p(r_c,k)} = \frac{u_0'(r_c)}{u_0(r_c)}, \tag{14}$$

and correspond to the requirement of completeness of the on-shell wave functions $u_p(r)$. Mathematically this corresponds to a one-parameter self-adjoint extension of the Hamiltonian for this partial wave.

We analyze first the simplest case with no potential which in momentum space corresponds to a contact interaction [24]. Since there is no potential the solutions for $r > r_c$ coincide with the asymptotic ones, see Eqs. (9) and (13). Using the relation between the on-shell and off-shell wave functions at $r = r_c$, Eq. (14), we get

$$K(p,k) = \frac{1}{k}\frac{k(\alpha_0 - r_c)\cos(kr_c) + \sin(kr_c)}{\cos(pr_c) + (r_c - \alpha_0)p\sin(pr_c)}. \tag{15}$$

Note that if $r_c \to 0$ the off-shellness disappears, i.e., we have $K(p,k) \to K(p,p)$. The momentum space analysis in

Ref. [24] yields the same conclusion. This result suggests that by removing the cutoff we may get rid of the unwanted off-shell ambiguities.

We turn now to the case of a potential with finite range. Recently, the finiteness and equivalence of the momentum and coordinate formulations of the renormalization problem for on-shell scattering [24] and the deuteron bound state [25] has been established. Here we extend those results to the off-shell case. To this end we follow the insight of previous works [16,17] and use the superposition principle of boundary conditions. The half off-shell wave function $u_p(r,k)$ can be written as

$$u_p(r,k) = v_p(r,k) - kK(p,k)w_p(r,k), \tag{16}$$

where $v_p(r,k)$ and $w_p(r,k)$ are two auxiliary wave functions fulfilling

$$\begin{aligned} -v_p''(r,k) + U(r)v_p(r,k) &= p^2 v_p(r,k) + (k^2 - p^2)\sin(kr), \\ v_p(r,k) &\to \sin(kr), \end{aligned} \tag{17}$$

and

$$\begin{aligned} -w_p''(r,k) + U(r)w_p(r,k) &= p^2 w_p(r,k), \\ w_p(r,k) &\to \cos(pr), \end{aligned} \tag{18}$$

respectively. Our aim is to show that for a singular potential the function $K(p,k)$ used in Eq. (9) is finite when the short distance cutoff is removed, $r_c \to 0$. We analyze here the case of a power like *attractive* potential, corresponding to the $^1S_0$ channel considered in the present paper and represented as $2\mu_{pn}V(r)R^2 \to -(R/r)^n$ where R is a short distance Van der Waals scale and $n = 5, 6, 7$ corresponds to NLO, N2LO, and N3LO, respectively (see Eq. (1) and Ref. [26] for explicit expressions for R). At short distances a WKB approximation applies [27,28] and one can show that for $r \ll R$ one has two independent regular solutions

$$\mathcal{C}(r) = \left(\frac{r}{R}\right)^{n/4} \cos\left[\frac{2}{n-2}\left(\frac{R}{r}\right)^{n/2-1}\right], \tag{19}$$

$$\mathcal{S}(r) = \left(\frac{r}{R}\right)^{n/4} \sin\left[\frac{2}{n-2}\left(\frac{R}{r}\right)^{n/2-1}\right], \tag{20}$$

fulfilling the Wronskian normalization $\mathcal{C}'(r)\mathcal{S}(r) - \mathcal{C}(r)\mathcal{S}'(r) = 1/R$. Note that these asymptotic solutions are both energy and momentum independent. In terms of these short distance solutions we must necessarily have at short distances

$$v_p(r,k) \to A(p,k)\mathcal{C}(r) + B(p,k)\mathcal{S}(r), \tag{21}$$

$$w_p(r,k) \to C(p,k)\mathcal{C}(r) + D(p,k)\mathcal{S}(r), \tag{22}$$

$$v_0(r) \to a\,\mathcal{C}(r) + b\,\mathcal{S}(r), \tag{23}$$

$$w_0(r) \to c\,\mathcal{C}(r) + d\,\mathcal{S}(r), \tag{24}$$

where $A(p,k)$, $B(p,k)$, $C(p,k)$, and $D(p,k)$ are suitable energy and momentum dependent normalization constants, and $a = A(0,0)$, $b = B(0,0)$, $c = C(0,0)$, and $d = D(0,0)$ the corresponding constants for the on-shell zero energy problem. From the above equations and the short distance boundary condition, Eq. (14), it is straightforward to obtain in the limit $r_c \to 0$ the result

$$kK(p,k) = \frac{\alpha_0 \mathcal{A}(p,k) + \mathcal{B}(p,k)}{\alpha_0 \mathcal{C}(p,k) + \mathcal{D}(p,k)} \tag{25}$$

with

$$\mathcal{A}(p,k) = bA(p,k) - aB(p,k), \tag{26}$$

$$\mathcal{B}(p,k) = cB(p,k) - dA(p,k), \tag{27}$$

$$\mathcal{C}(p,k) = bC(p,k) - aD(p,k), \tag{28}$$

$$\mathcal{D}(p,k) = cD(p,k) - dC(p,k), \tag{29}$$

which shows explicitly the finiteness of the result. Actually, using the subdominant short distance corrections to the wave functions we can show that the finite cut-off effect scales as $\mathcal{O}(r_c^{n/2-1})$ corresponding for $n = 5, 6, 7$ to a fast convergence. The previous off-shell relation is a straightforward generalization of the on-shell result found in Ref. [26]. Similarly to that case, the off-shell functions $\mathcal{A}(p,k)$, $\mathcal{B}(p,k)$, $\mathcal{C}(p,k)$, and $\mathcal{D}(p,k)$ depend by construction on the potential only. The remarkable feature is the explicit bilinear dependence on the scattering length, α_0. The generalization of the previous result to higher partial waves and coupled channels is quite straightforward but cumbersome.

We turn now to the numerical results. For details on potentials and parameter choices we refer to Ref. [24]. In Fig. 1 we show the results for the renormalized half-off shell K-matrix for a fixed value of the laboratory (lab) energy as a function of the c.m. momentum. We do so for NLO, N2LO, and N3LO, where the potential diverges as $1/r^5$, $1/r^6$, and $1/r^7$, respectively. We have checked the consistency between coordinate and momentum space NLO and N2LO results provided the same renormalization conditions are imposed. This confirms the adequacy of requesting the common boundary condition for both on-shell and off-shell states, Eq. (14). Besides the sharp cut-off method, we have also tried a gaussian cutoff, with similar results for off-shell momenta well below the cut-off range. Despite the strong short distance singularities

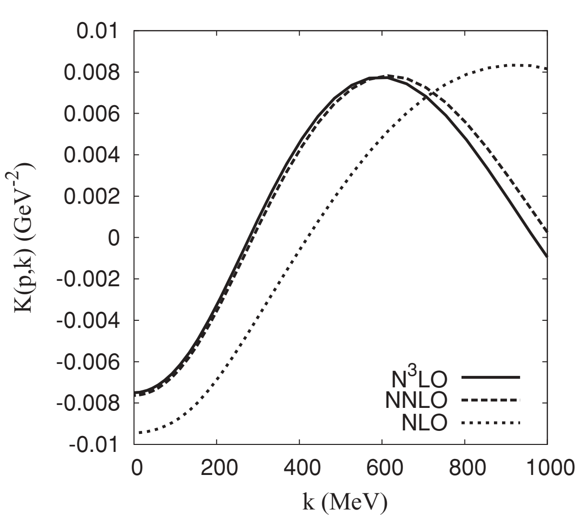

FIG. 1. Renormalized half off-shell K-matrix as a function of the c.m. momentum (in MeV) for $T_{\text{lab}} = 50$ MeV. The on-shell point corresponds to $p = 153.2$ MeV.

our renormalized K-matrix does not exhibit any pathological behavior and looks as smooth as other phenomenological and nonsingular potentials.

We summarize our points. We have shown that the renormalizability and finiteness of the off-shell scattering amplitude for power such as short distance singular potentials rests solely on purely on-shell information. The outgoing amplitudes are well behaved and soft despite the underlying short distance singularity being renormalized. This complies to the desirable expectation that after renormalization all short distance sensitivity has largely disappeared, possibly including off-shell ambiguities. Obviously, we cannot compare our results directly to any experimental quantity as off-shellness cannot be pinned down by definition. The impossibility of measuring off-shell effects directly has been emphasized in Ref. [29] mainly due to the freedom in defining the physical interpolating field (see also [30]). While renormalized chiral interactions might be phenomenologically tested by undertaking three-body, pp-bremsstrahlung, or nuclear matter calculations, it is natural to expect many difficulties. Our results suggest a viable and simpler alternative where the underlying two body singularities are tamed first through off-shell renormalization and the specific additional complications of the problem when more than two bodies are present can be tackled afterwards.

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